PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 59 (73), 1996, 39–48

SUFFICIENT CONDITIONS FOR OSCILLATION OF THE SOLUTIONS OF A CLASS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENT

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Communicated by Vojislav Marić

Abstract. Sufficient conditions are found for oscillation of the solutions of linear impulsive differential equations of first order with an advanced argument and oscillating coefficients.

Introduction. Impulsive differential equations with deviating argument are adequate mathematical models of numerous processes studied in physics, biology, electronics, etc. In spite of the great possibilities for their applications, the theory of these equations is developing rather slowly due to difficulties of technical and theoretical character which arise in their study.

In the last twenty years significantly increased the number of publications where the oscillation behaviour of solutions to functional-differential equations is studied. The bigger part of the works in that topic published before 1977 is given in [**5**]. In the monographs [**3**] and [**4**] published in 1991 and 1987, respectively, the oscillation and asymptotic properties of the solutions of various classes of functional-differential equations are studied.

The first paper devoted to the study of the oscillation properties of impulsive differential equations with retarded argument is [**2**].

In the present paper sufficient conditions are found for oscillation of the solutions of a linear impulsive differential equation with advanced argument

$$\begin{aligned} x'(t) + a(t)x(t) &= p(t)x(t+\tau), \quad t \neq t_k, \\ \Delta x(t_k) + a_k x(t_k) &= p_k x(t_k+\tau), \end{aligned} \tag{1}$$

AMS Subject Classification (1991): Primary 34A37

where the coefficients $p(t)$ and p_k may change their sign for $t \geq 0$, $k = 1, 2, \ldots$.

Preliminary notes. Firstly, we shall consider the linear impulsive differential equation of first order with advanced argument

$$\begin{aligned} x'(t) &= p(t)x(t+\tau), \quad t \neq t_k, \\ \Delta x(t_k) &= p_k x(t_k+\tau) \end{aligned} \tag{2}$$

and the corresponding inequalities

$$\begin{aligned} x'(t) &\geq p(t)x(t+\tau), \quad t \neq t_k, \\ \Delta x(t_k) &\geq p_k x(t_k+\tau) \end{aligned} \tag{3}$$

and

$$\begin{aligned} x'(t) &\leq p(t)x(t+\tau), \quad t \neq t_k, \\ \Delta x(t_k) &\leq p_k x(t_k+\tau) \end{aligned} \tag{4}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = x(t_k)$.

We introduce the following conditions:

H1. The constant τ is positive and the sequence $\{t_k\}_{k=1}^{\infty}$ is such that

$$0 = t_0 < t_1 < t_2 < \ldots, \qquad \lim_{k \to +\infty} t_k = +\infty.$$

H2. The function p: $\mathbf{R}_+ \to \mathbf{R}$ is piecewise continuous in $\mathbf{R}_+ = [0, +\infty)$ with points of discontinuity $\{t_k\}$, where it is continuous from the left.

Let $J = [\alpha, \beta) \subset \mathbf{R}_+$.

Definition 1. The function $x = \varphi(t)$ is called *solution* of the equation (2) in the interval J if:

1. $\varphi(t)$ is defined in $J_1 = [\alpha, \beta + \tau)$.
2. $\varphi(t)$ is absolutely continuous on each of the intervals $J_1 \cap (t_{k-1}, t_k]$, $k \in \mathbf{N}$.
3. $\varphi'(t) = p(t)\varphi(t+\tau)$ almost everywhere in $J \cap (t_{k-1}, t_k]$, $k \in \mathbf{N}$.
4. $\varphi(t_k^-) = \varphi(t_k)$, $\varphi(t_k^+) - \varphi(t_k) = p_k \varphi(t_k + \tau)$ for $t_k \in J$.

Analogously solutions of the equation (1), and inequalities (3) and (4) are defined.

Definition 2. The solution $x(t)$ of the equation (2) (or, of the inequalities (3), (4)) is said to be *regular*, if it is defined in some interval $[T_x, +\infty) \subset \mathbf{R}_+$ and $\sup\{|x(t)|\colon\ t \geq T\} > 0$ for each $T \geq T_x$.

Definition 3. The regular solution $x(t)$ of the inequality (3) is said to be *finally positive*, if there exists $T > 0$ such that $x(t) > 0$ for $t \geq T$.

