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USER'S GUIDE TO EQUIVARIANT METHODS IN COMBINATORICS

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1. Introduction

The purpose of this article is to present a user friendly and reasonably detailed exposition of ideas and tools of equivariant Topology which have proven to be useful in Combinatorics. This paper can also be viewed as a continuation of the chapter "Topological methods" prepared for the CRC Handbook of Combinatorics, [**57**]. Our intention is to develop an overall picture emphasizing a typical scheme of applications. Some technical details, which are often skipped in existing expositions, are isolated and discussed. On the technical side we emphasize the role of the so called *cohomological index theory.* We hope that the clarification and systematization of ideas will encourage a nonexpert to use freely these tools in his or her own research.

2. A bridge between Geometry and Topology

The chapter "Topological methods", [**57**], starts with a general hint how to use topological methods in combinatorial and geometric problems. The hint suggests that a typical application consists of two steps.

Step 1: The problem should be rephrased in topological terms.

This often means that the problem should give us a clue how to define a "natural" *configuration space* X and to interpret the question as a question about *coincidences of continuous maps on* X or *nonempty intersection of subsets of* X.

Step 2: The rephrased problem is solved by one of the standard topological techniques.

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Key words and phrases:

- **configuration space $X_{\mathcal{P}}$, test space $V_{\mathcal{P}}$ and test map:** Configuration space is the space of all "candidates" for a solution of a geometric problem $\mathcal{P}$. How far a "candidate" is from a solution is tested by a test map.
- **geometric, topological and combinatorial join:** These are operation often used in encoding the geometric data in a topological configuration space.
- **deleted join of simplicial complexes:** A special and particularly useful form of join.

2.1. How to solve it?

The reader remembers the book "How to solve it" by George Polya where the author collected and analyzed typical ideas and heuristics used in mathematical problem solving. Our goal is more modest. We concentrate on a single proof scheme which has been successfully applied in combinatorics and discrete geometry. This characteristic scheme provides a bridge between the problem itself and a topological question, typically a problem of the existence of an equivariant map, see section 3.

How does a typical combinatorial or geometric problem $\mathcal{P}$ look like? In combinatorics it often involves a construction or a proof of the existence of a combinatorial object (partition, graph, arrangement) possessing some special properties. Similarly, in combinatorial geometry one tries to establish the existence of a special geometric configuration of points, lines or other geometric objects satisfying some special conditions. In both cases, the desired configuration is naturally viewed inside a **configuration space** $X_{\mathcal{P}}$, of more general configurations naturally associated to the problem $\mathcal{P}$. Then one defines a **test space** $V_{\mathcal{P}}$, which is often an euclidean space and its subspace $Z_{\mathcal{P}}$. If we are on the right track the test space $V_{\mathcal{P}}$ comes together with a **test map** $f : X_{\mathcal{P}} \to V_{\mathcal{P}}$ and the original problem $\mathcal{P}$ is found to be equivalent to the question whether $\mathrm{Im}(f) \cap Z_{\mathcal{P}} \neq \emptyset$ i.e., if there exists $x \in X_{\mathcal{P}}$ such that $f(x) \in Z_{\mathcal{P}}$. The inner symmetries of the problem $\mathcal{P}$ show up at this stage and lead to a finite group G which acts naturally on all spaces $X_{\mathcal{P}}, V_{\mathcal{P}}$ and $Z_{\mathcal{P}}$, so that the test map $f : X_{\mathcal{P}} \to V_{\mathcal{P}}$ is G-equivariant. Finally, the assumption that $\mathrm{Im}(f) \cap Z_{\mathcal{P}} = \emptyset$ leads to a G-equivariant map $\hat{f} : X_{\mathcal{P}} \to V_{\mathcal{P}} \setminus Z_{\mathcal{P}}$ and the methods of the equivariant topology should help us reach a contradiction.

In order to make the scheme outlined here more concrete, we review various forms of join operations which are often used in constructions of configuration spaces associated to a problem $\mathcal{P}$. After that we review how a typical test space shows up and how this construction arises as an answer to a naive question "when two sets in R^d have a nonempty intersection".

2.2. Geometric, topological and combinatorial join

Let A and B be two convex sets in R^d such that the supporting affine subspaces $\mathrm{aff}(A)$ and $\mathrm{aff}(B)$ are in general position. If $\dim(\mathrm{aff}(A)) + \dim(\mathrm{aff}(B)) < d$, then for any $x \in \operatorname{conv}(A \cup B) \setminus (A \cup B)$ there exist unique $a \in A$, $b \in B$ and $t \in [0,1]$ so that $x = t\,a + (1-t)\,b$. More generally, for any two sets A and B which are not necessarily convex, if $\operatorname{aff}(A)$ and $\operatorname{aff}(B)$ satisfy the conditions above, then $\bigcup\{[a,b] \mid a \in A,\, b \in B\}$, the union of all line segments with end points in A and B is called **the geometric join** of spaces A and B. If A and B are spaces which are not necessarily embedded in some R^d, we need the concept of a **topological join,** $A * B$, [**12**], [**17**], [**47**], of spaces A and B, which permits us not only to extend the concept of a geometric join but to preserve the usual notation and intuition. If $A = |K|$, and $B = |L|$ are geometric realizations of abstract simplicial complexes, then $A * B \cong |K * L|$ where $K * L$ is an abstract simplicial complex, called the **combinatorial join** of complexes K and L. Assuming that K and L are vertex disjoint, a typical simplex $\omega \in K * L$ is described as the union $\omega = \alpha \cup \beta$, of two simplices $\alpha \in K$ and $\beta \in L$. One easily checks that $|K * L| \cong |K| * |L|$. This is a reason why, by usual simplification and abuse of language, we simply say **join** for all joins described above.

There is one more combinatorial join, defined for posets. If P and Q are posets then $P * Q$ is the set $P \cup Q$ with the order relation which extends the existing orders so that $\forall p \in P\, \forall q \in Q\; p \leq q$. If $\Delta(P)$ and $\Delta(Q)$ are the corresponding order complexes than obviously $\Delta(P * Q) \cong \Delta(P) * \Delta(Q)$ which means that this concept is a natural join operator for posets. Recall that the order complex $\Delta(P)$ of a poset P is the simplicial complex with simplices corresponding to finite chains in P.

