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A NOTE ON EQUIVARIANT EMBEDDINGS OF GRASSMANNIANS

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Abstract. Embeddings of Grassmannians realized by identification of a k-dimensional subspace of $\mathbb{F}^m$ with the orthogonal projection onto it are studied. It is shown that such an embedding has parallel second fundamental form and embeds the Grassmannian minimally into a hypersphere of certain Euclidean space of matrices. A result on holomorphic sectional curvature of the complex Grassmannian is also proved.

Since the seminal work of Tai [**11**] and Sakamoto [**10**], it has been known that projective spaces can be embedded isometrically and equivariantly into suitable Euclidean spaces of matrices with parallel second fundamental form, and such embeddings have been subsequently studied from different points of view (see [**9**], [**1**]). Since projective spaces are just Grassmannians of one-dimensional $\mathbb{F}$-lines it is natural to ask how much of that work on projective spaces one can generalize to arbitrary Grassmannians. In this note we show that one can construct the same kind of Veronese-type embeddings which will satisfy many properties common to projective spaces such as having the second fundamental form parallel. It will be also shown that such maps embed Grassmannians as minimal submanifolds in hyperspheres of certain radii. On the other hand, unlike the case of projective spaces these embeddings of Grassmannians of higher rank are not constant isotropic. At the end we produce a formula for geodesics and prove a theorem on the holomorphic sectional curvature of the complex Grassmannian.

Let $\mathbb{F}$ be one of the fields $\mathbb{R}$ (reals), $\mathbb{C}$ (complex numbers) or $\mathbb{H}$ (quaternions) with a natural inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$ and let $d = \dim_R \mathbb{F} \in \{1, 2, 4\}$. We denote by $\mathbb{F}^m$ the right vector space of column m-vectors with entries in $\mathbb{F}$ and by $G = G_k(\mathbb{F}^m)$ the Grassmannian of k-dimensional linear subspaces (through the origin) of $\mathbb{F}^m$. Let $\alpha(z, w) = \sum_{i=1}^m \bar{z}^i w^i$ be the standard $\mathbb{F}$-Hermitean symmetric inner product, where $z = (z^i)$, $w = (w^i) \in \mathbb{F}^m$, and let $U(m) = \{A \in M_m(\mathbb{F}) \,|\, AA^* = A^*A = I\}$

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be the set of $\mathbb{F}$-unitary matrices satisfying $\alpha(Az, Aw) = \alpha(z, w)$. Let $V_k(\mathbb{F}^m) = U(m)/U(m-k)$ be the Stiefel manifold of $\mathbb{F}$-unitary k-frames. Then there is a natural projection $\pi : V_k(\mathbb{F}^m) \to G_k(\mathbb{F}^m)$, $\mathbf{f} = (e_1, \ldots, e_k) \mapsto \mathrm{span}_F\{e_1, \ldots, e_k\}$ (see [**5**, Ch. 7] for details). Each unitary frame $\mathbf{f} = (z_1, \ldots, z_k) \in V_k(\mathbb{F}^m)$ is naturally identified with the $m \times k$ matrix $(z_1|\ldots|z_k)$ whose columns are formed by the vectors of the frame $\mathbf{f}$, and the group $U(k)$ acts on $V_k(\mathbb{F}^m)$ by multiplying a frame matrix on the right with an element in $U(k)$ (cf. [**8**, Example IX. 6.4]). Thus if $\mathbf{f}_1 = (z_1, \ldots, z_k)$ and $\mathbf{f}_2 = (w_1, \ldots, w_k)$ where $\mathbf{f}_2 = \mathbf{f}_1 A$, $A = (a^i_j) \in U(k)$, it follows that $w_r = \sum_i z_i a^i_r$ so that these two frames span the same subspace. As is well known, $G_k(\mathbb{F}^m) = U(m)/U(k) \times U(n)$ is a symmetric space of $\mathbb{F}$-dimension kn and of rank $\min(k, n)$ where $n = m - k$. Let $H(m) = \{A \in M_m(\mathbb{F}) \,|\, A^* = A\}$ be the set of $\mathbb{F}$-Hermitean matrices, where $A^* = \bar{A}^t$ denotes the conjugate transpose. Then $H(m)$ is given a Euclidean structure by the metric $g(A, B) = \frac{1}{2}\,\mathrm{Re}\,\mathrm{tr}_F(AB)$, whereupon it becomes a Euclidean space E^N of real dimension $N = m + dm(m-1)/2$. Note that Re is needed only in the quaternion case. We identify each k-dimensional $\mathbb{F}$-plane L with the operator of orthogonal projection P (equivalently, its matrix) onto L with respect to the inner product α. Namely, if $\mathbf{f} = (z_1, \ldots, z_k) \in \pi^{-1}(L)$ is any chosen unitary basis for L then