Definition 4. The regular solution $x(t)$ of the inequality (4) is said to be *finally negative*, if there exists $T > 0$ such that $x(t) < 0$ for $t \geq T$.

Definition 5. The regular solution $x(t)$ of the equation (2) is said to be *oscillatory,* if it changes its sign in the interval $[T, +\infty)$, where T is arbitrary number.

Main results. THEOREM 1. *Let the conditions* H1 *and* H2 *be fulfilled, and let there exist a sequence of non-intersected intervals* $J_n = (\xi_n, \eta_n]$ *with* $\eta_n - \xi_n = 2\tau$, *such that:*

1. *For each* $n \in \mathbf{N}$, $t \in J_n$ *and* $t_k \in J_n$,

$$p(t) \geq 0, \quad p_k \geq 0. \tag{5}$$

2. *There exists* $\nu_1 \in \mathbf{N}$ *such that for* $n \geq \nu_1$

$$\int\limits_{\xi_n}^{\xi_n+\tau} p(s)ds + \sum_{\xi_n < t_k \leq \xi_n+\tau} p_k \geq 1. \tag{6}$$

Then: 1. *The inequality* (3) *has no finally positive regular solution.*
2. *The inequality* (4) *has no finally negative regular solution.*
3. *Each regular solution of the equation* (2) *is oscillatory.*

Proof. First of all, we shall prove that the inequality (3) has no finally positive regular solution. Let us suppose the opposite, i.e., there exist a solution $x(t)$ of (3) and a number $T > 0$ such that $x(t)$ is defined for $t \geq T$ and $x(t) > 0$ for $t \geq T$.

Since $\xi_n \to +\infty$ as $n \to +\infty$, then there exists $\nu_0 \in \mathbf{N}$ such that $\xi_n > T$ for $n \geq \nu_0$. Then it follows from (3) and (5) that $x'(t) \geq 0$, $\Delta x(t_k) \geq 0$ for $t,\ t_k \in J_n$, i.e., $x(t)$ is nondecreasing function for $t \in J_n$, $n \geq \nu_0$.

Let $\nu = \max(\nu_0, \nu_1)$ and $n \geq \nu$.

Integrating (3) from $\xi_n + 0$ to $\xi_n + \tau + 0$, we obtain

$$x(\xi_n + \tau + 0) \geq x(\xi_n + 0) + \int\limits_{\xi_n}^{\xi_n+\tau} p(s)x(s+\tau)ds + \sum_{\xi_n < t_k \leq \xi_n+\tau} p_k x(t_k + \tau).$$

Since $x(s+\tau) \geq x(\xi_n + \tau + 0)$ for $s \in (\xi_n, \xi_n + \tau]$, then

$$x(\xi_n + \tau + 0)\Big\{ \int\limits_{\xi_n}^{\xi_n+\tau} p(s)ds + \sum_{\xi_n < t_k \leq \xi_n+\tau} p_k - 1\Big\} + x(\xi_n + 0) \leq 0. \tag{7}$$

It follows from (7) that for each $n \geq \nu$ the inequality

$$\int\limits_{\xi_n}^{\xi_n+\tau} p(s)ds + \sum_{\xi_n < t_k \leq \xi_n+\tau} p_k < 1$$

holds true, which contradicts (6).

In order to prove that (4) has no finally negative regular solution, it is enough to note that if $x(t)$ is a solution of (4), then $-x(t)$ is a solution of (3).

It follows from assertions 1 and 2 of Theorem 1 that the equation (2) has neither finally positive nor finally negative regular solution. Therefore, each regular solution of (2) is oscillatory. □

For the equation with constant coefficients and constant advance

$$\begin{aligned} x'(t) &= px(t+\tau), \quad t \neq t_k, \\ \Delta x(t_k) &= p_0 x(t_k+\tau), \end{aligned} \tag{8}$$

we obtain the following

COROLLARY 1. *Let $p \geq 0$, $p_0 \geq 0$, $\tau > 0$ and $p\tau + p_0 i(T_n, T_n + \tau] \geq 1$ for infinitely many numbers T_n with $\lim_{n\to+\infty} T_n = +\infty$, where $i(a,b]$ denotes the number of the points t_k lying in the interval $(a,b]$. Then each regular solution of the equation* (8)] *is oscillatory.*