2.3. Deleted join of a simplicial complex

Let K be a simplicial complex. Then the iterated join $K * \ldots * K$ of n copies of K is denoted by $K^{*(n)}$. More precisely, $K^{*(n)}$ is defined as the join $K_1 * \ldots * K_n$, of n vertex disjoint complexes isomorphic to K. If $J_i : K \to K_i$ denotes the isomorphism above, the typical simplex in $K^{*(n)}$ is of the form $J_1(\theta_1) \cup \ldots \cup J_n(\theta_n)$ where $\theta_i \in K$, but we will often simplify the notation and denote this simplex by $(\theta_1, \ldots, \theta_n)$. The n^{th} **deleted join** $K_\delta^{*(n)}$ of K is defined as the subcomplex of $K^{*(n)}$ defined by $K_\delta^{*(n)} := \{(\theta_1, \ldots, \theta_n) \mid \forall\, i \neq j\; \theta_i \cap \theta_j = \emptyset\}$.

The reader can supply the proof of the following simple and important proposition, see also [**33**], [**44**], [**45**], as a test of her/his understanding of the concepts and notations introduced above.

PROPOSITION 2.1 *Joins and deleted joins of simplicial complexes commute. In other words, if K and L are simplicial complexes than*

$$(K * L)_\delta^{*(n)} \cong K_\delta^{*(n)} * L_\delta^{*(n)}\,.$$

2.4. When do two sets in R^d have a nonempty intersection?

The initial geometric or combinatorial problem $\mathcal{P}$ is often reduced to a question about existence of equivariant maps in several steps. Some of these steps occur in majority of applications and deserve a brief comment. If A and B are two sets (spaces) in R^d then we find it convenient to rephrase the statement $A \cap B \neq \emptyset$ in a suitable language involving products or joins of spaces A and B. An elementary observation is that $A \times B$ is a subspace of $R^{2d} \cong R^d \times R^d$ and that the statement above is equivalent to $(A \times B) \cap \Delta \neq \emptyset$ where $\Delta \subset R^d \times R^d$ is the diagonal. Having in mind our emphasis on joins shown in previous sections, it would be useful to rephrase the last condition in these terms. Knowing that $A \times B \subset A * B$ where $(a, b) \in A \times B$ is identified with $\frac{1}{2}\,a + \frac{1}{2}\,b \in A * B$ we see that $A \cap B \neq \emptyset$ is equivalent to $(A * B) \cap \Delta \neq \emptyset$ where $\Delta \subset R^d \times R^d \subset R^d * R^d$ is the diagonal in $R^d * R^d$ viewed as the set of all points of the form $\frac{1}{2}\,x + \frac{1}{2}\,x$. It is convenient to view the topological join $R^d * R^d$ as a geometric join inside R^{2d+1}. It suffices to choose two embeddings $e_i : R^d \to R^{2d+1}$, $i = 1, 2$, so that $e_1(R^d)$ and $e_2(R^d)$ are two affine spaces in general position, and to identify $R^d * R^d$ with the geometric join $e_1(R^d) * e_2(R^d)$. If the diagonal Δ is identified with $\{\frac{1}{2}\,e_1(x) + \frac{1}{2}\,e_2(x)\}_{x \in R^d} \subset R^{2d+1}$, then

$$A \cap B \neq \emptyset \quad \Longleftrightarrow \quad (e_1(A) * e_2(B)) \cap \Delta \neq \emptyset\,.$$

The space R^{2d+1} is a typical test space $V_{\mathcal{P}}$ which arises if the group is Z_2 and Δ is the space $Z_{\mathcal{P}}$. More generally, qif we want to test whether $\bigcap_{i=1}^k A_i \neq \emptyset$ for a collection of spaces $\{A_i\}_{i=1}^k$, then we choose k general position embeddings $e_i : R^d \to R^D$ where $D := (d+1)k - 1$. The 'general position' requirement means that the affine spaces $L_i := e_i(R^d)\,(i = 1, \ldots, k)$ are supposed to span the ambient D-dimensional space R^D. For example one can identify R^D with the hyperplane $L := \mathrm{aff}\{v_j\}_{j \in I}$ where $\{v_j\}_{j \in I}$ is an orthonormal basis in $R^{D+1} \cong (R^{d+1})^{\oplus k}$ and $I = \{0, 1, \ldots, D\}$. Then the spaces L_i defined by $L_i := \mathrm{aff}\{v_j\}_{j \in I_i}$, where $I_i := \{j \in I | (d+1)(i-1) \leq j \leq (d+1)i - 1\}$, provide a very convenient collections of spaces $e_i(R^d)$. The natural group G of symmetries is the symmetric group S_k, or one of its subgroups, and it acts on the test space R^D by permuting affine subspaces $\{e_i(R^d)\}_{i=1}^k$. This action is in our special choice of embeddings inherited from the space $R^{D+1} \cong (R^{d+1})^{\oplus k}$ where S_k permutes the factors in this decomposition. The space $Z_{\mathcal{P}}$ is again the diagonal $\Delta := \{\frac{1}{k}\,e_1(x) + \ldots + \frac{1}{k}\,e_k(x)\}_{x \in R^d}$, and the target space $V_{\mathcal{P}} \setminus Z_{\mathcal{P}}$ for a G-equivariant test map is found to be G-homotopy equivalent to a G-sphere of dimension $(k-1)(d+1) - 1$.

2.5. Examples

We illustrate the general scheme of constructing *configuration spaces* and the corresponding *test maps* by a few examples. These examples should provide an initial evidence that the proof scheme outlined above has been successfully applied in diverse combinatorial geometric problems. The reader is invited to focus her attention at the general scheme, rather than on the technical details, and to observe the similarities and differences between the configuration spaces and test maps used in these examples. Some of these examples will be met again in section 4 for others the reader is suggested to consult the original sources.