$$P(v) = \sum_{r=1}^{k} z_r\, \alpha(z_r, v) = \left(\sum_{r=1}^{k} z_r z_r^*\right) v, \quad v \in \mathbb{F}^m. \tag{1}$$

This projection is nothing other but the Euclidean projection in $\mathbb{R}^{md} \cong \mathbb{F}^m$ onto real kd-dimensional plane L with respect to the inner product $\mathrm{Re}\,\alpha$. The following proposition follows directly from (1) (see also [**4**, p. 80]).

PROPOSITION 1. *P is an $\mathbb{F}$-linear map satisfying* (i) $P^2 = P$ (ii) $P^* = P$, *i.e.* $\alpha(Pz, w) = \alpha(z, Pw)$ (iii) $\mathrm{tr}_{\mathbb{F}} P = k$. *Conversely, any endomorphism P satisfying* (i)–(iii) *is a projection onto some k-dimensional $\mathbb{F}$-plane.*

Indeed, if $P \in \mathrm{End}(\mathbb{F}^m)$ such that $P^2 = P$, $P = P^*$ than P has eigenvalues 0 and 1. Note that according to [**11**, Appendix] such matrix can be diagonalized. Let $L = \{z \,|\, Pz = z\}$. Then $\mathrm{tr}_{\mathbb{F}} P = \dim_{\mathbb{F}} L = k$ and P is the projection onto L.

In terms of a unitary frame $\mathbf{f} = (z_1, \ldots, z_k)$ spanning L we have $P = (p_{ij})$ where $p_{ij} = \sum_{r=1}^{k} z^i_r \bar{z}^j_r$ = product of ith and jth row of the matrix $(z_1|\cdots|z_k)$. This identification of an $\mathbb{F}$-plane L with the orthogonal projection P_L onto it, gives rise to the embedding $\Phi : G_k(\mathbb{F}^m) \to H(m)$, $L \mapsto P_L$ with the image

$$\Phi(G_k(\mathbb{F}^m)) = \{P \in H(m) \,|\, P^2 = P,\ \mathrm{tr}_{\mathbb{F}} P = k\}.$$

The group $U(m)$ acts on the Stiefel manifold $V_k(\mathbb{F}^m)$ by left multiplication of each vector of a frame and thus acts naturaly on equivalence classes of frames in $G_k(\mathbb{F}^m)$. Since

$$\Phi(A[\mathbf{f}]) = \Phi([Az_1, \ldots, Az_k]) = \sum_{r=1}^{k} (Az_r)(Az_r)^* = \sum_{r=1}^{k} A z_r z_r^* A^* = A\Phi([\mathbf{f}])A^{-1}$$

for $A \in U(m)$, we see that the embedding Φ is $U(m)$-equivariant. We also see that the trace metric g is invariant with respect to the action of $U(m)$ on $H(m)$ by conjugation. Regarding the metrics on $U(m)/U(k) \times U(n) = G_k(\mathbb{F}^m)$, we know that there exists a unique (up to a constant factor) $U(m)$-invariant one, which is just a multiple of Killing form. This is true even in the reducible case of $G_2(\mathbb{R}^4)$ which has a 2-parametric family of $SO(4)$-invariant metrics [**6**]. Thus we can choose a constant multiple to make Φ an isometry. These facts enable us to do the computations locally, at a suitably chosen point. That point, which we will refer to as the origin of G, is taken to be $P_0 = \left(\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right)$. From now on all matrices in $H(m)$ will have the block form where the submatrix in the upper left corner is of order $k \times k$. By a dimension count, one sees that the tangent and normal spaces are given by

