

## ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF CERTAIN INTEGRAL OPERATORS

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**Abstract.** We find exact asymptotic behavior of positive and negative eigenvalues of the operator  $\int_{\Omega} k(x-y)a(y) \cdot dy$  where  $k$  is a real radial nonhomogenous function (satisfying some additional condition) and  $a$  is a continuous function changing sign on  $\Omega \subset R^m$ .

**1. Introduction.** In this paper we study the asymptotic eigenvalue behavior of integral operators defined by kernels of the form

$$(*) \quad k(x-y)a(y), \quad x, y \in \Omega.$$

Here  $\Omega \subset R^m$  is a bounded open set,  $a$  is a continuous function on  $\Omega$  and  $k$  is a sufficiently regular function. There are many results concerning asymptotic behavior of eigenvalues of the integral operator with the kernel of the form (\*). For  $a \equiv 1$ ,  $m = 1$  Widom [16] obtained exact asymptotic behavior of eigenvalues of the operator with kernel of the form (\*) if the function  $K(\xi) = \int_R e^{it\xi} k(t) dt$  is bounded, nonnegative and has sufficiently regular behavior when  $\xi \rightarrow +\infty$  ( $k$  is not necessary homogeneous). There is a similar result in [12].

In [15] Widom found the exact asymptotic behavior of eigenvalues of the operator with the kernel (\*) if  $a$  is a nonnegative bounded function when  $k$  satisfies some additional conditions. In the case  $k(x) = |x|^{-\alpha}$ ,  $a$  continuous strong positive function, Kac [7] obtained the exact asymptotic eigenvalue behavior by a probabilistic method using Karamata Tauberian Theorem. Cobos and Kühn in [3] found an upper bound on the eigenvalues of an operator with the kernel of the form

$$K(x, y) = L(x, y) \frac{(1 + |\ln \|x - y\||)^\gamma}{\|x - y\|^{N(1-\alpha)}}, \quad x, y \in \Omega \subset R^N$$

where  $L \in L^\infty(\Omega)$ . Birman, Solomjak and Kostometov in [1], [2], [8] found the exact asymptotic eigenvalues behavior of the operators with kernel of the form (\*) (and more general form) but with assumption that  $k$  is a homogeneous function. Some estimation of singular values of the integral operator with the kernel (\*) are given in [10]. In this paper we find exact asymptotic eigenvalue behavior of the operator generated by the kernel (\*), where  $a$  is continuous (and sign changing) and  $k$  is not a homogeneous function. The method is new and is based on a construction of a normal operator (whose spectrum is easily determined) and its connection with the starting convolution operator. As an application of this method we give the asymptotic formula for positive and negative eigenvalues of the following boundary problem

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) = \lambda a u$$

$$u|_{\partial D} = 0$$

where  $a \in C(\bar{\Omega})$  (and  $a$  is a sign changing function) and  $a_{ij} = a_{ji} \in C^1(\bar{\Omega})$ .

**1. Preliminaries.** Suppose  $\mathcal{H}$  is a complex Hilbert space and  $T$  is a compact operator on  $\mathcal{H}$ . The singular values of  $T$  ( $s_n(T)$ ) are the eigenvalues of  $(T^*T)^{1/2}$  (or  $(TT^*)^{1/2}$ ). The eigenvalues of  $(T^*T)^{1/2}$  arranged in a decreasing order and repeated according to their multiplicity, form a sequence  $s_1, s_2, s_3, \dots$  tending to zero. Denote the set of compact operators on  $\mathcal{H}$  by  $C_\infty$ . The operator  $T$  is a Hilbert Schmidt one ( $T \in C_2$ ) if  $(\sum_{n=1}^\infty s_n^2(T))^{1/2} = |T|_2 < \infty$ . If  $T$  is an integral operator on  $L^2(\Omega)$  defined by  $Tf(x) = \int_\Omega M(x, y)f(y)dy$ ,  $x \in \Omega \subset R^m$  and  $T \in C_2$  then [6]  $|T|_2^2 = \int_\Omega \int_\Omega |M(x, y)|^2 dx dy$ . Denote by  $\int_\Omega K(x, y) \cdot dy$  the integral operator on  $L^2(\Omega)$  with the kernel  $K$ . Let  $\mathcal{N}_t(T)$  be the singular value distribution function

$$\mathcal{N}_t(T) = \sum_{s_n(T) \geq t} 1 \quad (t > 0)$$

A positive function  $L$  is a slowly varying function on  $[b, \infty)$  if it is measurable and for each  $\lambda > 0$  the equality

$$\lim_{x \rightarrow +\infty} \frac{L(\lambda x)}{L(x)} = 1$$

holds. It is well known [13] that for every  $\gamma > 0$  we have

$$\lim_{x \rightarrow +\infty} x^\gamma L(x) = +\infty, \quad \lim_{x \rightarrow +\infty} x^{-\gamma} L(x) = 0.$$

In what follows we need some lemmas.

**LEMMA 1.** *Suppose  $L$  is a continuous slowly varying function such that  $\varphi(x) = x^{-r}L(x)$  and  $\psi(x) = x^rL(x)$  ( $r > 0$ ) are monotone for  $x \geq x_0$ , and*

$$(0) \quad \lim_{x \rightarrow +\infty} \frac{L(x(L(x))^{\pm 1/r})}{L(x)} = 1.$$

Then

$$\begin{aligned}\varphi^{-1}(y) &\sim \left(\frac{L(y^{-1/r})}{y}\right)^{1/r} & y \rightarrow 0+, \\ \psi^{-1}(y) &\sim \left(\frac{y}{L(y^{1/r})}\right)^{1/r} & y \rightarrow +\infty,\end{aligned}$$

where  $\varphi^{-1}$ ,  $\psi^{-1}$  are the inverses of  $\varphi$ ,  $\psi$ .

*Proof.* Directly follows from (0) by substitution. We observe the the function  $L(x) = \prod_{i=1}^s (\ln_{m_i}(x))^{d_i}$  ( $\ln_{m_i}(x) = \underbrace{\ln \ln \dots \ln x}_{m_i}$ ) satisfies the conditions of Lemma 1.