Equipartition of masses. The problem is [**41**] to check if a given measurable set $A \subset R^d$ can be partitioned into 2^k-measurable sets of equal measure, $A = \bigcup_{\omega \in 2^k} A_\omega$, by k hyperplanes H_i, $i = 1, \dots, k$; $d, k \in N$. A convenient configuration space is $X_\mathcal{P} = S^{d-1} \times \dots \times S^{d-1} = (S^{d-1})^k$ where each $a = (a_1, \dots, a_k) \in X_\mathcal{P}$ encodes a k-tuple of oriented hyperplanes $\{H_i^a\}_{i=1}^k$ so that H_i^a bisects A for every $i = 1, \dots, k$. We need 2^k scalar functions to test if $a = (a_1, \dots, a_k) \in X_\mathcal{P}$ determines an equipartition of A so the test space $V_\mathcal{P}$ is isomorphic to R^{2^k} while $Z_\mathcal{P} = \{0\}$. The appropriate group of symmetries turns out to be $G = Z_2 \oplus \dots \oplus Z_2 = (Z_2)^{\oplus(k)}$ and the problem is reduced to the question under which conditions there does not exist a $(Z_2)^{\oplus k}$-equivariant map $f : (S^{d-1})^k \to S^{2^k - 1}$.

Colored Tverberg problem. A set of $(k+1)$ colors is a collection $\mathcal{C} = \{C_0, \dots, C_k\}$ of disjoint subsets of R^d, $d \geq k$. A set $B \subset R^d$ is multicolored if it contains a point from each of the sets C_i; in this case $\mathrm{conv}(B)$ is called a rainbow simplex. The problem is to find the size t of each of the colors which guarantees that there always exist r multicolored sets A_i, $i = 1, \dots, r$ that are pairwise disjoint such that the corresponding rainbow simplexes $\sigma_i := \mathrm{conv}\,(A_i)$ have a nonempty intersection, $\bigcap_{i=1}^r \sigma_i \neq \emptyset$. The configuration space turns out to be the join $X_\mathcal{P} = \Delta_{r,t}^{*(k+1)}$ of "chess board" complexes $\Delta_{r,t} \cong [t]_\delta^{*(r)}$, while the test space $V_\mathcal{P}$ is isomorphic to $R^{r(d+1)-1}$. The role of the space $Z_\mathcal{P}$ is played by a d-dimensional subspace of $X_\mathcal{P}$ while the group of symmetries is the cyclic group $G = Z_r$.

The splitting necklace problem. The problem is to split a necklace consisting of $d+1$ beads of different color among q different people (thieves) with a minimal number of cuts, [**1**]. Equivalently but more precisely, the unit interval $[0,1]$ is expressed as a disjoint union of measurable sets $A_0, \dots, A_d$ representing beads of different color. What is the minimum number t of cuts of $[0,1]$ so that the remaining intervals $\{I_i\}_{i=0}^t$ can be rearranged in q disjoint groups, $T_j = \{I_k\}_{k \in P_j}$, $j = 1, \dots, q$ such that $\sum_{k \in P_j} m(I_k \cap A_j) = (1/q) m(A_j)$ for all j. It turns out ([**53**]) that the configuration space $X_\mathcal{P}$ of all partitions of $[0,1]$ and allocations of subintervals to q different persons coincides with the deleted join $X_\mathcal{P} = (\Delta^\Lambda)_\delta^{*(q)}$ of a Λ-dimensional simplex, $\Lambda = (q-1)(d+1)$. The spaces $V_\mathcal{P}$ and $Z_\mathcal{P}$ are the same as in the previous example while the group of symmetries is $G = Z_q$.

Higher dimensional analogs of nonplanarity. Some of the results in section 4 can be interpreted as higher dimensional analogues of nonplanarity of graphs $K_{3,3}$ and K_5. The following result can be also seen as an analogue of nonplanarity of the Kuratowski graph $K_{3,3}$, (proposition 4.1,where $K_{3,3}$ is replaced by $K_{6,6}$, R^2 by R^3 and a common point, viewed as a 0-dimensional transversal, is now a line, a 1-dimensional transversal, intersecting four edges of $K_{6,6}$ embedded in R^3.

Theorem 2.2 [**58**] *Let $K_{6,6} \to R^3$ be an embedding of a complete bipartite graph in R^3. Then there exist four vertex disjoint edges e_1, e_2, e_3, e_4 of this graph and a line $L \subset R^3$ so that $L \cap e_i \neq \emptyset$ for all $i = 1, \dots, 4$.*

A natural configuration space for this problem is the space of all lines in R^3 but the actual proof requires a mixed strategy involving a refinement of ideas from sections 3 and 4.

3. Equivariant maps and the index theory

This section is intended to be a fairly self contained and detailed exposition of elementary ideas and techniques of equivariant topology. Our goal is to develop the theory to the level necessary for a foundation of a *numerical index theory.* The reader is encouraged to take the axioms of the index theory for granted and see the applications in section 4.

Key words and phrases:

- **group action, G-spaces, equivariant maps:** Definition 3.5, Definition 3.3, [**5**], [**14**], [**16**].
- **E_n^G-spaces, fundamental poset $\mathcal{A}_G$, G-degrees of complexity:** Definition 3.5, Definition 3.3, [**37**]. The fundamental poset $\mathcal{A}_G$ is a collection of G-spaces organized in a hierarchy according to their complexities.
- **Numerical index $\mathrm{Ind}_G(X)$, Yang and Sarkaria inequality, index theorem:** Definition 3.13, Theorem 3.15, [**54**], [**21**], [**24**], [**29**]. Index is a numerical measure of complexity of a G-space X.

3.1. Group actions on spaces

Let G be a finite group. In many applications G is either a cyclic group or a product of such groups. The choice of a group is dictated by the symmetries that occur naturally in the problem.

Definition 3.1 A (left) action or operation of a group G on a topological space X is a collection of continuous maps $\{\phi_g\}_{g\in G}$ so that

(a) $\phi_g : X \to X$ is a homeomorphism for all $g \in G$

(b) $\phi_e = \mathrm{id}_X$ if e is the unit element of the group G.

(c) $\forall g, h \in G \quad \phi_{g\cdot h} = \phi_g \circ \phi_h$.

The action of an element $g \in G$ on $x \in X$, which is by definition the element $\phi_g(x)$, is often denoted by $g \cdot x$ or simply gx. The action is **free** if $gx = x$ for some $x \in X$ implies $g = e$.

Examples of "geometric" group actions: 1. $G = Z_2 = \{1, \omega\}$ and $X = R^1$. The homeomorphism $\phi_\omega : R^1 \to R^1$ is defined by $\phi_\omega(r) = -r$.

2. $G = D_n$ is the dihedral group viewed as a subgroup of all plane isometries which fix a regular n-gon and X the n-gon itself.

3. $G = A_5$ is the group of all symmetries of an icosahedron and X the icosahedron itself.