$$T_P G = \{X \in H(m) \mid XP + PX = X\}, \qquad T_P^{\perp} G = \{\xi \in H(m) \mid \xi P = P\xi\} \quad (2)$$

so that a representative tangent and normal vectors at the origin P_0 have the respective forms

$$X = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} Q & 0 \\ 0 & S \end{pmatrix}, \text{ where } B \in M_{n\times k}(\mathbb{F}), \ \ Q \in H(k), \ \ S \in H(n).$$

Let $\tilde{\nabla}$, ∇, D denote the Riemannian connections respectively in $H(m)$, $G_k(\mathbb{F}^m)$, and the normal bundle of $G_k(\mathbb{F}^m)$, in $H(m)$ and let σ and Λ_ξ denote the second fundamental form of Φ and the shape operator in the direction ξ. Then from the above as in [**9**], [**1**], one has

PROPOSITION 2. *The fol lowing formulas hold for any vector fields X, Y tangent to a Grassmannian and a normal vector field ξ:*

(i) $\nabla_X Y = 2(XY + YX)P + P(\tilde{\nabla}_X Y) + (\tilde{\nabla}_X Y)P$

(ii) $\sigma(X, Y) = (XY + YX)(I - 2P)$

(iii) $\Lambda_\xi X = (X\xi - \xi X)(I - 2P)$

(iv) $D_X \xi = \tilde{\nabla}_X \xi + 2P(\tilde{\nabla}_X \xi)P - (\tilde{\nabla}_X \xi)P - P(\tilde{\nabla}_X \xi)$.

Next we prove that the second fundamental form is covariantly constant.

PROPOSITION 3. *The second fundamental form of $G_k(\mathbb{F}^m)$ in $H(m)$ is parallel, i.e. $\nabla\sigma = 0$.*

Proof. Let X, Y, W denote arbitrary tangent vector fields and denote $X \circ Y := XY + YX$. From (2) it follows that

$$PWP = 0, \quad (I - 2P)P = -P, \quad (I - 2P)X = -X(I - 2P),$$
$$\sigma(X, Y)(I - 2P) = (I - 2P)\sigma(X, Y), \quad PXY = XYP$$

and thus also $P(X \circ Y) = (X \circ Y)P$. Note that since P denotes the position vector of $G_k(\mathbb{F}^m)$ we have $\tilde{\nabla}_W P = W$. Since $\tilde{\nabla}$ is a Euclidean connection (directional derivative), then $\tilde{\nabla}_W (X \circ Y) = (\tilde{\nabla}_W X) \circ Y + X \circ (\tilde{\nabla}_W Y)$ by the product rule.

Using these relations and Proposition 2(ii), we have first

$$\begin{aligned}\tilde\nabla_W(\sigma(X,Y)) &= [(\tilde\nabla_W X)\circ Y + X\circ(\tilde\nabla_W Y)](I-2P) - 2(X\circ Y)W\\ &= \sigma(\nabla_W X, Y) + \sigma(X,\nabla_W Y)\\ &\quad + [\sigma(W,X)\circ Y + X\circ\sigma(W,Y)](I-2P) - 2(X\circ Y)W\\ &= \sigma(\nabla_W X, Y) + \sigma(X,\nabla_W Y) - (X\circ Y)\circ W\end{aligned}$$

because by Proposition 2(ii) again

$$\begin{aligned}&[\sigma(W,X)\circ Y + X\circ\sigma(W,Y)](I-2P)\\ &\qquad = Y(W\circ X) - (W\circ X)Y + X(W\circ Y) - (W\circ Y)X\\ &\qquad = (X\circ Y)W - W(X\circ Y).\end{aligned}$$