LEMMA 2. Suppose the operator  $H \in C_\infty$  is such that for every  $\varepsilon > 0$  there exists a decomposition  $H = H'_\varepsilon + H''_\varepsilon$  ( $H'_\varepsilon, H''_\varepsilon \in C_\infty$ ) with following properties:

- 1° There exists  $\lim_{t \rightarrow 0+} (t/L(t^{-1/r}))^{1/r} N_t(H'_\varepsilon) = c(H'_\varepsilon)$
- 2°  $\overline{\lim}_{n \rightarrow \infty} \frac{n^r}{L(n)} s_n(H''_\varepsilon) < \varepsilon$ .

Then there exists  $\lim_{\varepsilon \rightarrow 0+} C(H'_\varepsilon) = c(H)$  and  $\lim_{t \rightarrow 0+} (t/L(t^{-1/r}))^{1/r} N_t(H) = c(H)$ . ( $L$  is a slowly varying function satisfying the conditions from Lemma 1).

Suppose  $T \in C_\infty$  is a selfadjoint operator on  $\mathcal{H}$ ,  $\{\lambda_n^+(T)\}$ ,  $\{\lambda_n^-(T)\}$  are the sequences of its positive and negative eigenvalues. Denote by  $\mathcal{N}_t^\pm(T)$  ( $t > 0$ ) the corresponding eigenvalue distribution functions  $\mathcal{N}_t^\pm(T) = \sum_{\pm \lambda_n^\pm(T) \geq t} 1$ .

LEMMA 3. Let  $H', H'' \in C_\infty$  and  $H = H' + H''$ . If

$$\begin{aligned}\lim_{t \rightarrow 0+} (t/L(t^{-1/r}))^{1/r} \mathcal{N}_t^\pm(H') &= C_\pm(H') \\ \lambda_n^\pm(H'') &= o(L(n)/n^r)\end{aligned}$$

then  $\lim_{t \rightarrow 0+} (t/L(t^{-1/r}))^{1/r} \mathcal{N}_t^\pm(H) = C_\pm(H')$ .

*Proof.* Lemma 2 and Lemma 3 can be proved by a slight modification of the proof of Ky-Fan theorem [6].

LEMMA 4. If  $H$  and  $K$  are positive compact operators such that

$$(1) \quad s_n(H - K) = o(L(n)/n^r)$$

then for  $\theta \in (0, 1)$

$$(2) \quad s_n(H^\theta - K^\theta) = o((L(n)/n^r)^\theta).$$

*Proof.* In [11] the following inequality

$$(3) \quad \sum_{k=1}^n s_k^\beta(H^\theta - K^\theta) \leq C(\theta, \beta) \cdot n^{1-\theta} \left( \sum_{k=1}^n s_k^\beta(H - K) \right)^\theta \quad (\theta, \beta \in (0, 1))$$

is proved.

From (1) it follows that for any  $\varepsilon > 0$  there exist  $n_\varepsilon \in N$  such that

$$s_n(H - K) < \varepsilon L(n)/n^r \quad \text{for any } n > n_\varepsilon.$$

Then from (3) we obtain

$$ns_n^\beta(H^\theta - K^\theta) \leq C(\theta, \beta)n^{1-\theta} \left( \sum_{k=1}^{n_\varepsilon} s_k^\beta(H - K) + \varepsilon^\beta \sum_{k=n_\varepsilon+1}^n \left( \frac{L(k)}{k^r} \right)^\beta \right)^\theta.$$

As a consequence of the last inequality we have

$$(4) \quad \left( \frac{n^r}{L(n)} \right)^\theta s_n(H^\theta - K^\theta) \leq (C(\theta, \beta))^{1/\beta} \left[ n_\varepsilon s_1^\beta(H - K) \frac{n^{\beta r - 1}}{(L(n))^\beta} + \varepsilon^\beta \frac{n^{\beta r - 1}}{(L(n))^\beta} \sum_{k=n_\varepsilon+1}^n \left( \frac{L(k)}{k^r} \right)^\beta \right]^{\theta/\beta}$$

Let  $\beta$  be a fixed number such that  $0 < \beta < \min\{1, 1/r\}$ . From (4) we get

$$\overline{\lim}_{n \rightarrow \infty} \left( \frac{n^r}{L(n)} \right)^\theta s_n(H^\theta - K^\theta) \leq (C(\theta, \beta))^{1/\beta} \left[ \varepsilon^\beta \overline{\lim}_{n \rightarrow \infty} \frac{n^{\beta r - 1}}{(L(n))^\beta} \sum_{k=2}^n \left( \frac{L(k)}{k^r} \right)^\beta \right]^{\theta/\beta}.$$

Since

$$\sum_{k=2}^n \left( \frac{L(k)}{k^r} \right)^\beta \leq \int_1^n \left( \frac{L(x)}{x^r} \right)^\beta dx \leq D \frac{(L(n))^\beta}{n^{1-\beta r}}$$

(here  $D$  depends only on  $\beta$  and  $r$ ) we obtain

$$\overline{\lim}_{n \rightarrow \infty} (n^r/L(n))^\theta s_n(H^\theta - K^\theta) \leq (C(\theta, \beta))^{1/\beta} (\varepsilon^\beta D)^{\theta/\beta}$$

Because  $\varepsilon > 0$  is arbitrary, we get

$$\lim_{n \rightarrow \infty} (n^r/L(n))^\theta s_n(H^\theta - K^\theta) = 0.$$

The relation (2) is proved.

**2. Main result.** Suppose  $k$  and  $k_1$  are even realvalued functions from  $C^\infty(\mathbb{R} \setminus \{0\})$  sufficiently rapidly decreasing to infinity (for example  $k$  and  $k_1$  have compact support) satisfying the following conditions

1°  $k(x) = k_1 * k_1(x) (= \int_{\mathbb{R}} k_1(t)k_1(x-t)dt)$ ,  $k(x) \rightarrow 0$ ,  $k_1(x) \rightarrow 0$   $x \rightarrow +\infty$ .

2° The function  $\mathcal{K}(\xi) = \int_{\mathbb{R}} e^{i\xi t} k(t)dt$  is a positive decreasing on  $(0, \infty)$  and

$$\mathcal{K}(\xi) = \xi^{-r} L(\xi)(1 + o(1)) \quad (r > 0).$$

Let  $m \in C[-1, 1]$  ( $m(x) \in R$ , for  $x \in [-1, 1]$ ). Consider the operator  $A$  on  $L^2(-1, 1)$  defined by

$$Af(x) = \int_{-1}^1 k(x-y)m(y)f(y)dy.$$

The operator  $\int_{-1}^1 k(x-y) \cdot dy$  is positive (by property 2°) and the operator  $f \rightarrow mf$  is selfadjoint. Hence their product (i.e. operator  $A$ ) has only real eigenvalues. If  $m \geq 0$  on  $[-1, 1]$  then all eigenvalues of  $A$  are positive. In that case denote by  $\{\lambda_n\}$  the eigenvalues sequence of  $A$ . Here the eigenvalues are repeated according to their multiplicity.