4. $G \subset O(n, R)$ any finite subgroup of the group of all orthogonal matrices and X any subset of R^n invariant with respect to this group.

5. Choose $X = S^{n-1}$ to be the unit sphere in the previous example.

Note that none of these actions is free. However, in examples $1-3$ free actions arise if a suitable point (the point fixed by the action) is removed from X.

The examples above are "natural" examples arising from geometric considerations and they are closely linked with the definition of the group itself. There is a second circle of examples which are more of combinatorial nature.

Examples of "combinatorial" group actions: 1. Let Y be a topological space and $X := Y^n$ its n-th cartesian power. Then the symmetric group $G = S_n$ of all permutations "in n letters" acts on the space X by permuting the coordinates i.e., for $\sigma : [n] \to [n], \quad \phi_\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Obviously any subgroup $G \subset S_n$ also acts on X.

2. Let $Y = X^{*(n)} := X * \ldots * X$ be the join of n copies of the space X. Then S_n acts on this space by permuting the n-copies of X. Let K be a simplicial complex and $X = |K|$ its geometric realization. Then the deleted join $K_\delta^{*(n)} \subset X^{*(n)}$ is obviously invariant with respect to the action of the group S_n. This is a key example and we usually restrict our attention to the action of the cyclic subgroup $Z_n \subset S_n$ which cyclically permutes the coordinates (simplices) in the deleted join.

3.2. Fundamental poset and G-degrees of complexity

We have provided a basic supply of spaces with group actions which is more than sufficient for our purposes. If the group G is known, then a space X with an action of a group G is called a G-space. Given two G-spaces X and Y, a continuous map $f : X \to Y$ is called G-equivariant if for any $g \in G$ and $x \in X, \quad f(g \cdot x) = g \cdot f(x)$ or in the original notation $\phi_g \circ f = f \circ \phi_g$. Informally speaking, an equivariant map f mirrors the space X in Y in which case we may say that the space X is of smaller G-complexity than the space Y or that X is dominated by the space Y. Of course an equivariant map does not have to be $1-1$ so this relation of dominance is slightly more subtle than it may appear on the first sight. Many combinatorial problems are reduced to the question whether a G-space X is of "smaller complexity" than a G-space Y i.e., if there is a G-equivariant map $f : X \to Y$.

Definition 3.2 We say that a G-space X is dominated by a G-space Y, and denote this by $X \leq_G Y$, if there exists a G-equivariant map $f : X \to Y$. We say also that X is of smaller G-complexity then Y or that X is G-reducible to Y.

The relation $\leq_G$ is reflexive and transitive but obviously not antisymmetric. The relation $X \sim_G Y \Leftrightarrow X \leq_G Y$ and $Y \leq_G X$ is an equivalence relation. Assuming that we restrict our attention to a reasonably big "universe" $\mathcal{U}_G$ of G-spaces, e.g.

the class of all finite G-CW-complexes, we denote by $[X] := \{Y \in \mathcal{U}_G \,|\, Y \sim_G X\}$ the corresponding equivalence class. The universe $\mathcal{U}_G$ is in the usual language, [**31**], a small, full subcategory of the category G-Top of all G-topological spaces. We may here and there, if it doesn't lead to a confusion, instead of an equivalence class $[X]$ write X and vice versa. A simple but theoretically important is the following concept.

Definition 3.3 Let $\mathcal{A}_G := \{[X] \,|\, X \in \mathcal{U}_G\}$ be the set of all $\sim_G$ equivalence classes and let $\leq_G$ be the induced relation i.e., $[X] \leq_G [Y] \Leftrightarrow X \leq_G Y$. The relation $\leq_G$ is reflexive, transitive and antisymmetric and the poset $\mathcal{A}_G$ is the poset of "G-degrees of complexity" or simply the poset of G-degrees. If $X \leq_G Y$ and $[X] \neq [Y]$ we write $X <_G Y$.

Example 3.4 Let $G = Z_2$ and let S^n be the n-dimensional sphere with the G-action defined by $\omega(x) = -x$. Then obviously $S^{n-1} \leq_G S^n$ since the map sending S^{n-1} to the equator of S^n is obviously Z_2-equivariant. On the other hand the well known Borsuk-Ulam theorem says that there does not exist a Z_2-equivariant map $f : S^n \to S^{n-1}$ which in our notation means that $S^{n-1} <_G S^n$.

Definition 3.5 A free G-space X is of type E_n^G or simply a E_n-space if it satisfies the following conditions:

(i) X is a finite CW-complex equipped with a G-invariant CW-structure,

(ii) X is $(n-1)$-connected,

(iii) X is n-dimensional i.e., the cells of top dimension are n-dimensional.

Proposition 3.6 *E_n^G-spaces exist.*

Proof: The join $G * \ldots * G = G^{*(n+1)}$ of $n+1$ copies of the group G viewed as a 0-dimensional simplicial complex is an example. The property (ii) is easily checked by induction. Alternatively it can be shown that this complex is (lexicographically) shellable, [**12**].

Proposition 3.7 *For any two E_n-spaces X and Y, both $X \leq_G Y$ and $Y \leq_G X$ i.e., there exist G-equivariant maps between any two E_n-spaces. In particular, the equivalence class $[E_n^G] := \{X \,|\, X \sim_G G^{*(n+1)}\}$ is described as the class containing all E_n^G-spaces.*

Proof: The proof is by the standard inductive procedure, [**16**]. An elementary and detailed exposition can be found in [**52**].

Proposition 3.8 *Every n-dimensional, free G-CW-complex X can be completed to a E_n^G-space. As a consequence* $\dim(X) \leq n \Rightarrow X \leq_G E_n^G$.

Proof: The proof is again standard, [**52**], [**16**]. The idea of the proof is to "kill" homotopy groups in dimensions $< n$ by adding new free G-cells of dimension at most n.

Corollary 3.9 *If X is n-dimensional, free G-CW-complex then $X \leq_G Y$ for any $(n-1)$-connected free G-CW-complex Y.*

Remark 3.10 Clearly any two G-spaces X and Y which have a G-fixed point belong to the same $\sim_G$-equivalence class. So the relation $\leq_G$ is certainly more informative if restricted to the class of free G-spaces. On the other hand if EG is a natural "limit" of spaces E_n^G, i.e., it is a G-free, contractible CW-complex, then we can replace X by $X' := X \times EG$. The new space X' is always free, albeit infinitely dimensional CW-complex. In case X is free itself the new space X' and X have the same G-homotopy type which implies that $X \sim_G X'$.

