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# ON INDEPENDENT VERTICES AND EDGES OF BELT GRAPHS

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**Abstract**. Let $m(G,k)$ and $n(G,k)$ be the number of distinct $k$-element sets of independent edges and vertices, respectively, of a graph $G$. Let $h, p_1, p_2, \dots, p_h$ be positive integers. For each selection of $h, p_1, p_2, \dots, p_h$ we construct two graphs $N = N_h(p_1, p_2, \dots, p_h)$ and $M = M_h(p_1, p_2, \dots, p_h)$, such that $m(N,k) = m(M,k)$ and $n(N,k) = n(M,k)$ for all but one value of $k$. The graphs $N$ and $M$ correspond respectively to a normal and a Möbius-type belt.

## Introduction

In this paper we are concerned with finite graphs without loops, multiple and directed edges. If $G$ is such a graph, then $m(G,k)$ and $n(G,k)$ denote the number of its distinct $k$-element independent edge and vertex sets, respectively, $k = 0, 1, 2, \dots$. Recall that $m(G,O) = n(G,O) = 1$, $m(G,1) =$ number of edges of $G$, $n(G,1) =$ number of vertices of $G$. Independent edge and vertex sets of graphs and, in particular, the invariants $m(G,k)$ and $n(G,k)$ were much investigated in graph theory [**1**–**16**].

Nonisomorphic graphs $G_1$ and $G_2$ for which the equalities

$$m(G_1,k) = m(G_2,k) \tag{1}$$

and

$$n(G_1,k) = n(G_2,k) \tag{2}$$

hold for all $k \geq 0$ are easily constructed; a pair of such graphs is depicted in Fig. 1.

Nonisomorphic graphs for which the equalities (1) and (2) are obeyed for all but one value of $k$ are found even easier. The simplest examples of this kind are the two 2-vertex graphs and the two connected 3-vertex graphs. However, all such examples known until now consist of graphs with small number of vertices.

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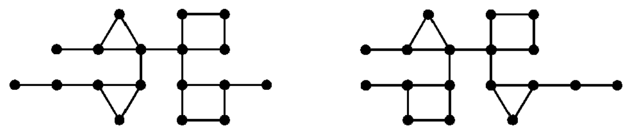

$G_1$ $G_2$

Fig. 1

The invariants $m(G,k)$, $k = 1,2,\ldots$, are highly intercorrelated and a number of their "collective" properties is known: for instance, they form a unimodal sequence [**16**] and their distribution is asymptotically normal [**3**]. All the zeros of the matching polynomial (see below) are real-valued [**1**, **4**, **5**]. Numerous pairs of graphs were found for which if the relation $m(G_1,k) > m(G_2,k)$ holds for some $k$, then $m(G_1,k) \geq m(G_2,k)$ holds for all $k$ [**6**, **13**, **14**].

Similar, but not fully analogous, properties of the numbers $n(G,k)$ have also been deduced [**7–12**].

In view of the above, one may ask if, for graphs $G$ with sufficiently large number of vertices, $m(G,k^*)$ is determined by $m(G,k)$, $k \geq 1$ , $k \neq k^*$. We show here that speculations in such a direction are not very promising. We namely have the following result.

THEOREM 1. *There exist graphs $G_1$ and $G_2$ with arbitrarily many vertices, for which both equalities* (1) *and* (2) *hold for all $k$, $k \geq 0$, $k \neq k^*$, but are both violated for a certain $k = k^*$.*

Theorem 1 is an immediate consequence of our main result, namely Theorem 2. In order to be able to formulate it we have to define the belt graphs.

## Belt graphs and statement of the main result

Let $h$ be a positive integer. Let $p_1, p_2, \ldots, p_h$ be positive integers. Define $r_0 = 0$, $r_1 = p_1$, $r_2 = p_1 + p_2, \ldots, r_h = p_1 + p_2 + \ldots + p_h$. For brevity, instead of $r_h$ we shall write $r$.

Let $P_{r+1}$ be the $(r+1)$-vertex path whose vertices are labeled consecutively by $u_0, u_1, \ldots, u_r$. Let $P'_{r+1}$ be another copy of the $(r+1)$-vertex path, whose vertices are labeled by $v_0, v_1, \ldots, v_r$.

*Definition* 1. The graph $L = L_h(p_1, p_2, \ldots, p_h)$ is obtained from $P_{r+1}$ and $P'_{r+1}$ by joining (by means of a new edge) the vertices $u_{r_i}$ and $v_{r_i}$, $i = 0, 1, \ldots, h$.