**THEOREM 1.** *If  $m \in C[-1, 1]$ ,  $m(x) \geq 0$  for  $x \in [-1, 1]$  and*

$$s_n \left( \int_0^2 k(x+y) \cdot dy \right) = o\left(\frac{L(n)}{n^r}\right) \quad (5)$$

$$s_n \left( \int_0^2 k_1(x+y) \cdot dy \right) = o\left(\sqrt{\frac{L(n)}{n^r}}\right) \quad (6)$$

then

$$(7) \quad \lambda_n(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^r.$$

Prior to the proof of Theorem 1, we prove the following lemma.

**LEMMA 5.** *If  $m \in C[-1, 1]$ ,  $m(x) \geq 0$  for  $x \in [-1, 1]$  and (5) is valid, then*

$$s_n \left( \int_{-1}^1 k(x-y)m(y) \cdot dy \right) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^r.$$

*Proof.* From [4, Theorem 1] it follows

$$s_n \left( \int_{\Delta} k(x-y) \cdot dy \right) \sim L(n) \left( \frac{n\pi}{|\Delta|} \right)^{-r}$$

where  $\Delta$  is an interval and  $|\Delta|$  is its length. Divide interval  $[-1, 1]$  into  $N$  intervals

$$\Delta_i = \left[ -1 + \frac{2}{N}(i-1), -1 + \frac{2i}{N} \right], \quad i = 1, 2, \dots, N$$

and denote by  $x_i$  the midpoint of  $\Delta_i$ . Suppose  $\varepsilon > 0$  is given and  $N$  is large enough such that  $|m(y) - m(x_i)| < \varepsilon$  for  $y \in \Delta_i$ . Then

$$(8) \quad \left| \sum_{j=1}^N (m(y) - m(x_j)) \chi_{\Delta_j}(y) \right| < \varepsilon$$

for each  $y \in [-1, 1]$  ( $\chi_{\Delta_j}$  is the characteristic function of the interval  $\Delta_j$ ).

The operator  $A$  might be represented in the form  $A = B_N + D_N + E_N$  where

$$\begin{aligned} B_N f(x) &= \int_{-1}^1 k(x-y) \left[ \sum_{j=1}^N (m(y) - m(x_j)) \chi_{\Delta_j}(y) \right] f(y) dy \\ D_N f(x) &= \sum_{j=1}^N (m(x_j) \chi_{\Delta_j}(x)) \int_{\Delta_j} k(x-y) f(y) dy \\ E_N f(x) &= \sum_{\substack{i \neq j \\ i, j=1}}^N (m(x_j) \chi_{\Delta_j}(x)) \int_{\Delta_j} k(x-y) f(y) dy \end{aligned}$$

Since

$$m(x_j) \chi_{\Delta_i}(x) \int_{\Delta_j} k(x-y) \cdot dy : L^2(\Delta_j) \rightarrow L^2(\Delta_i)$$

then by (5) for  $i \neq j$  it follows

$$s_n \left( m(x_j) \chi_{\Delta_i}(x) \int_{\Delta_j} k(x-y) \cdot dy \right) = o\left(\frac{L(n)}{n^r}\right)$$

and hence

$$(9) \quad s_n(E_N) = o(L(n)/n^r).$$

From (8) we have

$$(10) \quad s_n(B_N) < C \cdot \varepsilon L(n)/n^r$$

where the constant  $C$  does not depend on  $n$  and  $\varepsilon$ . Let  $A_j^N : L^2(\Delta_j) \rightarrow L^2(\Delta_j)$  ( $j = 1, 2, 3, \dots, N$ ) be a linear operator defined by

$$A_j^N f(x) = m(x_j) \int_{\Delta_j} k(x-y) f(y) dy.$$

Then  $D_N$  is a direct sum of the operators  $A_j^N$ ; hence

$$(11) \quad \mathcal{N}_t(D_N) = \sum_{j=1}^N \mathcal{N}_t(A_j^N)$$

Since

$$s_n(A_j^N) \sim m(x_j)L(n) \left( \frac{n\pi}{|\Delta_j|} \right)^{-r} \quad (n \rightarrow \infty)$$

then

$$(12) \quad \mathcal{N}_t(A_j^N) \sim (m(x_j))^{1/r} \frac{|\Delta_j|}{\pi} \left( \frac{L(t^{-1/r})}{t} \right)^{1/r} \quad (t \rightarrow 0^+).$$

From (11) and (12) it follows

$$(13) \quad \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t(D_N) = \frac{1}{\pi} \sum_{j=1}^N (m(x_j))^{1/r} |\Delta_j|.$$

From (9), (10), (13) and Lemma 2 we obtain

$$(14) \quad \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t(A) = \int_{-1}^1 \frac{1}{\pi} (m(x))^{1/r} dx \left( = \lim_{N \rightarrow \infty} \frac{1}{\pi} \sum_{j=1}^N (m(x_j))^{1/r} |\Delta_j| \right).$$

Putting  $t = s_n(A)$  in (14) we get

$$n \left( \frac{s_n(A)}{L(1/(s_n(A))^{1/r})} \right)^{1/r} \rightarrow \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx.$$

Let  $\mu_n = (s_n(A))^{-1/r}$ , then we have

$$\mu_n^r L(\mu_n) \sim n^r \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^{-r}.$$

Applying Lemma 1 we get

$$\mu_n \sim \frac{n}{(L(n))^{1/r}} \cdot \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^{-1} \text{ i.e. } s_n(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^r.$$

Lemma 5 is proved.