Let us formulate now a key technical result which permits us to decide in some cases that $X \not\leq_G Y$ for two G-spaces X and Y.

THEOREM 3.11 *Let X be a $(n-1)$-connected G-CW-complex and Y a G-CW-complex of dimension at most $n-1$. Assume that the action of the group G on these spaces is free. Under these conditions $X \not\leq_G Y$. In other words if the connectedness of X is greater or equal to the dimension of Y then there does not exist a G-equivariant map $f : X \to Y$.*

Proof: Since by proposition 3.8 an $(n-1)$-dimensional, free G-CW-complex can be completed to an E_{n-1}^G-space, which means that $Y \leq_G E_{n-1}^G$, without loss of generality we can assume that Y itself is a E_{n-1}^G-space. Assume that there exists an equivariant map $f : X \to E_{n-1}^G$. Since f is an equivariant map for each nontrivial subgroup of G, let us assume that G is a cyclic group of order m, $G \cong Z_m$. By proposition 3.9 there exists an equivariant map $g : E_{n-1}^G \to X$. Moreover, since X is $(n-1)$-connected, we conclude that $g_* : H_{n-1}(E_{n-1}^G; Z) \to H_{n-1}(X; Z)$ is a trivial map. In order to reach a contradiction, let us show that the composition map $h_* = f_* \circ g_* : H_{n-1}(E_{n-1}^G; Z) \to H_{n-1}(E_{n-1}^G; Z)$ cannot be trivial. First of all we observe that $H_{n-1}(E_{n-1}^G; Z)$ has no torsion since E_{n-1}^G is $(n-1)$-dimensional and $(n-2)$-connected. The composition map $h = f \circ g : E_{n-1}^G \to E_{n-1}^G$ can be assumed to be cellular. Let us compute the Lefschetz number $L(h)$ of this map, [**18**], by using the cellular structure first and than on the level of homology. Since $G \cong Z_m$ is a cyclic group we claim that $L(h)$ is divisible by m. Indeed, if e is a k-dimensional cell in E_{n-1}^G, $\{\omega^j \cdot e\}_{j=1}^m$ its orbit and $\hat{h}_k$ the map induced by h on the level of cellular chains, then $\hat{h}_k(e) = \Sigma_{j=1}^m x_j \cdot \omega^j \cdot e + \textit{cells from other orbits}$, and the contribution to the Lefschetz number of this orbit is $m \cdot x_1$. On the other hand the homology of E_{n-1}^G is nontrivial only in dimension 1 and maybe in dimension $n-1$ so we have $L(h) = 1 \pm t$ where $t = \mathrm{Tr}\{h_* : H_{n-1}(E_{n-1}^G; Z) \to H_{n-1}(E_{n-1}^G; Z)\}$ is the trace of the corresponding linear map. So t is an integer which is congruent to ± 1 modulo m which means that it must be nonzero, a contradiction.

COROLLARY 3.12 *For all n, $E_{n-1}^G < E_n^G$. As a consequence, for every finite group G, the poset $\mathcal{A}_G$ contains a chain $\{[E_n^G]\}_{n\in N}$ which has the order type of natural numbers. The ideal $\mathcal{B}_G$ in $\mathcal{A}_G$ described by $\mathcal{B}_G := \{[X] \in \mathcal{A}_G \,|\, \exists\, n \in N\, [X] \leq_G E_n^G\}$ is characterized as the set of equivalence classes of finite, free G-CW-complexes.*

3.3. Index functions and the Sarkaria inequality

A fundamental problem is to compare G-complexities of two G-spaces X and Y. An integer valued function measuring the complexity of G-spaces is usually called an index. There are many forms of the index function and some of them are not integer valued, see [**21**], [**24**], [**29**], [**54**]. Once we defined this function and established its basic properties, the reader should feel encouraged to use these properties as some sort of axioms of the complexity function Ind_G. This should allow him to compare G-complexity of spaces and prove the corresponding geometric results even if the consistency of these axioms is not his primary concern.

Definition 3.13 Let $\mathrm{Ind}_G : \mathcal{B}_G \to N$ be a function defined by

$$\mathrm{Ind}_G(X) := \min\{n \in N \mid X \leq_G E_n^G\}$$

In other words, $\mathrm{Ind}_G(X) > n$ is equivalent to nonexistence of a G-equivariant map $f : X \to E_n^G$.

THEOREM 3.14 *The index function* Ind_G *has the following properties:*

(1) (Naturality) $\mathrm{Ind}_G(E_n^G) = n$,

(2) (Monotonicity) $X \leq_G Y \Longrightarrow \mathrm{Ind}_G(X) \leq_G \mathrm{Ind}_G(Y)$

(3) (Index theorem) $\mathrm{Ind}_G(L) \geq \mathrm{Ind}_G(K) - \mathrm{Ind}_G(\Delta(Q_L)) - 1$ *under the assumption that K is a free G-simplicial complex, L its G-subcomplex and $\Delta(Q_L)$ the order complex of the poset $Q_L := P_K \setminus P_L$ where P_K is the poset* $(K, \subset)$ (see theorem 3.15).

The first two properties are obvious consequences of definitions. The third is reffered to as the index theorem or the Sarkaria inequality and it is a versatile tool for proving combinatorial and geometric result which can be reduced to questions of (non)existence of equivariant maps. There exist more complex index functions and the corresponding index theorems, see [**24**].

Recall that every simplicial complex K defines a poset $P_K := (K \setminus \{\emptyset\}, \subset)$ consisting of all nonempty simplices in K ordered by the containment relation. For an arbitrary poset P, the simplicial complex of all chains in P is called the order complex and denoted by $\Delta(P)$. It is well known, [**12**], that $\Delta(P_K)$ is the first barycentric subdivision $\hat{K}$ of K.