Let us suppose, in addition, that the equation (8) *is τ-periodic, i.e., there exists $n \in \mathbf{N}$ such that $t_{k+n} = t_k + \tau$, $k \in \mathbf{Z}$, or, equivalently $i(t, t+\tau] \equiv n$. Then each regular solution of the equation* (8) *is oscillatory if $p\tau + p_0 n \geq 1$.*

In the proof of Theorem 2 below, we shall use the following

LEMMA 1. *Let the sequence $\{x_n\}$ be defined by the equalities*

$$x_0 = 1, \quad x_n = f(x_{n-1}), \ n \in \mathbf{N}, \tag{9}$$

where $f(x) = e^{bx}(1+qx)$ and $b \geq 0$, $\quad q \geq 0$, $\quad be + q > 1$. Then the sequence $\{x_n\}$ is increasing and unbounded.

Proof. In order to prove that $\{x_n\}$ is increasing sequence it is enough to show that

$$f(x) > x, \quad x \geq 0. \tag{10}$$

Case 1. Let $q \geq 1$. Then (10) holds true since $f(x) \geq 1 + qx > x$, $x \geq 0$.

Case 2. Let $0 \leq q < 1$. The inequality (10) is valid for $0 \leq x \leq (1-q)^{-1}$ since $f(0) = 1 > 0$ and $e^{bx} > 1 \geq x/(1+qx)$ for $0 < x \leq (1-q)^{-1}$. Since $e^x \geq ex$, $x \in \mathbf{R}$, it follows that for $x > (1-q)^{-1}$ we obtain the estimate

$$\frac{f(x)}{x} \geq \frac{bex(1+qx)}{x} \geq be\left(1 + \frac{q}{1-q}\right) = \frac{be}{1-q} > 1,$$

which implies (10).

If we suppose that the sequence $\{x_n\}$ is bounded from above, then $\{x_n\}$ is convergent and $\lim_{n\to+\infty} x_n = x_* > 0$. Then it follows from (9) that $x_* = f(x_*)$, which contradicts (10). □

THEOREM 2. *Let the conditions* H1 *and* H2 *be fulfilled, and let there exist a sequence of non-intersected intervals* $J_n = (\xi_n, \eta_n]$ *with* $\eta_n - \xi_n \geq 2\tau$ *such that:*

1. *For each* $n \in \mathbf{N}$, $t \in J_n$ *and* $t_k \in J_n$

$$p(t) \geq 0, \quad p_k \geq 0. \tag{11}$$

2. *There exist constants* $b \geq 0$, $q \geq 0$ *and an integer* $\nu_1 > 0$ *such that for each* $n \geq \nu_1$ *and* $t \in (\xi_n, \eta_n - \tau]$, *the inequalities*

$$\int\limits_t^{t+\tau} p(s)ds \geq b, \qquad \sum_{t<t_k\leq t+\tau} p_k \geq q, \tag{12}$$

$$be + q > 1 \tag{13}$$

hold true.

3. *There exist a constant* $\delta > 0$ *and an integer* $\nu_2 > 0$ *such that for each* $n \geq \nu_2$ *there exists* $t_n^* \in (\xi_n, \xi_n + \tau]$ *such that*

$$B_n(t_n^*)C_n(t_n^*) \geq \delta, \tag{14}$$

where $B_n(t_n^*) = \int\limits_{\xi_n}^{t_n^*} p(s)ds + \sum\limits_{\xi_n<t_k\leq t_n^*} p_k$, $C_n(t_n^*) = \int\limits_{t_n^*}^{\xi_n+\tau} p(s)ds + \sum\limits_{t_n^*<t_k\leq \xi_n+\tau} p_k$.

4. $\lim\limits_{n\to+\infty}(\eta_n - \xi_n) = +\infty$.

Then: 1. *The inequality* (3) *has no finally positive regular solution.*
2. *The inequality* (4) *has no finally negative regular solution.*
3. *Each regular solution of the equation* (2) *is oscillatory.*

Proof. 1. Let us suppose that the inequality (3) has a solution $x(t)$ such that for sufficiently large T we have $x(t) > 0$ for $t \geq T$.

Since $\xi_n \to +\infty$ as $n \to +\infty$, then there exists $\nu_3 > 0$ such that $\xi_n > T$ for $n \geq \nu_3$. Therefore, it follows from (3) and (11) that $x(t)$ is nondecreasing function in J_n, $n \geq \nu_3$.