Observe that the system  $\{\varphi_n\}_{n=1}^{\infty}$ , where  $\varphi_n(x) = \sin n\pi(1+x)/2$ , is an orthonormal basis of  $L^2(-1, 1)$ . Let

$$H(x, y) = \sum_{n=-\infty}^{\infty} [k(x-y+4n) - k(x+y+4n+2)] \quad (15)$$

$$H_1(x, y) = \sum_{n=-\infty}^{\infty} [k_1(x-y+4n) - k_1(x+y+4n+2)] \quad (16)$$

Consider the operators  $B$  and  $B_1$  acting on  $L^2(-1, 1)$ , defined by

$$Bf(x) = \int_{-1}^1 H(x, y)f(y)dy, \quad B_1f(x) = \int_{-1}^1 H_1(x, y)f(y)dy.$$

By a direct computation we get

$$B\varphi_n = K(n\pi/2)\varphi, \quad B_1\varphi_n = \sqrt{K(n\pi/2)}\varphi_n \quad n = 1, 2, 3, \dots$$

and therefore  $B = B_1^2$ .

Let  $K$  and  $K_1$  be linear operators on  $L^2(-1, 1)$  defined by

$$Kf(x) = \int_{-1}^1 k(x-y)f(y)dy, \quad K_1f(x) = \int_{-1}^1 k_1(x-y)f(y)dy$$

( $K$  and  $K_1$  are positive operators because  $\hat{k} > 0$  and  $\hat{k}_1 > 0$ , [16]).

LEMMA 6. *If the functions  $k$  and  $k_1$  satisfy the conditions of Theorem 1, then*

$$(17) \quad s_n(K - K_1^2) = o(L(n)/n^r)$$

*Proof.* From (5), (6), (15), (16) and the definition of the operators  $B$  and  $B_1$  it follows  $B = K + R_1$  and  $B_1 = K_1 + R_2$  where  $R_1$  and  $R_2$  are compact operators such that

$$(18) \quad s_n(R_i) = o(L(n)/n^r) \quad i = 1, 2.$$

As we have  $K - K_1^2 = -R_1 + R_2B_1 + B_1R_2 - R_2^2$  (because  $B = B_1^2$ ), then using (18), equalities

$$s_n(B_1) = O\left(\sqrt{L(n)/n^r}\right), \quad s_n(B) = O\left(\sqrt{L(n)/n^r}\right)$$



and the properties of singular values from [6], we get

$$s_n(K - K_1^2) = o(L(n)/n^r),$$

proving (17).

Now, we prove Theorem 1. Let  $M: L^2(-1, 1) \rightarrow L^2(-1, 1)$  be a linear operator defined by  $Mf(x) = m(x)f(x)$  (Since  $m \geq 0$  we have  $M \geq 0$ ). Obviously, we have  $A = KM$ . Then

$$\lambda_n(A) = \lambda_n(KM) = \lambda_n(M^{1/2}KM^{1/2}) = s_n(M^{1/2}KM^{1/2})$$

(because  $M^{1/2}KM^{1/2} \geq 0$ ). By Lemma 6 and Ky Fan Theorem [6] we conclude

$$s_n(M^{1/2}KM^{1/2}) \sim s_n(M^{1/2}K_1^2M^{1/2}) = \lambda_n((K_1M^{1/2})^*K_1M^{1/2}) = s_n^2(K_1M^{1/2}).$$

So

$$(19) \quad \lambda_n(A) \sim s_n^2(K_1M^{1/2}).$$

Let  $L_1(\xi) = \sqrt{L(\xi)}$ ,  $m_1(\xi) = \sqrt{m(\xi)}$  and  $r_1 = r/2$ . It can be easily verified that if the function  $L(x)x^{-r_1}$  satisfies the same conditions.

From Lemma 5 we obtain

$$s_n(K_1M^{1/2}) \sim \frac{L_1(n)}{n^{r_1}} \left( \frac{1}{\pi} \int_{-1}^1 (m_1(x))^{1/r_1} dx \right)^{r_1} = \sqrt{\frac{L(n)}{n^r}} \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^{r/2}.$$

Combining this with (19) we get

$$(20) \quad \lambda_n(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{-1}^1 (m(x))^{1/r} dx \right)^r.$$

The theorem is proved.

*Remark 1.* By substituting a variable in the eigenvalue relation  $A\varphi = \lambda\varphi$ , from (20), we get ( $m \geq 0$  on  $[a, b]$ ,  $m \in C[a, b]$ ):

$$\lambda_n \left( \int_a^b k(x-y)m(y) \cdot dy \right) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_a^b (m(x))^{1/r} dx \right)^r$$

*Remark 2.* A similar asymptotic formula is valid in the case when the integration domain is some Jordan measurable set  $\Omega$ . From Lemma 2, Lemma 5 and the asymptotic relation (19) we get

$$(21) \quad \lambda_n \left( \int_{\Omega} k(x-y)m(y) \cdot dy \right) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega} (m(x))^{1/r} dx \right)^r.$$

(see [4, Theorem 4]), where  $m \in c(\Omega)$ ,  $m \geq 0$  on  $\Omega$  and  $k$  satisfies conditions of the Theorem 1.

Let  $k$  and  $k_1$  satisfy conditions 1° and 2° from the begining of this section.

**THEOREM 2.** *Let  $m \in c[-1, 1]$  be a real valued function changing sign on  $[-1, 1]$  and let the functions  $k$  and  $k_1$  satisfy conditions (5) and (6) of Theorem 1. Denote by  $\lambda_1^+ \geq \lambda_2^+ \geq \dots > 0$  the positive and by  $-\lambda_1^- \leq -\lambda_2^- \leq \dots < 0$  the negative eigenvalues of  $A$  (repeated according to their multiplicity). Then we have*

$$\lambda_n^+(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega^+} (m(x))^{1/r} dx \right)^r, \quad \lambda_n^-(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega^-} (-m(x))^{1/r} dx \right)^r,$$

where  $\Omega^+ = \{x \in [-1, 1]: m(x) > 0\}$ ,  $\Omega^- = \{x \in [-1, 1]: m(x) \leq 0\}$ .

Before the proof of Theorem 2, we prove the following Lemma.