THEOREM 3.15 [The index theorem; Sarkaria inequality] *Let K be a finite and free G-simplicial complex. In other words, K is a simplicial complex equipped with a free action of a finite group G acting simplicially on K. Let L be a G-invariant subcomplex. Let P_K be the associated poset and P_L the subposet associated to L. Let $Q_L := P_K \setminus P_L$ be the complementary subposet and $\Delta(Q_L)$ the associated order complex. Then*

$$\mathrm{Ind}_G(L) \geq \mathrm{Ind}_G(K) - \mathrm{Ind}_G(\Delta(Q_L)) - 1\,.$$

Proof: A simplicial complex K and its first barycentric subdivision $\Delta(P_K)$ are obviously simplicially equivalent as G-complexes i.e., $K \sim_G \hat{K}$. The reader can easily check this fact by repeating the usual proof of equivalence of a simplicial complex K and its barycentric subdivision $\hat{K}$ using the fact that this equivalence is constructed inductively a simplex at a time so it can be easily made equivariant. Let $f_1 : K \to \hat{K}$ be the simplicial equivalence obtained by this procedure. The following step is to observe that there exists a natural G equivariant map $f_2 : \Delta(P_K) \to \Delta(P_L) * \Delta(Q_L)$. Indeed, each chain $c = \{c_o < \ldots < c_k\}$ in P_K is decomposed into a disjoint union of two chains, $c' := c \cap P_L$ and $c'' := c \cap Q_L$. Finally, let us choose, as in the proof of proposition 3.6, E^G_m and E^G_n to be simplicial complexes which implies that $E^G_m * E^G_n \cong E^G_{m+n+1}$.

Let $\mathrm{Ind}_G(L) = m$ and $\mathrm{Ind}_G(\Delta(Q_L)) = n$. Then there exist two G-equivariant maps $g : \Delta(P_L) \to E^G_m$ and $h : \Delta(Q_L) \to E^G_n$ which by the equivariant simplicial approximation theorem can be assumed to be simplicial. Than the map $g * h : \Delta(P_L) * \Delta(Q_L) \to E^G_m * E^G_n \cong E^G_{m+n+1}$ is well defined. The map $(g * h) \circ f_2 \circ f_1$ is G-equivariant which implies $\mathrm{Ind}_G(K) \leq m + n + 1$ and the theorem follows.

Remark 3.16 Theorem 3.15 is an example of a result from a noble family of index theorems. Very general forms of index functions and the corresponding index theorems are known; a good example is Fadell-Husseini index theorem [**24**]. An early index function and an inequality of the type above is found by Yang, [**54**]. Early ideas appeared apparently for the first time in connection with the embeddability problem of simplicial complexes in papers of A. Flores and E.R. van Kampen. They proved, (see [**7**]), that there is no continuous one-to-one map from σ^{2k}_{k-1}, the $(k-1)$-dimensional skeleton of the $(2k)$-dimensional simplex into the euclidean space $R^{2(k-1)}$. Karanbir Sarkaria virtuously demonstrated the power of the deleted join technique in many geometric and combinatorial problems, [**43**], [**44**], [**45**].

4. Nonembeddability of complexes and Tverberg type theorems

We collect some of the most interesting applications of the technique described in previous sections. We restrict our attention to some low dimensional special cases which nevertheless tell the whole story about the general case. The reader will find general formulations and additional information in the reviews [**7**], [**33**], [**57**] and in the original sources [**10**], [**43**], [**52**], [**60**].

Key words and phrases:

- **Kuratowski's nonplanar graphs K_5 and $K_{3,3}$:** The famous Kuratowski's graphs are examples of 2-nonembeddable complexes.
- **Tverberg theorems and q-nonembeddable simplicial complexes:** A complex K is q-nonembeddable in R^d if for any continuous map $f : K \to R^d$, there exist q points $x_1, \ldots, x_q$, belonging to q disjoint faces of K so that $f(x_1) = \ldots = f(x_q)$.

4.1. Kuratowski's nonplanar graphs

PROPOSITION 4.1 *The complete bipartite graph* $K_{3,3} := [3] * [3]$ *is not embeddable in* R^2. *More precisely, for any continuous map* $f : K_{3,3} \to R^2$ *there exist two disjoint edges* e_1 *and* e_2 *of this graph so that* $f(e_1) \cap f(e_2) \neq \emptyset$.

Proof: The configuration space $X_{\mathcal{P}}$ encoding all disjoint, ordered pairs of edges in $K_{3,3}$ is the deleted join $(K_{3,3})^{*(2)}_{\delta}$ of two copies of $K_{3,3}$. The test space $V_{\mathcal{P}}$ is the space R^5 defined in the section 2 and $Z_{\mathcal{P}} := \Delta$ is the diagonal. The map $f : K_{3,3} \to R^2$ combined with the two embeddings $e_1, e_2 : R^2 \to R^5$ defined in section 2 leads to two maps $h_i := e_i \circ f : K_{3,3} \to V_i\,, i = 1, 2$, where $V_i := e_i(R^2) \subset R^5$. The *test map* $F : (K_{3,3})^{*(2)}_{\delta} \to V_1 * V_2 \subset R^5$ is defined by $F(t\,x + (1-t)\,y) := t\,h_1(x) + (1-t)\,h_2(y)$. Since the join and the deleted join operation commute, proposition 2.1,$(K_{3,3})^{*(2)}_{\delta} \cong ([3]^{*(2)})^{*(2)}_{\delta} \cong ([3]^{*(2)}_{\delta})^{*(2)} \cong (S^1)^{*(2)} \cong S^3$. Note that the group $G = Z_2$ naturally arises in this problem as the group which interchanges two copies of $K_{3,3}$ in $(K_{3,3})^{*(2)}_{\delta}$ and two subspaces V_1 and V_2 of R^5 leaving each point in Δ fixed. Following the general pattern outlined in section 2 we should turn our attention now to the space $V_{\mathcal{P}} \setminus Z_{\mathcal{P}} = R^5 \setminus \Delta$. This space can be deformed Z_2-equivariantly to the space $\Delta^{\perp} \setminus 0$ which is just a punctured copy of R^3 which is radially Z_2-equivariantly deformed into a unit sphere $S(\Delta^{\perp}) \cong S^2$. Finally, according to the general scheme described in section 2,there arises a Z_2-equivariant map $F' : S^3 \to S^2$.

This is in contradiction with either the Borsuk–Ulam theorem or, knowing that S^n is a $E^{Z_2}_n$-space, with our more general theorem 3.11.

PROPOSITION 4.2 *The complete graph* K_5 *(a clique of five elements) is not embeddable in* R^2.