On the other hand, by Proposition 2(iii)

$$\Lambda_{\sigma(X,Y)}W = [W\sigma(X,Y) - \sigma(X,Y)W](I-2P) = W(X\circ Y) + (X\circ Y)W = (X\circ Y)\circ W.$$

Thus, $\tilde\nabla_W(\sigma(X,Y)) = \sigma(\nabla_W X, Y) + \sigma(X,\nabla_W Y) - \Lambda_{\sigma(X,Y)}W$. By comparing normal parts on both sides we conclude that

$$D_W(\sigma(X,Y)) = \sigma(\nabla_W X, Y) + \sigma(X,\nabla_W Y) \quad \text{i.e. } \nabla_W\sigma = 0. \quad \square$$

Let us next compute the mean curvature vector $\mathbf{H}$ of the embedding Φ. We designate the following range for the indices: $1 \le a, b \le n$, $1 \le r, s \le k$, $u, v \in \{1, i, j, k\}$ where i, j, k denote the imaginary units of $\mathbb{H}$. Let E^u_{ar} denote the matrix $E^u_{ar} = \begin{pmatrix} 0 & M^{u\,*}_{ar} \\ M^u_{ar} & 0 \end{pmatrix}$ where all entries of M^u_{ar} are zeros except (a,r) entry which equals u. Then a direct computation shows that

$$g(E^u_{ar}, E^v_{bs}) = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}\,(E^u_{ar}\, E^v_{bs}) = \epsilon\,\mathrm{Re}(uv)\,\delta_{ab}\delta_{rs}$$

where $\epsilon = 1$ if $u = v = 1$ and $\epsilon = -1$ otherwise. Therefore $\{E^u_{ar}\}$ forms an ortonormal basis of $T_{P_0}G$. Hence

$$\begin{aligned}\mathbf{H}_{P_0} &= \frac{1}{dkn}\sum_{a,r,u}\sigma(E^u_{ar}, E^u_{ar}) = \frac{2}{dkn}\sum_{a,r,u}(E^u_{ar})^2(I-2P_0)\\ &= \frac{2}{kn}\begin{pmatrix} nI_k & 0 \\ 0 & kI_n\end{pmatrix}\begin{pmatrix} -I_k & 0 \\ 0 & I_n\end{pmatrix} = \frac{2}{kn}\begin{pmatrix} -nI_k & 0 \\ 0 & kI_n\end{pmatrix} = \frac{2}{kn}[kI-(k+n)P_0].\end{aligned}$$

Thus by equivariancy of Φ we have

$$\mathbf{H} = \frac{2}{kn}[kI - mP] = -\frac{2m}{kn}\,[P - \frac{k}{m}I]. \tag{3}$$

PROPOSITION 4. *The Grassmannian $\Phi(G_k(\mathbb{F}^m))$ is minimal submanifold of the hypersphere $S^{N-1}_{kI/m}\left(\sqrt{kn/2m}\right)$ and up to a translation, Φ gives an embedding by eigenfunctions from a single eigenspace.*

Proof. One readily computes

$$g\Big(P-\frac{k}{m}I,\ P-\frac{k}{m}I\Big)=\frac{1}{2}\mathrm{Re}\,\mathrm{tr}\,\Big(P^2-\frac{2k}{m}P+\frac{k^2}{m^2}I\Big)=\frac{kn}{2m}$$

so that $G_k(\mathbb{F}^m)$ belongs to the said sphere, and since the mean curvature vector $\mathbf{H}$ is parallel to the radius vector $P-\frac{k}{m}I$, the embedding is minimal into that sphere. Moreover, using the Beltrami formula $\Delta P=-(\dim G)\mathbf{H}$ ($\Delta=$ the Laplacean on G) and (3) we get

$$\Delta\,[P-(k/m)\,I]=\Delta P=-dkn\,\mathbf{H}=2md\,[P-(k/m)\,I],$$

so that the component functions of the immersion vector Φ translated to $(k/m)\,I$, are eigenfunctions of the Laplacean corresponding to the eigenvalue $\lambda=2md$. Thus, this embedding is of 1-type [**1**]. Note that for any $a=$const there exist a sphere centered at aI and of appropriate radius containing $\Phi(G_k(\mathbb{F}^m))$, but this Grassmannian is minimal only in the sphere centered at $(k/m)I$. $\square$