**LEMMA 7.** *If  $\Omega_1$  and  $\Omega_2$  are measurable sets ( $\Omega_i \subset [-1, 1]$ ,  $i = 1, 2$ ) such that  $\Omega_1 \cap \Omega_2 = \emptyset$ , then the singular values of the operator  $C: L^2(\Omega_1) \rightarrow L^2(\Omega_2)$  defined by*

$$Cf(x) = \int_{\Omega_1} k_1(x-y)f(y)dy$$

have the following property

$$(22) \quad \lim_{n \rightarrow \infty} \sqrt{n^r/L(n)} s_n(C) = 0$$

*Proof.* It is enough to prove (22) in the case when  $\Omega_1$  and  $\Omega_2$  are intervals. Then (22) follows from  $\Omega_1 \cap \Omega_2 = \emptyset$  and (6). Denote by  $P$  and  $Q$  orthoprojectors on  $L^2(-1, 1)$  defined by

$$Pf(x) = \chi_{\Omega^+}(x)f(x), \quad Qf(x) = \chi_{\Omega^-}(x)f(x), \quad P + Q = I$$

and

$$m_+(x) = \begin{cases} m(x); & x \in \Omega^+ \\ 0; & x \in \Omega^- \end{cases} \quad m_-(x) = \begin{cases} 0; & x \in \Omega^+ \\ -m(x); & x \in \Omega^- \end{cases}$$

The functions  $m_+$  and  $m_-$  are continuous on  $[-1, 1]$  and  $m = m^+ - m^-$ . Denote by  $M_+$ ,  $M_-$  the operators acting on  $L^2(-1, 1)$  defined by  $M_{\pm}f(x) = m_{\pm}(x)f(x)$ . The operators  $M_{\pm}$  are positive and  $M = M_+ - M_-$ . Clearly  $M_+ = PM$  and  $M_- = -QM$ .

**LEMMA 8.**

- a)  $s_n(QK^{1/2}M_+K^{1/2}P) = o(L(n)/n^r)$       c)  $s_n(QK^{1/2}M_-K^{1/2}P) = o(L(n)/n^r)$   
b)  $s_n(QK^{1/2}M_-K^{1/2}Q) = o(L(n)/n^r)$       d)  $s_n(PK^{1/2}M_-K^{1/2}P) = o(L(n)/n^r)$ .

*Proof.* We prove here a). Other equalities can be proved similarly. From  $QK^{1/2}M_+K^{1/2}P = QK^{1/2}PMK^{1/2}P$  we get

$$s_{2n}(QK^{1/2}M_+K^{1/2}P) \leq s_n(QK^{1/2}P) \cdot s_n(MK^{1/2}P).$$

Since

$$s_n(K^{1/2}) \sim \sqrt{\frac{L(n)}{(n\pi/2)^r}}$$

[4, Theorem 1] it is enough to prove that

$$(23) \quad s_n(QK^{1/2}P) = o\left(\sqrt{L(n)/n^r}\right).$$

From Lemma 6 and Lemma 4 (for  $\theta = 1/2$ ) it follows

$$(24) \quad s_n(K^{1/2} - K_1) = o\left(\sqrt{L(n)/n^r}\right).$$

Now, consider the operator

$$QK_1Pf(x) = \chi_{\Omega^-}(x) \int_{-1}^1 k_1(x-y)\chi_{\Omega^+}(y)f(y)dy \quad (: L^2(\Omega^+) \rightarrow L^2(\Omega^-)).$$

Since  $\Omega^+ \cap \Omega^- = \emptyset$ , by Lemma 7 we have

$$(25) \quad s_n(QK_1P) = o\left(\sqrt{L(n)/n^r}\right).$$

Then (23) follows from (24), (25) and the properties of the singular values [6].

By Lemma 8, a) and c) we get

$$(26) \quad \begin{aligned} |\lambda_n(QK^{1/2}M_+K^{1/2}P + (QK^{1/2}M_-K^{1/2}P)^*)| &= o(L(n)/n^r) \\ |\lambda_n(QK^{1/2}M_-K^{1/2}P + (QK^{1/2}M_-K^{1/2}P)^*)| &= o(L(n)/n^r). \end{aligned}$$

Let

$$\begin{aligned} S &= QK^{1/2}M_+K^{1/2}P + (QK^{1/2}M_+K^{1/2}P)^* + QK^{1/2}M_+K^{1/2}Q \\ &\quad - PK^{1/2}M_-K^{1/2}P - (QK^{1/2}M_-K^{1/2}P)^* - QK^{1/2}M_-K^{1/2}P. \end{aligned}$$

The operator  $S$  is selfadjoint. From (26) and Lemma 8 b) and d) it follows

$$(27) \quad |\lambda_n(S)| = o(L(n)/n^r)$$

If  $\{\lambda_n^+(S)\}$  and  $\{-\lambda_n^-(S)\}$  are sequences of positive and negative eigenvalues of  $S$  ( $\lambda_1^+ \geq \lambda_2^+ \geq \dots > 0$ ,  $\lambda_1^- \geq \lambda_2^- \geq \dots > 0$ ) then from (27) it follows

$$(28) \quad |\lambda_n^\pm(S)| = o(L(n)/n^r).$$

*Proof of Theorem 2.* The operator  $A = KM$  and the selfadjoint operator  $A_1 = K^{1/2}MK^{1/2}$  have the same positive and negative eigenvalues. Since  $P + Q = I$ , then

$$\begin{aligned} A_1 &= K^{1/2}M_+K^{1/2} - K^{1/2}M_-K^{1/2} \\ &= (P + Q)K^{1/2}M_+K^{1/2}(P + Q) - (P + Q)K^{1/2}M_-K^{1/2}M_-K^{1/2}(P + Q). \end{aligned}$$

After simplification we get

$$(29) \quad A_1 = PK^{1/2}M_+K^{1/2}P - QK^{1/2}M_-K^{1/2}Q + S.$$

From (28) and (29) and Lemma 3 we get

$$\begin{aligned} \lim_{t \rightarrow 0^+} \mathcal{N}_t^\pm(A_1) \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \\ = \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t^\pm(PK^{1/2}M_+K^{1/2}P - QK^{1/2}M_-K^{1/2}Q). \end{aligned}$$

Since  $\mathcal{N}_t^\pm(A) = \mathcal{N}_t^\pm(A_1)$  and the operators  $PK^{1/2}M_+K^{1/2}P$  and  $QK^{1/2}M_-K^{1/2}Q$  are positive and orthogonal, we get

$$\begin{aligned} \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t^+(A) &= \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t(PK^{1/2}M_+K^{1/2}P) \\ \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t^-(A) &= \lim_{t \rightarrow 0^+} \left( \frac{t}{L(t^{-1/r})} \right)^{1/r} \mathcal{N}_t(QK^{1/2}M_-K^{1/2}Q). \end{aligned}$$