We set $c_0 = 1$, $c_n = f(c_{n-1})$, $n \in \mathbf{N}$, where $f(x) = e^{bx}(1 + qx)$.

Since the sequence $\{c_n\}$ is increasing and unbounded by virtue of Lemma 1, there exists $m \in \mathbf{N}$ such that

$$c_m > 1/\delta. \tag{15}$$

By means of condition 4 of Theorem 2 there exists $\nu_4 > 0$ such that

$$\eta_n - \xi_n > (m+1)\tau \tag{16}$$

for $n \geq \nu_4$.

Let $\nu = \max(\nu_1, \nu_2, \nu_3, \nu_4)$. Then for each $n \geq \nu$ the solution $x(t)$ is a nondecreasing function in J_n and conditions (12), (14) and (16) are fulfilled.

It follows from (3) that

$$x'(t) \ge p(t)x(t), \quad t \ne t_k,$$
$$\Delta x(t_k) \ge p_k x(t_k)$$

for $t,\ t_k \in (\xi_n, \eta_n - \tau]$, $n \ge \nu$, and by means of the Theorem of impulsive differential inequalities [**1**, Theorem 2.3] we have

$$x(t+\tau+0) \ge x(t+0)\exp\Big(\int_t^{t+\tau} p(s)ds\Big) \prod_{t<t_k\le t+\tau} (1+p_k).$$

Therefore, for each $n \ge \nu$ and $t \in (\xi_n, \eta_n - \tau]$,

$$\frac{x(t+\tau+0)}{x(t+0)} \ge \exp\Big(\int_t^{t+\tau} p(s)ds\Big)\Big(1 + \sum_{t<t_k\le t+\tau} p_k\Big) \ge e^b(1+q) = f(1) = c_1.$$

Repeating the above procedure, we arrive at

$$\frac{x(t+\tau+0)}{x(t+0)} \ge c_m \tag{17}$$

for $t \in (\xi_n, \eta_n - m\tau]$, $n \ge \nu$.

Since $\xi_n + \tau < \eta_n - m\tau$, it follows that (17) holds true for each $n \ge \nu$ and $t = t_n^* \in (\xi_n, \xi_n + \tau]$, i.e.,

$$\frac{x(t_n^*+\tau+0)}{x(t_n^*+0)} \ge c_m. \tag{18}$$

On the other hand, integrating (3) from $\xi_n + 0$ to $t_n^* + 0$, we obtain the inequality

$$x(t_n^*+0) \ge x(\xi_n+0) + \int_{\xi_n}^{t_n^*} p(s)x(s+\tau)ds + \sum_{\xi_n<t_k\le t_n^*} p_k x(t_k+\tau),$$

which implies

$$x(t_n^*+0) \ge x(\xi_n+\tau+0)B_n(t_n^*). \tag{19}$$

Analogously, from the inequality

$$x(\xi_n+\tau+0) \ge x(t_n^*+0) + \int_{t_n^*}^{\xi_n+\tau} p(s)x(s+\tau)ds + \sum_{t_n^*<t_k\le \xi_n+\tau} p_k x(t_k+\tau)$$

we obtain

$$x(\xi_n+\tau+0) \ge x(t_n^*+\tau+0)C_n(t_n^*). \tag{20}$$

Thus, it follows from (14), (19) and (20) that

$$\frac{x(t_n^* + \tau + 0)}{x(t_n^* + 0)} \le \frac{1}{\delta}. \tag{21}$$

Now (18) and (21) imply $1/\delta \ge c_m$, which contradicts (15).

The proof of the assertions 2 and 3 is carried out as in Theorem 1. □

Remark 1. Condition (12) is fulfilled if we suppose that

$$\liminf_{t\to+\infty} \int\limits_t^{t+\tau} p(s)ds > b, \qquad \liminf_{t\to+\infty} \sum_{t<t_k\le t+\tau} p_k > q$$

for $t \in \cup_{n=1}^{\infty}(\xi_n, \eta_n - \tau]$.