A few remarks are in order here. Proposition 4 can be thought of as a particular case of a general theory by which a compact (irreducible) homogeneous Riemannian manifold can be isometrically minimally immersed into a Euclidean sphere by eigenfunctions of its l^{th} eigenspace - the so called l^{th} standard embedding (see [**12**], [**14**]), but our results give natural and explicit embedding which is in the line of ideas of [**11**], [**9**] for projective spaces. Also, our Proposition 3 is consistent with a work of Ferus [**2**], [**3**], where it was proved that the equivariant embeddings of Grassmannians and other R-symmetric spaces constructed in [**7**], [**13**] have parallel second fundamental form as symmetric submanifolds. Our embeddings differ from those of [**7**] by a motion of the ambient space composed with a homothety. Our approach does not use the Lie algebra techniques and is more direct, relying on the natural identification of a plane and the projection operator onto it. Moreover, this approach seems to be more suitable for a study of submanifolds of Grassmannians as we illustrate next by example of geodesics.

Let $P(s)\subset G_k(\mathbb{F}^m)$ be a curve parametrized by its arclength and let $\dot P$ denote the derivative with respect to s. For a geodesic we have $\nabla_{\dot P}\dot P=0$ and therefore $\tilde\nabla_{\dot P}\dot P=\sigma(\dot P,\dot P)=2\dot P^2(I-2P)$ by Proposition 2(ii). Since $\tilde\nabla$ is nothing other then the directional derivative, we get the equation of a geodesic as follows:

$$\ddot P=2\dot P^2(I-2P),\qquad \mathrm{Re}\,\mathrm{tr}\,\dot P^2=2. \tag{4}$$

This equation, or the equivalent one, $\ddot PP=P\ddot P$ (indicating that $\ddot P$ is normal), should play an important role in the study of geodesics of a Grassmannian.

One can use Proposition 2 and the Gauss equation $R(X,Y)Z=\Lambda_{\sigma(Y,Z)}X-\Lambda_{\sigma(X,Z)}Y$ for a submanifold of $H(m)$ to find the curvature tensor of a Grassmannian as $R(X,Y)Z=[[X,Y],Z]$, where $[X,Y]=XY-YX$. This expression differs by a sign from the familiar expression for curvature tensor of a symmetric space since our tangent vectors are Hermitean symmetric rather than Hermitean skew. Thus when X, Y are unit mutually perpendicular tangent vectors, the sectional curvature has the form

$$K(X,Y) = \frac{1}{2} \operatorname{Re}\operatorname{tr}(X^2Y^2 + YX^2Y - 2(XY)^2). \tag{5}$$

Let us now pay closer attention to the complex Grassmannian $G_k(\mathbb{C}^m)$ and its holomorphic curvature. For $X = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} \in T_{P_0}G$ one defines the complex structure J by $JX = \begin{pmatrix} 0 & (iB)^* \\ iB & 0 \end{pmatrix} = \begin{pmatrix} 0 & -iB^* \\ iB & 0 \end{pmatrix}$ which equals $i\,(I-2P_0)\,X$ where $i = \sqrt{-1}$. By equivariancy then $JX = i\,(I-2P)\,X$ at every point.The holomorphic sectional curvature in the direction of a unit vector X is defined by $K_H(X) = K(X, JX)$. Using this we give a new proof of some results of Wong [**15**].