Applying Lemma 1, we see that positive (negative) eigenvalues of  $A$  have the same asymptotic behavior as the eigenvalues of the operator

$$PK^{1/2}M_+K^{1/2}P \quad (QK^{1/2}M_-K^{1/2}Q).$$

Eigenvalues of the operator  $PK^{1/2}M_+K^{1/2}P$  are equal to eigenvalues of the operator  $K^{1/2}M_+K^{1/2}: L^2(\Omega^+) \rightarrow L^2(\Omega^+)$ . The operator  $K^{1/2}M_+K^{1/2}$  is positive and its eigenvalues are equal to the eigenvalues of the operator  $KM_+: L^2(\Omega^+) \rightarrow L^2(\Omega^+)$ . So,  $\lambda_n^+(A) \sim \lambda_n(KM_+)$  and from Theorem 1 (Remark 2) we obtain

$$\lambda_n(KM_+) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega^+} (m + (x))^{1/r} dx \right)^r.$$

Therefore

$$\lambda_n^+(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega^+} (m(x))^{1/r} dx \right)^r.$$

Similarly

$$\lambda_n^-(A) \sim \frac{L(n)}{n^r} \left( \frac{1}{\pi} \int_{\Omega_-} (-m(x))^{1/r} dx \right)^r.$$

Theorem 2 is proved.

**Multidimensional case.** Let  $k_0$  and  $k_0^1$  be functions from  $C^\infty(0, \infty)$  rapidly enough decreasing on infinity and tending to zero (or having compact support). Let  $k(t) = k_0(\|t\|)$ ,  $h(t) = k_0^1(\|t\|)$  and  $k = h * h$  ( $t \in R^m$ ). It is well known [14] that

$$\int_{R^m} e^{ixy} k(y) dy = \frac{(2\pi)^{m/2}}{\|x\|^{(m-2)/2}} \int_0^\infty k_0(\varrho) \cdot \varrho^{m/2} J_{m/2-1}(\varrho\|x\|) d\varrho,$$

where  $J_\lambda$  is the Bessel function. Suppose that the function

$$K_0(\xi) = \frac{(2\pi)^{m/2}}{\xi^{(m-2)/2}} \int_0^\infty k_0(\varrho) \varrho^{m/2} J_{m/2-1}(\varrho\xi) d\varrho$$

is positive decreasing and

$$K_0(\xi) = \xi^{-r} \frac{L(\xi)}{\xi} (1 + o(1)), \quad \xi \rightarrow \infty,$$

where  $L$  is a slowly varying function from Lemma 1. Then

$$\int_{R^m} e^{it\xi} h(t) dt = \sqrt{K_0(\xi)} \quad (\text{because } k = h * h).$$

Let  $I = [-1, 1]$  and  $a \in C(I^m)$ . Now, consider the operator  $A: L^2(I^m) \rightarrow L^2(I^m)$  defined by

$$Af(x) = \int_{I^m} k(x-y) a(y) f(y) dy$$

**THEOREM 3.** *If*

(30)

$$(a) \quad s_n \left( \int_{[0,2]^m} k_0(\sqrt{(x_1 \pm y_1)^2 + \dots + (x_m \pm y_m)^2}) \cdot dy \right) = o\left(\frac{L(n^{1/m})}{n^{r/m}}\right)$$

$$(b) \quad s_n \left( \int_{[0,2]^m} k_0^1(\sqrt{(x_1 \pm y_1)^2 + \dots + (x_m \pm y_m)^2}) \cdot dy \right) = o\left(\sqrt{\frac{L(n^{1/m})}{n^{r/m}}}\right)$$

holds for all the combinations of + and - except for the one with all signs -, then

$$s_n \left( \int_{I_m} k(x-y) \cdot dy \right) \sim \left( \frac{2}{\pi d_0} \right)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

Furthermore, if  $a(x) \geq 0$  ( $x \in I^m$ ), then

$$\lambda_n \left( \int_{I^m} k(x-y)a(y) \cdot dy \right) \sim \frac{1}{(\pi d_0)^r} \left( \int_{I^m} (a(x))^{m/r} dx \right)^{r/m} \frac{L(n^{1/m})}{n^{r/m}}$$

where  $d_0 = \frac{2}{\sqrt{\pi}} \left( \Gamma \left( 1 + \frac{m}{2} \right) \right)^{1/m}$ .

Before the proof of Theorem 3, we prove the following Lemma.

LEMMA 9. Consider all possible numbers  $\sum_{k=1}^m n_k^2$ , where  $n_k \in \mathbf{N} \cup \{0\}$ ,  $k = 1, 2, \dots, m$ . If we arrange these numbers in a nondecreasing order  $\lambda'_1 \leq \lambda'_2 \leq \lambda'_3 \leq \dots$  then

$$\lambda'_n \sim \frac{n^{2/m}}{C_m^{2/m}}, \quad \text{where } C_m = \frac{\pi^{m/2}}{2^m \Gamma(1 + m/2)}$$

*Proof of Lemma 9.* Let  $N$  be some fixed positive integer. Denote by  $r_1$  and  $r_2$  the smallest and the largest values  $n$  such that  $\lambda'_n = N^2$ . It is known [9] that

$$r_1 = \frac{\pi^{m/2} N^m}{2^{m-1} m \Gamma(m/2)} + o(N^m), \quad r_2 = \frac{\pi^{m/2} N^m}{2^{m-1} m \Gamma(m/2)} + o(N^m).$$

Since  $r_1 \leq n \leq r_2$ , we get

$$n = \frac{\pi^{m/2} N^m}{2^{m-1} m \Gamma(m/2)} + o(N^m),$$

and therefore  $N \sim C_m^{-1/m} n^{1/m}$ . The statement of Lemma 9 then follows from  $\lambda'_n = N^2$ .

