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A REMARK ON THE PARTIAL SUMS IN HARDY SPACES

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Abstract. We prove that a function f, analytic in the unit disc, belongs to the Hardy space H^1 if and only if

$$\sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| = O(\log n) \quad (n \to \infty),$$

where $s_j f$ are the partial sums of the Taylor series of f. As a corollary we have that, for $f \in H^1$,

$$\sum_{j=0}^{n} \frac{1}{j+1} \|f - s_j f\| = o(\log n),$$

The analogous facts for L^1 do not hold.

For a function f analytic in the unit disc D let

$$P_n f = \frac{1}{A_n} \sum_{j=0}^{n} \frac{1}{j+1} s_j f \qquad (n = 0, 1, 2, \dots),$$

where

$$A_n = \sum_{j=0}^{n} \frac{1}{j+1}$$

and $s_j f$ are the partial sums of the Taylor series of f,

$$s_j f(z) = \sum_{k=0}^{j} \hat{f}(k) z^k.$$

As usual, we denote by H^1 the space of those functions f, analytic in D, such that

$$\|f\| = \sup_{r<1} I(f, r) < \infty,$$

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where

$$I(f,r)=\int_0^{2\pi}|f(re^{it})|dt/2\pi.$$

For the properties of H^1 see [**1**] and [**2**].

It is well known that $\|s_n f\| \le \text{const.}\, A_n\|f\|$ and that A_n is "best possible". (Note that A_n behaves like $\log n$ as $n\to\infty$.) A direct consequence is that, for $n\ge 2$,

$$\frac{1}{A_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j f\| \le C\|f\| \qquad (f\in H^1,\ n\ge 0). \tag{1}$$

where C is an absolute constant. In this note we prove, by using an inequality of Hardy and Littlewood, that (1) can be improved to get that

$$\frac{1}{A_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j f\| \le C\|f\| \qquad (f\in H^1,\ n\ge 0). \tag{2}$$

Moreover, we prove the following characterization of the space H^1.

THEOREM 1. *For a function f analytic in D the following assertions are equivalent.*

(i) *f belongs to H^1;*

(ii) $\sup_n \frac{1}{A_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j f\| < \infty$;

(iii) $\sup_n \|P_n f\| < \infty$.

Remark. It follows from the proof that the quantities occuring in (ii) and (iii) are "proportional" to the original norm in H^1; in particular there holds (2).

Before proving the theorem we give some immediate consequences and also consider the analogous facts in the Lebesgue space $L^1 = L^1(\partial D)$.

THEOREM 2. *If $f\in H^1$, then*

$$\lim_n \frac{1}{A_n}\sum_{j=0}^{n}\frac{1}{j+1}\|f - s_j f\| = 0 \tag{3}$$

and, consequently,

$$\lim_n \frac{1}{A_n}\sum_{j=0}^{n}\frac{1}{j+1}\|s_j f\| = \|f\|. \tag{4}$$

Proof. It is easy to verify that (3) holds when f is a polynomial. Then, the result is deduced in a standard way from (2) and the fact that the polynomials are dense in H^1 (cf. [**1**]). □

COROLLARY 1. *If $f\in H^1$, then*

$$\liminf_{n\to\infty}\|f - s_n f\| = 0. \tag{5}$$

In fact, one can prove somewhat more: for each $\varepsilon > 1$ there is a sequence $\{k_n\}_{n=0}^{\infty}$ of integers such that $\lim_n \|f - s_{k_n} f\| = 0$ and $n^{\varepsilon} \le k_n \le (n+1)^{\varepsilon}$ for sufficiently large n. We omit the easy proof.