Proof: This proposition illustrates nicely the use of Sarkaria inequality. Let us suppose that there exists an embedding $f : K_5 \to R^2$. Let σ^4 be a 4-dimensional simplex so K_5 can be identified with the 1-dimensional skeleton σ^4_1. We shall show that for every continuous map $f : \sigma^4_1 \to R^2$ there exist two disjoint edges e_1 and e_2 in σ^4_1 so that $f(e_1) \cap f(e_2) \neq \emptyset$. The configuration space is $X_{\mathcal{P}} := (\sigma^4_1)^{*(2)}_{\delta}$. The test space is as in the first example, after the same sequence of reductions, found to be the sphere S^2 with the antipodal action of the group Z_2. Unlike the first example it is not clear how to estimate $\mathrm{Ind}_G(X_{\mathcal{P}})$ directly. Fortunately we can view $X_{\mathcal{P}} = (\sigma^4_1)^{*(2)}_{\delta} =: L$ as a subcomplex of the complex $K := (\sigma^4)^{*(2)}_{\delta} \cong (\{\mathrm{pt.}\}^{*(5)})^{*(2)}_{\delta} \cong (\{\mathrm{pt.}\}^{*(2)}_{\delta})^{*(5)} \cong [2]^{*(5)} \cong S^4$. Following the notation of theorem 3.15 we observe that the complementary poset Q_L consists of all pairs of disjoint faces (θ_1, θ_2) in $(\sigma^4)^{*(2)}_{\delta}$ with the property that either $\dim(\theta_1) \geq 2$ or $\dim(\theta_2) \geq 2$. Of course, for dimensional reasons, only one of these conditions can be fulfilled. As a consequence, there is a well defined map $h : Q_L \to \{a_1, a_2\}$, $h(\theta_1, \theta_2) = a_i \Leftrightarrow \dim(\theta_i) \geq 2$, of posets where $\{a_1, a_2\}$ is a two element poset consisting of two incomparable elements. From here we deduce that there is a Z_2-equivariant map $\Delta(h) : \Delta(Q_L) \to \Delta(\{a_1, a_2\}) \cong S^0$ which implies $\mathrm{Ind}_G(\Delta(Q_L)) = 0$. Sarkaria inequality yields

$\mathrm{Ind}_G(X_{\mathcal{P}}) > 2$. Finally, $\mathrm{Ind}_G(X_{\mathcal{P}}) > \mathrm{Ind}_G(S^2)$ implies that there does not exist an equivariant map $f : X_{\mathcal{P}} \to S^2$ which is a contradiction.

4.2. Tverberg type theorems and q-nonembeddability

The following three examples illustrate R^3-analogues of the first two statements. The reader will find in reviews [**7**], [**33**], [**57**] general forms of these statements and other related information. All these statements are offsprings or relatives of the well known Tverberg theorem.

PROPOSITION 4.3 *Every collection C of nine points in R^3 can be partitioned into three nonempty disjoint sets $C = C_1 \cup C_2 \cup C_3$, so that* $\mathrm{conv}(C_1) \cap \mathrm{conv}(C_2) \cap \mathrm{conv}(C_3) \neq \emptyset$ *.*

Proof: The statement above can be reformulated as follows. Let σ^8 be a 8-dimensional simplex. A choice of the set $C \subset R^3$ automatically leads to a linear map $L : \sigma^8 \to R^3$ where the vertices of σ^8 a mapped, via L, onto the set C. We are supposed to show that there exist three disjoint faces θ_1, θ_2 and θ_3 of σ^8 so that $L(\theta_1) \cap L(\theta_2) \cap L(\theta_3) \neq \emptyset$. The reader should note that the proof does not use the fact that L is a linear map and the same conclusion is achieved if $L : \sigma^8 \to R^3$ is any continuous map. A correct configuration space in this problem is $X_{\mathcal{P}} := (\sigma^8)^{*(3)}_{\delta}$. The target space is $(R^3)^{*(3)}$ which is, according to the analysis given in section 2, contained in the test space $V_{\mathcal{P}} = R^{11}$. The diagonal Δ plays the role of the space $Z_{\mathcal{P}}$ and, following the usual scheme, the problem is reduced to the question whether there exists an equivariant map from $X_{\mathcal{P}}$ to $V_{\mathcal{P}} \setminus \Delta$. Since $X_{\mathcal{P}} = (\sigma^8)^{*(3)}_{\delta} \cong (\{\mathrm{pt.}\}^{*(9)})^{*(3)}_{\delta} \cong (\{\mathrm{pt.}\}^{*(3)}_{\delta})^{*(9)} \cong E_8^{Z_3}$ we observe that $\mathrm{Ind}_{Z_3}(X_{\mathcal{P}}) = 8$. The target space is easily found to be Z_3-homotopy equivalent to the sphere S^7 equipped with a free Z_3-action. Hence, $\mathrm{Ind}_{Z_3}(V_{\mathcal{P}} \setminus Z_{\mathcal{P}}) = 7$ and we reached a contradiction.

The theorem above shows that a partition into three parts always exists if the size of the set C is 9. The following theorem says that we have a more precise statement if we add two more points.

PROPOSITION 4.4 *Let C be a collection of* 11 *points in R^3. Then there exist three pairwise disjoint subsets $C_i \subset C$, $i = 1, 2, 3$, each of size* 3 *so that* $\mathrm{conv}(C_1) \cap \mathrm{conv}(C_2) \cap \mathrm{conv}(C_3) \neq \emptyset$ *.*