*Proof of Theorem 3.* Introduce the function  $H(x, y)$  ( $x = (x_1, \dots, x_m)$ ,  $y = (y_1, \dots, y_m)$ ) by the following system of recurrent relations

$$\begin{aligned} k_1(t_1, \dots, t_{m-1}) &= \sum_{n_m \in \mathbf{Z}} [k(t_1, t_2, \dots, t_{m-1}, x_m - y_m + 4n_m) \\ &\quad - k(t_1, t_2, \dots, t_{m-1}, x_m + y_m + 4n_m + 2)] \\ k_2(t_1, \dots, t_{m-1}) &= \sum_{n_{m-1} \in \mathbf{Z}} [k_1(t_1, \dots, t_{m-2}, x_{m-1} - y_{m-1} + 4n_{m-1}) \\ &\quad - k_1(t_1, \dots, t_{m-2}, x_{m-1}, y_{m-1} + 4n_{m-1} + 2)] \\ &\vdots \\ k_{m-1}(t_1) &= \sum_{n_2 \in \mathbf{Z}} [k_{m-2}(t_1, x_2 - y_2 + 4n_2) - k_{m-2}(t_1, x_2 + y_2 + 4n_2 + 2)] \end{aligned}$$

$$H(x, y) = \sum_{n_1 \in \mathbf{Z}} [k_{m-1}(x_1 - y_1 + 4n_1) - k_{m-1}(x_1 + y_1 + 4n_1 + 2)].$$

The functions  $k_0$  and  $k_0^1$  are chosen so that all these series converge. By a direct computation we get

$$\int_{I^m} H(x, y) \varphi_{n_1 n_2 \dots n_m}(y) dy = K_0 \left( \frac{\pi}{2} \sqrt{n_1^2 + n_2^2 + \dots + n_m^2} \right) \varphi_{n_1 n_2 \dots n_m}(x)$$

where

$$\varphi_{n_1 n_2 \dots n_m}(x) = \prod_{i=1}^m \sin \frac{n_i \pi (1 + x_i)}{2}.$$

From (30) a) by Ky Fan theorem [6] it follows

$$(31) \quad s_n \left( \int_{I^m} k(x-y) \cdot dy \right) \sim s_n \left( \int_{I^m} H(x, y) \cdot dy \right)$$

Singular values of the operator  $\int_{I^m} H(x, y) dy$  are

$$K_0 \left( \frac{\pi}{2} \sqrt{n_1^2 + n_2^2 + \dots + n_m^2} \right) = s_{n_1 \dots n_m}$$

and therefore

$$n_1^2 + \dots + n_m^2 = \left( \frac{2}{\pi} K_0^{-1}(s_{n_1 \dots n_m}) \right)^2$$

( $K_0^{-1}$  is the inverse function of  $K_0$ ).

Let the sequence  $\{s_{n_1 n_2 \dots n_m}\}$  be arranged in a nonincreasing order  $s_1 \geq s_2 \geq s_3 \geq \dots$ . Put  $\lambda'_n = \left( \frac{2}{\pi} K_0^{-1}(s_n) \right)^2$ . From Lemma 9 we obtain  $\lambda'_n \sim c_m^{-2/m} n^{2/m}$  i.e.

$$K_0^{-1}(s_n) \sim \frac{\pi}{2} C_m^{-1/m} n^{1/m}.$$

Since  $K_0(\xi) \sim L(\xi) \cdot \xi^{-r}$  ( $\xi \rightarrow +\infty$ ) we get  $s_n \sim \left( \frac{2}{\pi d_0} \right)^r \frac{L(n^{1/m})}{n^{r/m}}$ . From (31) it follows

$$s_n \left( \int_{I^m} k(x-y) \cdot dy \right) \sim \left( \frac{2}{\pi d_0} \right)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

The first part of Theorem 3 is proved.

*Remark 3.* By substitution as in the one-dimensional case we get

$$s_n \left( \int_{[-a, a]^m} k(x-y) \cdot dy \right) \sim \left( \frac{2a}{\pi d_0} \right)^r \frac{L(n^{1/m})}{n^{r/m}}.$$

If  $\Delta$  is some cube in  $R^m$  then

$$s_n \left( \int_{\Delta} k(x-y) \cdot dy \right) \sim \frac{|\Delta|^{r/m} L(n^{1/m})}{(\pi d_0)^r n^{r/m}},$$

where  $|\Delta|$  denotes the measure of  $\Delta$ . Analogously as in Theorem 1 (see [4, Theorem 4]) we get (for  $a \in C(\Delta)$ ,  $a(x) \geq 0$  on  $\Delta$ ):

$$\lambda_n \left( \int_{\Delta} k(x-y)a(y) \cdot dy \right) \sim \frac{1}{(\pi d_0)^r} \left( \int_{\Delta} (a(x))^{m/r} dx \right)^{r/m} \frac{L(n^{1/m})}{n^{r/m}}$$

and more generally

$$\lambda_n \left( \int_{\Omega} k(x-y)a(y) \cdot dy \right) \sim \frac{1}{(\pi d_0)^r} \left( \int_{\Omega} (a(x))^{m/r} dx \right)^{r/m} \frac{L(n^{1/m})}{n^{r/m}},$$

where  $\Omega$  is a Jordan measurable set in  $R^m$  ( $a \in C(\bar{\Omega})$ ,  $a \geq 0$  on  $\Omega$ ).

We formulate now a multidimensional variant of Theorem 2. (The proof is carried out in the same way as in Theorem 2). Assume that  $a \in C(I^m)$  and that  $a$  changes sign.

**THEOREM 4.** *The following*

$$s_n \left( \int_{[0,2]^m} k_0(\sqrt{(x_1 \pm y_1)^2 + \cdots + (x_m \pm y_m)^2}) \cdot dy \right) = o\left(\frac{L(n^{1/m})}{n^{r/m}}\right)$$

$$s_n \left( \int_{[0,2]^m} k_0^1(\sqrt{(x_1 \pm y_1)^2 + \cdots + (x_m \pm y_m)^2}) \cdot dy \right) = o\left(\sqrt{\frac{L(n^{1/m})}{n^{r/m}}}\right)$$

holds for all the combinations of + and - except for the one with all signs -, when

$$\lambda_n^+(A) \sim \frac{1}{(\pi d_0)^r} \left( \int_{\Omega^+} (a(x))^{m/r} dx \right)^{r/m} \frac{L(n^{1/m})}{n^{r/m}}$$

$$\lambda_n^-(A) \sim \frac{1}{(\pi d_0)^r} \left( \int_{\Omega^-} (-a(x))^{m/r} dx \right)^{r/m} \frac{L(n^{1/m})}{n^{r/m}}.$$

Here  $\Omega^+ = \{x \in I^m : a(x) > 0\}$ ,  $\Omega^- = \{x \in I^m : a(x) \leq 0\}$  and  $\lambda_n^+(A)$  and  $\lambda_n^-(A)$  have the same meaning as in Theorem 2.

*Remark 4.* The similar asymptotic formulas are valid if  $I^m$  is substituted by some Jordan measurable set  $\Omega \subset R^m$ .