The case of L^1. The space H^1 can be realized, via the Poisson integral, as the subspace of $L^1 = L^1(\partial D)$ cosisting of those $f \in L^1$ for which $\hat{f}(j) = 0$ for $j < 0$, where $\hat{f}$ is the Fourier transformation of f. However, not one of the relations (2), (3), (4), (5) is valid in L^1, and this follows from the fact that there is a function $f \in L^1$ such that $\lim_n \|f - s_n f\| = \infty$; such an example is given by

$$f(w) = \sum_j (\log j)^{-1/2} \cos jt \qquad (w = e^{it}).$$

Since the sequence $\{(\log j)^{-1/2}\}$ is convex, the function belongs to L^1 ([**2**], Theorem 4.1). Furthermore, using the standard technique, one shows that $\|f - s_n f\| \ge$ const. $(\log n)^{1/2}$. We omit the details.

It should be noted that inequality (1) is the best possible in L^1 in the sense that $\log n$ cannot be replaced by any $\psi(n)$ (independent of f) such that $\psi(n) = o(\log n)$ $(n \to \infty)$. To see this we take f to be the Poisson kernel,

$$f(w) = \frac{l - r^2}{|w - r|^2} \qquad (|w| = 1,\ \ 0 < r < 1),$$

then let r tend to l and use the norm estimate for the Dirichlet kernel.

Let h^1 denote the class of harmonic functions satisfying the condition $\|f\| = \sup_{r<1} I(f, r) < \infty$. The Poisson integral provides an isometric isomorphism of L^1 into h^1 (cf. [**1**]). Using Fejer's theorem one shows, by summation by parts, that if $f \in h^1$, then $\sup_n \|P_n f\| < \infty$, where P_n is extended to harmonic function in the obvious way. Conversely, it follows from the proof of Theorem l that if f is harmonic in D and $\sup_n \|P_n f\| < \infty$, then $f \in h^1$.

Proof of Theorem 1. That (ii) implies (iii) is obvious. To prove that (i) implies (ii) let $f \in H^1$ and for fixed $n \ge 2$ and $w \in D$ define the function $g \in H^1$ by

$$g(z) = (1 - rz)^{-1} f(rwz) \qquad (|z| \le 1),$$

where $r = 1 - 1/n$. We have $g(z) = \sum_{j=0}^{\infty} s_j f(w) r^j z^j$. Applying Hardy's inequality (cf. [**1**]) we get

$$\sum_{j=0}^{\infty} \frac{1}{j+1} |s_j f(w)| r^j = \sum_{j=0}^{\infty} \frac{1}{j+1} |\hat{g}(j)| \le \pi \|g\|.$$

Since $r^j = (1 - 1/n)^j \ge c$ for $0 \le j \le n$, where $c > 0$ is an absolute constant, we have

$$\sum_{j=0}^{n} \frac{1}{j+1} |s_j f(w)| \le (\pi/c)\|g\| = (1/2c) \int_0^{2\pi} |1 - re^{it}|^{-1} |f(rwe^{it})| dt.$$

Integrating this inequality over the circle $|w| = 1$ we find

$$\sum_{j=0}^{n} \frac{1}{j+1} \|s_j f\| \leq (1/2c)\|f\| \int_0^{2\pi} |1 - re^{it}|^{-1} dt,$$

where we have used Fubini's theorem. Finally, using the familiar estimate

$$\int_0^{2\pi} |1 - re^{it}|^{-1} dt \leq C \log \frac{1}{1-r} = C \log n,$$

we see that (2) holds and therefore we have proved that (i) implies (ii).

Let f be analytic in D. From the uniform convergence of $s_n f$ on compact sets it follows that $P_n f \to f$ $(n \to \infty)$ uniformly on compact subsets of D. Assuming that $\|P_n f\| \leq 1$ for each n we have $I(P_n f, r) \leq 1$ for all n and $r < 1$. This implies, via the uniform convergence of $P_n f$ on the circle $|z| = r$, that $I(f, r) \leq 1$ for every $r < 1$, which means that $\|f\| < 1$. Thus we have proved that (iii) implies (i), and this completes the proof. $\square$

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