Proof: The reader who went through the proof of proposition 4.2 will hopefully have the same pleasure again following this proof. The index theorem 3.15 is here as effective and elegant as before. The configuration space associated to this problem is based on the 2-dimensional skeleton σ^{10}_2 of a 10-dimensional simplex σ^{10}. Precisely, $X_{\mathcal{P}} = (\sigma^{10}_2)^{*(3)}_{\delta}$. The simplicial complex $L := X_{\mathcal{P}}$ is naturally seen as a subcomplex of the complex $K := (\sigma^{10})^{*(3)}_{\delta}$. The usual analysis shows that $K \cong (\{\mathrm{pt.}\}^{*(11)})^{*(3)}_{\delta} \cong (\{\mathrm{pt.}\}^{*(3)}_{\delta})^{*(11)} \cong [3]^{*(11)} \cong E_{10}^{Z_3}$. Consequently, $\mathrm{Ind}_{Z_3}(K) = 10$. The poset Q_L complementary to the poset P_L in P_K consists of all ordered triples $(\theta_1, \theta_2, \theta_3) \in (\sigma^{10})^{*(3)}_{\delta}$ such that for some $i \in \{1, 2, 3\}$, the size of the set θ_i is at least 4. Since the total size $\sum_{i=1}^{3}|\theta_i| \leq 11$ we observe that at least one but not more than two of these simplices can have the size greater or equal 4. Let us define "the type" function T by $T(\theta_1,\theta_2,\theta_3) := (\epsilon_1,\epsilon_2,\epsilon_3)$ where $\epsilon_i = 0$ or 1 depending on whether $|\theta_i| \leq 3$ or not. It follows that the set of all types can be identified with the poset $R := \mathcal{P}([3]) \setminus \{\emptyset,[3]\}$ where $\mathcal{P}([3])$ is the power set of the three element set $[3]$. Alternatively R is described as the face poset of the boundary of a triangle which implies that $\Delta(R) \cong S^1 \cong E_1^{Z_3}$. The map $T : Q_L \to R$ is obviously monotone and induces a simplicial Z_3-equivariant map $\Delta(T) : \Delta(Q_L) \to \Delta(R)$. Hence, $\mathrm{Ind}_{Z_3}(\Delta(Q_L)) \leq 1$. An application of Sarkaria inequality leads us to the conclusion that $\mathrm{Ind}_{Z_3}(X_{\mathcal{P}}) \geq 8$. Let's turn our attention now to the target space. As in the proof of the previous proposition this space $V_{\mathcal{P}} \setminus Z_{\mathcal{P}}$ is found to have the same Z_3-homotopy type as the sphere $S^7 \sim_{Z_3} E_7^{Z_3}$. We conclude that there does not exist a Z_3-equivariant map from $X_{\mathcal{P}}$ to $V_{\mathcal{P}} \setminus Z_{\mathcal{P}}$ which leads to a contradiction proving the theorem.

As in the previous proposition, if we increase the number of points in the set C we can guarantee existence of a collection of subsets satisfying some additional constraints. This time points in C are colored with three colors and the triangles C_i, $i = 1,2,3$, are supposed to have vertices colored with all three colors.

PROPOSITION 4.5 *A collection of five red, five blue and five white points in 3-space always contains three vertex pairwise disjoint triangles formed by points of different color which have a nonempty intersection.*

Proof: For each of the colors we choose a five element set $[5]$. The choice of colored points in R^3 is equivalent to a choice of a map $O : [5] \sqcup [5] \sqcup [5] \to R^3$, from a disjoint sum of three copies of $[5]$ to R^3. A triangle with vertices of different colors is naturally encoded by a 2-simplex from the simplicial complex $[5]^{*(3)}$. A choice of three, vertex disjoint, multicolored triangles is equivalent to a choice of a 8-dimensional simplex in $([5]^{*(3)})^{*(3)}_{\delta}$. Hence, the configuration space in this case is $X_{\mathcal{P}} = ([5]^{*(3)})^{*(3)}_{\delta} \cong ([5]^{*(3)}_{\delta})^{*(3)}$. The target space is as before Z_3-homotopy equivalent to S^7.

As before we are supposed to show that $\mathrm{Ind}_{Z_3}(X_{\mathcal{P}}) \geq 8$. There are two possibilities to achieve this goal. We can show that $X_{\mathcal{P}}$ is 7-connected and then invoke the theorem 3.11. For this it would suffice, in light of the Küneth formula for joins, [**17** p. 218], to show that the 2-dimensional simplicial complex $\Delta_{3,5} := [5]^{*(3)}_{\delta}$ is 1-connected. This is indeed the case, [**60**], [**13**], [**52**]. The complex $\Delta_{3,5}$ is known under the name "chess board" complex and it has an amusing and interesting history, [**26**], [**11**] [**60**], [**52**], [**13**], [**56**].

Alternatively we can, as observed in [**33**], use the index theorem 3.15 as follows. By taking a quick look at the complex $L := X_{\mathcal{P}} = ([5]^{*(3)})^{*(3)}_{\delta}$ we see that it can be embedded as a subcomplex of the complex $K = ((\sigma^4)^{*(3)})^{*(3)}_{\delta}$ by embedding the 0-dimensional complex $[5]$ into a simplex σ^4 spanned by 5 vertices. It is convenient to view simplices in $(\sigma^4)^{*(3)}$ as subsets of the set $[3] \times [5]$. Since $(\sigma^4)^{*(3)} \cong \{\mathrm{pt.}\}^{*(15)}$, by the usual procedure we find out that $K \cong [3]^{*(15)}$ which is 13-connected so we conclude that $K \sim_{Z_3} E_{14}^{Z_3}$ and $\mathrm{Ind}_{Z_3}(K) = 14$. The poset $Q_L = P_K \setminus P_L$ consists of all triples $(\theta_1, \theta_2, \theta_3) \in P_K$ with the property that $\exists i \, \exists j \, |\theta_i \cap (\{j\} \times [5])| \geq 2$. Let $T(\theta_1, \theta_2, \theta_3) := (\epsilon_1, \epsilon_2, \epsilon_3)$ where $\epsilon_i := \{j \in [3] : |\theta_i \cap (\{j\} \times [5])| \geq 2\}$. The triple of sets $(\epsilon_1, \epsilon_2, \epsilon_3)$ belongs to the face poset P_D of the simplicial complex $D := (\sigma^2)^{*(3)}$ which inherits the Z_3-action so that the map $T : Q_L \to D$ is monotone and Z_3-equivariant. Let $T := \mathrm{Image}(T) \subset D$. Since for every $(\epsilon_1, \epsilon_2, \epsilon_3) \in T$, $\bigcap_{i=1}^3 \epsilon_i = \emptyset$, we observe that the longest chain in T is of length 6 which implies that $\Delta(T)$ is a 5-dimensional Z_3-complex. It follows from proposition 3.8 that $\mathrm{Ind}_{Z_3}(\Delta(T)) \leq 5$. The Z_3-equivariant map $\Delta(Q_L) \to \Delta(T)$ shows that $\mathrm{Ind}_{Z_3}(\Delta(Q_L)) \leq 5$. By Sarkaria inequality $\mathrm{Ind}_{Z_3}(L) \geq \mathrm{Ind}_{Z_3}(K) - \mathrm{Ind}_{Z_3}(\Delta(Q_L)) - 1 \geq 14 - 5 - 1 = 8$ and a contradiction follows.

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