**3. An example.** The function

$$G_\alpha(x) = \frac{2^{(2-m-\alpha)/2} K_{(m-\alpha)/2}(|x|)}{\pi^{m/2} \Gamma\left(\frac{\alpha}{2}\right) |x|^{(m-\alpha)/2}} \quad (x \in R^m, \alpha > 0)$$

( $K_r$  is McDonald function) satisfies the following conditions [14]

$$1^\circ \quad G_\alpha \in L^1(R) \quad \text{and} \quad \int_{R^m} e^{itx} G_\alpha(t) dt = (1 + |x|^2)^{-\alpha/2};$$

$$G_\alpha(x) \sim \begin{cases} \Gamma\left(\frac{m-\alpha}{2}\right) / 2^\alpha \pi^{m/2} \Gamma\left(\frac{\alpha}{2}\right) |x|^{\alpha-m}; & 0 < \alpha < m \\ 1/2^{m-1} \pi^{m/2} \Gamma\left(\frac{m}{2}\right) \ln \frac{1}{|x|}; & \alpha = m \\ \Gamma\left(\frac{\alpha-m}{2}\right) / 2^m \Gamma\left(\frac{\alpha}{2}\right); & \alpha > m \end{cases} \quad \text{for } x \rightarrow 0^+$$

and

$$G_\alpha(x) \sim \frac{|x|^{(\alpha-m-1)/2}}{2^{(m+\alpha-1)/2} \pi^{(m-1)/2} \Gamma\left(\frac{\alpha}{2}\right)} e^{-|x|} \quad \text{for } |x| \rightarrow +\infty.$$

Let  $m = \alpha = 2$ ,  $k(x) = G_2(x) \left( = \frac{1}{2\pi} K_0(x) \right) h(x) = G_1(x)$ . Since

$$\int_{R^2} e^{itx} G_2(t) dt = \frac{1}{1 + |x|^2}$$

then  $K_0(\xi) = 1/(1 + \xi^2) \sim 1/\xi^2$  ( $\xi \rightarrow +\infty$ ). The functions  $G_1$  and  $G_2$  satisfy the conditions of Theorem 4.

If  $\Omega \subset R^2$  is a Jordan measurable set,  $a \in c(\bar{\Omega})$  and  $a$  changes sign, then by Theorem 4 (Remark 4) we get

$$\begin{aligned} \lambda_n^+ \left( \int_{\Omega} G_2(x-y) a(y) dy \right) &\sim \frac{1}{4\pi n} \int_{\Omega^+} a(x) dx \\ \lambda_n^- \left( \int_{\Omega} G_2(x-y) a(y) dy \right) &\sim \frac{1}{4\pi n} \int_{\Omega^-} (-a(x)) dx \end{aligned}$$

( $\Omega^+ = \{x \in \Omega: a(x) > 0\}$ ,  $\Omega^- = \{x \in \Omega: a(x) \leq 0\}$ ).

Since  $K_0(x) = -\ln|x| + \varphi_1(x) \ln|x| + \varphi_2(x)$  [14], where  $\varphi_1$  and  $\varphi_2$  are even entire functions and  $\varphi_1(0) = 0$ , from theorem of Birman and Solomjak [2] and Ky Fan follows

$$(33) \quad \begin{aligned} \lambda_n^+ \left( \int_{\Omega} \frac{-\ln|x-y|}{2\pi} a(y) dy \right) &\sim \frac{1}{4\pi n} \int_{\Omega^+} a(x) dx \\ \lambda_n^- \left( \int_{\Omega} \frac{-\ln|x-y|}{2\pi} a(y) dy \right) &\sim \frac{1}{4\pi n} \int_{\Omega^-} (-a(x)) dx. \end{aligned}$$

Now, as an illustration, we show how the asymptotic behavior of the positive and negative eigenvalues for an elliptic boundary problem, can be deduced from (33).

Let  $D$  be a bounded simple connected domain in  $R^2$  with smooth boundary and  $a_{ij} = a_{ji} \in C^1(\bar{D})$ . Assume that the differential expression

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right)$$

is uniformly elliptic on  $D$ . Let  $0 < \mu_1^+ \leq \mu_2^+ \leq \dots$  and  $0 > -\mu_1^- \geq -\mu_2^- \geq \dots$  be the positive and negative eigenvalues for the following boundary problem

$$-\sum_{i,j=1}^2 \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_j} \right) = \lambda a(x)u, \quad u|_{\partial D} = 0$$

THEOREM 5. [5] *The following asymptotic formula is valid*

$$(\mu_n^\pm)^{-1} \sim \frac{1}{4\pi n} \int_{\Omega^\pm} \frac{|a(x)|}{\begin{vmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{vmatrix}} dx$$

where  $\Omega^+ = \{x \in D: a(x) > 0\}$  and  $\Omega^- = \{x \in D: a(x) \leq 0\}$ .

*Proof.* Applying the procedure for the reduction of elliptic differential expression to the canonical form, it is obviously enough to find the asymptotic behavior of eigenvalues for the following boundary problem

$$(34) \quad -\Delta u = \lambda a \cdot u, \quad u|_{\partial D} = 0.$$

To prove Theorem 5 it is enough to prove that the problem (34) has the following asymptotic behavior of eigenvalues

$$(35) \quad (\mu_n^\pm)^{-1} \sim \frac{1}{4\pi n} \int_{\Omega^\pm} |a(x)| dx.$$

Firstly, consider the case when  $D$  is the unit disc.

Then, (34) can be written in integral form

$$(36) \quad -u(z) = \lambda \int_D G(z, \xi) a(\xi) u(\xi) dA(\xi)$$

where  $G(z, \xi) = \frac{1}{2\pi} \ln \left| \frac{z-\xi}{1-z\bar{\xi}} \right|$  is Green function for the Laplace operator on the unit disc;  $dA(\xi) = dpdq$ ,  $\xi = p + iq$ . Since the operator

$$\int_D -\ln |1 - z\bar{\xi}| dA(\xi) \quad (\cdot: L^2(D, dA))$$

is a nuclear one, then

$$s_n \left( \int_D -\ln|1 - z\bar{\xi}| \cdot dA(\xi) \right) = o(n^{-1})$$

and (35) follows from (33), (34) and Ky Fan Theorem.

The general case, when  $D$  is an arbitrary domain, is reduced to the just considered one by a conformal mapping into the unit disc.

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