*Definition* 2. The graph $N = N_h\ (p_1, p_2, \ldots, p_h)$ is obtained from $L_h(p_1, p_2, \ldots, p_h)$ by identifying the vertices $u_0$ and $v_0$ with $u_r$ and $v_r$, respectively. We say that $N$ is a *normal belt graph.*

*Definition* 3. The graph $M = M_h(p_1, p_2, \ldots, p_h)$ is obtained from $L_h(p_1, p_2, \ldots, p_h)$ by identifying the vertices $u_0$ and $v_0$ with $v_r$ and $u_r$, respectively. We say that $M$ is a *Möbius-type belt graph.*

In Fig. 2 is depicted a graph $L$ as well as two pairs of belt graphs. From these examples the meaning of the parameters $h, p_1, p_2, \ldots, p_h$ is clear: The graph $L = L_h(p_1, p_2, \ldots, p_h)$ is composed of $h$ linearly arranged circuits of sizes $(2p_i + 2)$, $i = 1, 2, \ldots, h$. The cyclomatic number of $L$ is $h$. The graphs $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$ are formed of $h$ cyclically arranged circuits of sizes $(2p_i + 2)$, $i = 1, 2, \ldots, h$. Their cyclomatic numbers are $h + 1$. Their number of vertices is $2r$.

The graphs $N$ can be viewed as representing normal (two-sided) belts. The graphs $M$, on the other hand, correspond to Möbius-type (one-sided) belts.

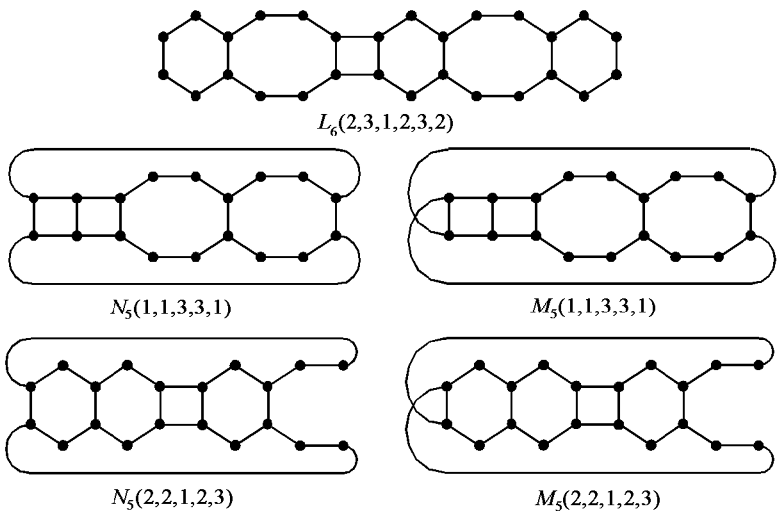


Fig. 2

If $r$ is even, then $N$ is a bipartite graph whereas $M$ is non-bipartite. If $r$ is odd, then $N$ is non-bipartite and $M$ is bipartite. Recall that $r$ is odd if and only if among the parameters $p_1, p_2, \ldots, p_h$ odd values occur odd number of times. In other words: $r$ is odd if and only if among the circuits forming $N$ and $M$ there is an odd number of circuits whose size is divisible by 4.

Our main result reads now as follows.

Theorem 2. *Let $h, p_1, p_2, \ldots, p_h$ be arbitrary positive integers. Let $r = p_1 + p_2 + \cdots + p_h$. Let $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$ be belt graphs in the sense of Definitions* 2 *and* 3. *Then:*

(a) $m(N, k) = m(M, k)$ *and* $n(N, k) = n(M, k)$ *for* $0 \le k \le r - 1$,

(b) $m(N, r) = m(M, r) + (-1)^r 2$ *and* $n(N, r) = n(M, r) + (-1)^r 2$

(c) $m(N, k) = m(M, k) = n(N, k) = n(M, k) = 0$ *for* $k > r$.

The proof of Theorem 2 requires some preparations.

### Matching and independence polynomials of belt graphs

The generating functions associated with the invariants $m(G,k)$ and $n(G,k)$ are

$$\alpha(G)=\alpha(G,x)=\sum_{k\geq 0} m(G,k)x^k \quad \text{and} \quad \omega(G)=\omega(G,x)=\sum_{k\geq 0} n(G,k)x^k$$

Because $G$ is supposed to be finite, both $\alpha(G)$ and $\omega(G)$ are polynomials; we call them matching [**1**, **2**, **4**, **5**, **12**] and independence polynomial [**1**, **10**–**12**], respectively. (The matching polynomial is usually defined [**1**, **2**, **4**, **5**, **15**] in a slightly different, but fully equivalent, way; the definition given above is convenient for the present considerations.) These polynomials obey the following recurrence relations [**1**, **4**]:

Lemma 1. (a) *Let $e=(u,v)$ be an edge of $G$, connecting the vertices $u$ and $v$. Let $N_v$ be the set of vertices of $G$, consisting of the vertex $v$ and the first neighbors of $v$. Then*

$$\alpha(G)=\alpha(G-e)+x\alpha(G-u-v); \qquad \omega(G)=\omega(G-v)+x\omega(G-N_v)$$

(b) *If $u$ is the only neighbor of $v$, then*

$$\alpha(G)=\alpha(G-u)+x\alpha(G-u-v); \qquad \omega(G)=\omega(G-u)+x\omega(G-u-v)$$

Lemma 2. *Let $v'$ and $v''$ be two distinct vertices of the graph $G$ and let $e'=(u',v')$ and $e''=(u'',v'')$ be two distinct edges of $G$.*

(a) *If $e'$ and $e''$ are independent, then*

$$\begin{aligned}\alpha(G)=\alpha(G-e'-e'')+x[\alpha(G-e'-u''-v'')+\alpha(G-e''-u'-v')]\\ +x^2\alpha(G-u'-v'-u''-v'')\end{aligned}$$

(b) *If $e'$ and $e''$ are not independent, then*

$$\alpha(G)=\alpha(G-e'-e'')+x[\alpha(G-u'-v')+\alpha(G-u''-v'')]$$

(c) *If $v'$ and $v''$ are independent, then*

$$\begin{aligned}\omega(G)=\omega(G-v'-v'')+x[\omega(G-v'-N_{v''})+\omega(G-v''-N_{v'})]\\ +x^2\omega(G-N_{v'}-N_{v''})\end{aligned}$$

(d) *If $v'$ and $v''$ are not independent, then*

$$\omega(G)=\omega(G-v'-v'')+x[\omega(G-N_{v'})+\omega(G-N_{v''})]$$

Needless to say that Lemma 2 is obtained by a two-fold application of the recurrence relations given in Lemma 1a.

By applying Lemma 2a to the edges $(u_{r-1}, u_r)$ and $(v_{r-1}, v_r)$ of the belt graphs $N = N_h(p_1, p_2, \ldots, p_h)$ and $M = M_h(p_1, p_2, \ldots, p_h)$ and by recalling that in $N, u_r \equiv u_0$, $v_r \equiv v_0$, whereas in $M, u_r \equiv v_0$ and $v_r \equiv u_0$, we arrive at

$$\begin{aligned}\alpha(N) &= \alpha(L - u_r - v_r) + x[\alpha(L - u_0 - u_r - u_{r-1} - v_r) \\ &\quad + \alpha(L - v_0 - v_r - v_{r-1} - u_r)] + x^2\alpha(L - u_0 - u_r - u_{r-1} - v_0 - v_r - v_{r-1}) \quad (3) \\ \alpha(M) &= \alpha(L - u_r - v_r) + x[\alpha(L - u_0 - v_r - v_{r-1} - u_r) \\ &\quad + \alpha(L - v_0 - u_r - u_{r-1} - v_r)] + x^2\alpha(L - u_0 - u_r - u_{r-1} - v_0 - v_r - v_{r-1}) \quad (4)\end{aligned}$$

Here $L$ stands for the graph $L_h(p_1, p_2, \ldots, p_h)$. It is assumed that the vertices of $L$, $N$ and $M$ are labeled in accordance with Definitions 1, 2 and 3.

Bearing in mind that the graphs $L - u_0 - u_r - u_{r-1} - v_r$ and $L - v_0 - v_r - v_{r-1} - u_r$ as well as $L - u_0 - v_r - v_{r-1} - u_r$ and $L - v_0 - u_r - u_{r-1} - v_r$ are isomorphic, we obtain from (3) and (4):

$$\alpha(N) - \alpha(M) = 2x[\alpha(L - u_0 - u_r - u_{r-1} - v_r) - \alpha(L - u_0 - v_r - v_{r-1} - u_r)] \quad (5)$$

The vertices $u_{r-1}$ and $v_{r-1}$ in $N$ and $M$ may be independent, but need not. It is easy to see that $u_{r-1}$ and $v_{r-1}$ are adjacent if $p_h = 1$ and are independent if $p_h > 1$.

Suppose first that $p_h > 1$. Applying Lemma 2c to the vertices $u_{r-1}$ and $v_{r-1}$ and performing calculations fully analogous to those leading to eq. (5), we obtain

$$\begin{aligned}\omega(N) - \omega(M) = 2x[&\omega(L - u_0 - u_r - u_{r-1} - u_{r-2} - v_r - v_{r-1}) \\ &-\omega(L - u_0 - v_r - v_{r-1} - v_{r-2} - u_r - u_{r-1})]\end{aligned} \quad (6)$$

The precisely same formula (6) holds also in the case $p_h = 1$.