Remark 2. In the case when the equation (2) is without impulse effect ($p_k \equiv 0$, $k \in \mathbf{N}$) the conditions (12) and (13) are reduced to

$$\int\limits_t^{t+\tau} p(s)ds \ge b > e^{-1}, \quad t \in (\xi_n, \eta_n - \tau],\ n \ge \nu_1$$

and this condition implies that condition (14) is fulfilled with $\delta = b^2/4$.

In the case when $p_k \ne 0$ we cannot derive (14) as consequence of (12) and (13). In fact, if $p(t) \equiv 0$, $t_k = k\tau$, then conditions (12) and (13) are reduced to $p_k \ge q > 1$ and $B_n(t)C_n(t) \equiv 0$ in this case, and obviously, the condition (14) is not satisfied.

Remark 3. If we have $p > 0$ in the equation (8) with constant coefficients, then (14) is fulfilled since $B_n(t_n^*)C_n(t_n^*) \ge p^2\tau^2/4$.

Corollary 2. *If $p > 0$, $p_0 \ge 0$, $\tau > 0$ and $p\tau e + p_0 \liminf_{t\to+\infty} i(t, t+\tau] > 1$, then each regular solution of the equation (8) is oscillatory.*

If, in addition, the equation (8) is τ-periodic and $i(t, t+\tau] \equiv n \in \mathbf{N}$, then each regular solution of the equation (8) is oscillatory if $p\tau e + p_0 n > 1$.

Let us consider now the equation (1) together with the corresponding inequalities

$$\begin{aligned} x'(t) + a(t)x(t) &\ge p(t)x(t+\tau), \quad t \ne t_k, \\ \Delta x(t_k) + a_k x(t_k) &\ge p_k x(t_k + \tau) \end{aligned} \tag{22}$$

and

$$\begin{aligned} x'(t) + a(t)x(t) &\le p(t)x(t+\tau), \quad t \ne t_k, \\ \Delta x(t_k) + a_k x(t_k) &\le p_k x(t_k + \tau). \end{aligned} \tag{23}$$

We introduce the following conditions:

H3. The function a: $\mathbf{R}_+ \to \mathbf{R}$ is piecewise continuous in $\mathbf{R}_+$ with points of discontinuity $\{t_k\}$, where it is continuous from the left.

H4. $1-a_k>0$, $k\in\mathbf{N}$.

We set into the equation (1) (or, into the inequalities (22), (23))

$$x(t)\equiv\varphi(t)z(t)\equiv\exp\Big(-\int\limits_0^t a(s)ds\Big)\prod_{0<t_k<t}(1-a_k)z(t). \tag{24}$$

Making use of the relations

$$\begin{aligned} x'(t)&=-a(t)\varphi(t)z(t)+\varphi(t)z'(t),\quad t\neq t_k,\\ \Delta x(t_k)&=-a_k\varphi(t_k)z(t_k)+\varphi(t_k^+)\Delta z(t_k),\end{aligned}$$

we obtain

$$\begin{aligned} z'(t)&=p(t)\exp\Big(-\int\limits_t^{t+\tau}a(u)du\Big)\prod_{t\le t_j<t+\tau}(1-a_j)z(t+\tau),\quad t\neq t_k,\\ \Delta z(t_k)&=p_k\exp\Big(-\int\limits_{t_k}^{t_k+\tau}a(u)du\Big)\prod_{t_k<t_j<t_k+\tau}(1-a_j)z(t_k+\tau).\end{aligned} \tag{25}$$

Since $1-a_k>0$, then $\varphi(t)>0$ and the oscillatory properties of the equations (1) and (25) coincide. Moreover, the directions of the inequalities in (22) and (23) are preserved after the change of the variable (27). Thus, applying Theorems 1 and 2 to the equation (25) we can obtain the oscillatory results for the equation (1).

The following theorems hold true:

THEOREM 3. *Let the conditions* H1–H4 *be fulfilled and let there exist a sequence of non-intersected intervals $J_n=(\xi_n,\eta_n]$ with $\eta_n-\xi_n=2\tau$, such that:*