Proposition 5. (i) $K_H(X) = 2\ \operatorname{tr} X^4$. (ii) *The holomorphic sectional curvature of $G_k(\mathbb{C}^m)$ for a direction tangent to a geodesic is constant along the geodesic.* (iii) *The holomorphic sectional curvature is $1/r$-pinched, where $r = \min(k,n)$ is the rank of $G_k(\mathbb{C}^m)$. More precisely, $4/r \le K_H \le 4$.*

Proof. (i) Since $(I-2P)X = -X(I-2P)$, $(I-2P)^2 = I$ and $JX = i(I-2P)$, from (5) we get

$$K(X,JX) = \frac{1}{2}\operatorname{tr}(X^2(JX)^2 + (JX)X^2(JX) - 2\,X(JX)X(JX)) = 2\ \operatorname{tr} X^4.$$

(ii) Let $X = \dot{P}(s)$ be a unit tangent vector to a geodesic. Then $\nabla_X X = 0$, and

$$\begin{aligned} X\,K_H(X) &= 2X(\operatorname{tr} X^4) = 2\ \operatorname{tr} \tilde{\nabla}_X(X^4) = 8\ \operatorname{tr}[(\tilde{\nabla}_X X)X^3] \\ &= 8\ \operatorname{tr}[\sigma(X,X)X^3] = 16\,g(\sigma(X,X),X^3) = 0. \end{aligned}$$

Note that for a tangent vector X and a positive integer k, X^{2k} is normal and X^{2k-1} is tangent to the Grassmannian by (2); in particular, X^3 is tangent, thus perpendicular to $\sigma(X,X)$.

(iii) For an arbitrary complex matrix $Z = (z_i^j)$ denote its Euclidean square norm by $\|Z\|^2 = \sum_{i,j} |z_i^j|^2$. Let $X = \begin{pmatrix} 0 & B^* \\ B & 0 \end{pmatrix} \in T_{P_0}G$ be unit, i.e.

$$\frac{1}{2}\ \operatorname{tr}(B^*B + BB^*) = \sum |b_i^j|^2 = \|B\|^2 = 1.$$

Then $X^4 = \begin{pmatrix} C^2 & 0 \\ 0 & D^2 \end{pmatrix}$, where $C = B^*B \in H(k)$, $D = BB^* \in H(n)$ and $\operatorname{tr} C^2 = \operatorname{tr} D^2$. Suppose now that, for example, $n \le k$. Then $K_H(X) = 2\ \operatorname{tr} X^4 = 2\ \operatorname{tr}(C^2 + D^2) = 4\,\|D\|^2$. By Cauchy–Schwarz inequality $\|D\|^2 \le \|B\|^2\|B^*\|^2 = 1$ and hence $K_H(X) \le 4$. Let $\mathbf{b}_j$ denote the jth row of the matrix B, and let α_k denote the standard Hermitean inner product on $\mathbb{C}^k$. Then

$$\|D\|^2 = \sum_{i,j=1}^{n} |\alpha_k(\mathbf{b}_i,\mathbf{b}_j)|^2 \ge \sum_{j=1}^{n} |\alpha_k(\mathbf{b}_j,\mathbf{b}_j)|^2 = \sum_{j=1}^{n} \|\mathbf{b}_j\|^4 \ge \frac{1}{n}\Big(\sum_{j=1}^{n} \|\mathbf{b}_j\|^2\Big)^2 = \frac{1}{n}.$$

For the last inequality we used $\|\mathbf{b}_i\|^2\|\mathbf{b}_j\|^2 \le \frac{1}{2}(\|\mathbf{b}_i\|^4 + \|\mathbf{b}_j\|^4)$. Therefore $K_H(X) \ge 4/n = 4/r$, and by equivariancy these inequalities hold everywhere. □

We should remark that the parts (ii) and (iii) were known to Wong [**15**, Th. 15 and Corollaries 1, 2], also [**16**], where the results are stated without proofs. Our proof is different than the one that Wong had in mind, since it uses projection operators rather than his geometrical idea of angles between neighboring planes in $\mathbb{C}^m$.

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