At this point it is convenient to introduce the abbreviate notation:

$$L - u_0 - u_r - u_{r-1} - \ldots - u_{r-i} - v_r - v_{r-1} - \ldots - v_{r-i+1} \equiv L_i' \quad (7)$$

$$L - u_0 - v_r - v_{r-1} - \ldots - v_{r-i} - u_r - u_{r-1} - \ldots - u_{r-i+1} \equiv L_i'' \quad (8)$$

Then equations (5) and (6) can be rewritten in a more compact form:

$$\alpha(N) - \alpha(M) = 2x[\alpha(L_1') - \alpha(L_1'')] \quad (5a)$$

$$\omega(N) - \omega(M) = 2x[\omega(L_2') - \omega(L_2'')] \quad (6a)$$

## Proof of Theorem 2

LEMMA 3. *For* $1 \leq i \leq r-2$, $\alpha(L_i') - \alpha(L_i'') = -x[\alpha(L_{i+1}') - \alpha(L_{i+1}'')]$.

*Proof.* Notice that in the graph $L_i'$ the vertex $v_{r-i}$ is of degree one. The same is true for the vertex $u_{r-i}$ in $L_i''$. Then Lemma 1b is applicable, yielding

$$\alpha(L_i') = \alpha(L - u_0 - u_r - u_{r-1} - \ldots - u_{r-i} - v_r - v_{r-1} - \ldots - v_{r-i}) + x\alpha(L_{i+1}'')$$
$$\alpha(L_i'') = \alpha(L - u_0 - u_r - u_{r-1} - \ldots - u_{r-i} - v_r - v_{r-1} - \ldots - v_{r-i}) + x\alpha(L_{i+1}'')$$

From the above relations Lemma 3 is deduced straightforwardly. □

LEMMA 4. $\alpha(L_1') - \alpha(L_1'') = (-1)^r x^{r-1}$.

*Proof.* From Lemma 3,

$$\alpha(L_1') - \alpha(L_1'') = -x[\alpha(L_2') - \alpha(L_2'')] = \ldots = (-x)^{r-2}[\alpha(L_{r-1}') - \alpha(L_{r-1}'')]$$

From (7) is seen that $L_{r-1}'$ consists of two vertices ($v_0$ and $v_1$), connected by an edge. Therefore, $\alpha(L_{r-1}') = 1 + x$. From (8) follows that $L_{r-1}''$ consists of two disconnected vertices ($v_0$ and $u_1$). Therefore, $\alpha(L_{r-1}'') = 1$. □

LEMMA 5. *For* $1 \leq i \leq r-2$, $\omega(L_i') - \omega(L_i'') = -x[\omega(L_{i+1}') - \omega(L_{i+1}'')]$

Proof is fully analogous to the proof of Lemma 3.

LEMMA 6. $\omega(L_2') - \omega(L_2'') = (-1)^r x^{r-1}$.

*Proof.* From Lemma 5,

$$\omega(L_2') - \omega(L_2'') = -x[\omega(L_3') - \omega(L_3'')] = \ldots = (-x)^{r-3}[\omega(L_{r-1}') - \omega(L_{r-1}'')]$$

Lemma 6 follows from taking into account that $\omega(L_{r-1}') = 1 + 2x$ and $\omega(L_{r-1}'') = 1 + 2x + x^2$. □

Combining the formulas (5a) and (6a) with Lemmas 4 and 6 we arrive at:

LEMMA 7. $\alpha(N) - \alpha(M) = (-1)^r 2x^r$ *and* $\omega(N) - \omega(M) = (-1)^r 2x^r$. □

*Proof of Theorem* 2. The statements (a) and (b) of Theorem 2 are just another way of expressing the result of Lemma 7. It thus remains only to show that also the statements (c) in Theorem 2 are valid.

A graph with $2r$ vertices has at most $r$ independent edges. Therefore, $m(N,k)$ and $m(M,k)$ must be zero for $k > r$.

One of the graphs $N$ and $M$ is bipartite. Denote this has graph by $G'$ and color its vertices in the usual manner. This graph has equal number ($= r$) of vertices of each color. Each group of equally colored vertices forms an independent vertex set of cardinality $r$, hence $n(G', r) = 2$. Evidently, $n(G', k) = 0$ for $k > r$. For the other graph, say $G''$, which is non-bepartite, it cannot be $n(G'', r) > 0$. Therefore, it must be $n(G'', k) = 0$ also for $k > r$.

By this also part (c) of Theorem 2 is verified. □

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