1. *For each $n\in\mathbf{N}$, $t\in J_n$ and $t_k\in J_n$*

$$p(t)\ge 0,\quad p_k\ge 0.$$

2. *There exists $\nu_1\in\mathbf{N}$ such that for $n\ge\nu_1$*

$$\begin{aligned}&\int\limits_{\xi_n}^{\xi_n+\tau}p(s)\exp\Big(-\int\limits_s^{s+\tau}a(u)du\Big)\prod_{s\le t_j<s+\tau}(1-a_j)ds\\ &+\sum_{\xi_n<t_k\le\xi_n+\tau}p_k\exp\Big(-\int\limits_{t_k}^{t_k+\tau}a(u)du\Big)\prod_{t_k<t_j<t_k+\tau}(1-a_j)\ge 1.\end{aligned}$$

Then: 1. *The inequality* (22) *has no finally positive regular solution.*

2. *The inequality* (23) *has no finally negative regular solution.*

3. *Each regular solution of the equation* (1) *is oscillatory.*

THEOREM 4. *Let the conditions* H1–H4 *be fulfilled and let there exist a sequence of non-intersected intervals* $J_n = (\xi_n, \eta_n]$ *with* $\eta_n - \xi_n \geq 2\tau$, *such that:*

1. *For each* $n \in \mathbf{N}$, $t \in J_n$ *and* $t_k \in J_n$,

$$p(t) \geq 0, \quad p_k \geq 0.$$

2. *There exist constants* $b \geq 0$, $q \geq 0$ *and an integer* $\nu_1 > 0$ *such that for each* $n \geq \nu_1$ *and* $t \in (\xi_n, \eta_n - \tau]$ *the following inequalities are valid:*

$$\int_t^{t+\tau} p(s) \exp\Big(-\int_s^{s+\tau} a(u)du\Big) \prod_{s \leq t_j < s+\tau} (1-a_j)ds \geq b,$$

$$\sum_{t < t_k \leq t+\tau} p_k \exp\Big(-\int_{t_k}^{t_k+\tau} a(u)du\Big) \prod_{t_k < t_j < t_k+\tau} (1-a_j) \geq q,$$

$$be + q > 1.$$

3. *There exist a constant* $\delta > 0$ *and an integer* $\nu_2 > 0$ *such that for each* $n \geq \nu_2$ *there exists* $t_n^* \in (\xi_n, \eta_n + \tau]$ *such that* $\tilde{B}_n(t_n^*)\tilde{C}_n(t_n^*) \geq \delta$, *where*

$$\begin{aligned}
\tilde{B}_n(t_n^*) = &\int_{\xi_n}^{t_n^*} p(s) \exp\Big(-\int_s^{s+\tau} a(u)du\Big) \prod_{s \leq t_j < s+\tau} (1-a_j)ds \\
&+ \sum_{\xi_n < t_k \leq t_n^*} p_k \exp\Big(-\int_{t_k}^{t_k+\tau} a(u)du\Big) \prod_{t_k < t_j < t_k+\tau} (1-a_j), \\
\tilde{C}_n(t_n^*) = &\int_{t_n^*}^{\xi_n+\tau} p(s) \exp\Big(-\int_s^{s+\tau} a(u)du\Big) \prod_{s \leq t_j < s+\tau} (1-a_j)ds \\
&+ \sum_{t_n^* < t_k \leq \xi_n+\tau} p_k \exp\Big(-\int_{t_k}^{t_k+\tau} a(u)du\Big) \prod_{t_k < t_j < t_k+\tau} (1-a_j).
\end{aligned}$$

4. $\lim\limits_{n \to +\infty} (\eta_n - \xi_n) = +\infty$.

Then: 1. *The inequality* (22) *has no finally positive regular solution.*
2. *The inequality* (23) *has no finally negative regular solution.*
3. *Each regular solution of the equation* (1) *is oscillatory.*

For the equation

$$\begin{aligned}
x'(t) + ax(t) &= px(t+\tau), \quad t \neq t_k, \\
\Delta x(t_k) + a_0 x(t_k) &= p_0 x(t_k+\tau),
\end{aligned} \tag{26}$$

the following assertion holds true:

COROLLARY 3. *If* $p > 0$, $p_0 \geq 0$, $\tau > 0$, $a_0 < 1$, $i(t, t+\tau] \equiv n \in \mathbf{N}$ *and* $ep\tau e^{-a\tau}(1-a_0)^n + p_0 n e^{-a\tau}(1-a_0)^{n-1} > 1$, *then each regular solution of the equation* (26) *is oscillatory.*

Acknowledgement. The present investigation was supported by the Bulgarian Ministry of Education, Science and Technologies under Grant MM–511.

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