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ON THE ASYMPTOTIC BEHAVIOUR OF TWO SEQUENCES RELATED BY A CONVOLUTION EQUATION

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Abstract We analyse analyse the relation between the asymptotic behaviour of two sequences $\{a(n)\}$ and $\{b(n)\}$ related by the system of equations $nb(n) = a * b(n)$, where $*$ denotes convolution. This type of relation appears in studying discrete infinitely divisible laws and more recently in risk theory. In Hawkes and Jenkins (1978) the authors considered this relation and obtained the asymptotic behaviour of $b(n)$ in the cases where $a(n) \to \alpha$, or $\dfrac{1}{n}\sum_{k=0}^{n} a(k) \to \alpha$, where $\alpha > 0$. We consider the case $\alpha = 0$ and consider O-analogues.

1. Introduction Sequences $\{a(n)\}$ and $\{b(n)\}$ related by the system of equations

$$nb(n) = a * b(n), \quad n = 0, 1, 2, \dots \tag{1.1}$$

where $a * b(n) = \sum_{k=0}^{n} a(k)b(n-k)$ denotes the convolution of the sequences, often occur in the area of mathematics. And when they do one is interested in relating the asymptotic behaviour of the two sequences. For a survey of some of the situations in which (1.1) occurs, see Hawkes and Jenkins (1978), Wright (1967a,b), Embrechts and Hawkes (1982), B.G. Hansens (1988). More recently equation (1.1) appears in risk models and given $b(n)$, the sequence $a(n)$ defined by (1.1) is called the De Pril transform of $\{b(n)\}$, see De Pril (1989), Dhaene and Sundt (1994).

Throughout the paper we assume that $a(0) = 0$ and $a(j) \geq 0$ and $b(0) = 1$. Without loss of generality (cf. Wright (1967a)) we may and do assume that $b(n) > 0$ for all sufficiently large values of n. In their paper Embrechts and Hawkes (1982) used subexponential sequences to study (1.1) and assumed that $\sum_i b(i) < \infty$. Hawkes and Jenkins (1978) considered the case where $a(n) \to \alpha$ where $0 < \alpha \leq \infty$, see theorem B below. In the present paper we consider the case $\alpha = 0$ and hereby complete the results of Hawkes and Jenkins. In analysing (1.1) generating functions will be very useful.

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Recall that for a sequence $\{u(n)\}$ the generating function $U(x)$ is defined by $U(x) = \sum_i u(i)x^i$. If $A(x)$, $B(x)$ and $D(x)$ denote the generating functions of the sequences $\{a(n)\}, \{b(n)\}$ and $\{nb(n)\}$ it follows from (1.1) that

$$D(x) = xB'(x) = A(x)B(x). \tag{1.2}$$

It is well known that the asymptotic behaviour of a sequence is determined by the asymptotic behaviour (as x tends to 1) of its generating function. The simple relation (1.2) will be fully exploited to relate the asymptotic behaviour of $\{a(n)\}$ to that of $\{b(n)\}$.

2. Results. Being interested in asymptotic behaviour we consider several types of 'regular' asymptotic behaviour. Recall that a measurable and positieve function $f(x)$ is regularly varying at infinity and with index β if for each $t > 0$, $\lim f(tx)/f(x) = t^{\beta}$. Notation: $f \in RV_{\beta}$. If $\beta = 0$, $f(x)$ is called slowly varying (SV). The class ORV of O-regularly varying functions (see e.g. Aljančić et. al. (1977)) consists of those measurable and positive functions for which $\limsup f(tx)/f(x) < \infty$, for all $t > 0$.

A sequence $\{a(n)\}$ is in RV or in ORV if the function $f(x) := a([x])$ is, see Bojanić and Seneta (1973). A subclass of SV is the class Π of de Haan (1970): $f(x)$ is in the class $\Pi(L(x))$ with auxiliary function $L(x)$ if for each $t > 0$, $\lim(f(tx) - f(x))/L(x) = \log(t)$.

It is well known that in this case $L(x) \in$ SV. Finally, a nondecreasing function $f(x)$ is in the class $\text{O}\Pi(L)$ if for each $t > 0$, $\limsup |f(tx) - f(x)|/L(x) < \infty$. In this case automatically L is in the class ORV. This class of functions is related to the class AB(L) of Geluk and de Haan (1987). For sequences $\{a(n)\}$, similar classes can be defined by considering the function $f(x) = a([x])$. The previous classes of functions are very useful when one wants to relate the asymptotic behaviour of a function to that of its Laplace transform. Recall that the Laplace transform $F(s) = L(f)(s)$ of a function $f(x)$ is defined by. $L(f)(s) = F(s) = s\int_0^{\infty} e^{-sx} f(x)dx$. For a sequence $\{a(n)\}$ we define the generating function $A(x)$ is before and for $x \geq 0$, we define the function $f_a(x) := a(0) + a(1) + \cdots + a([x])$. An easy calculation shows that $L(f_a)(s) = A(e^{-s})$. The next results will be used frequently; the proofs can be found e.g. in Geluk and de Haan (1987), Bingham et. al (1987). As usual $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$, and all $\sim$, O(1), o(1) and other limit statements deal with asymptotic behaviour as $x \to \infty$.

THEOREM A. *Assume $f(x)$ is nondecreasing and $F(s)$ is finite for all $s > 0$. Let $g(x) \in RV_{\beta}$, $\beta \geq 0$ and let $L(x) \in SV$. Then*

(i) $f(x) \sim g(x)$ *iff* $F(1/x) \sim \Gamma(1 + \beta)g(x)$,

(ii) $f(x) = \text{O}(1)g(x)$ *iff* $F(1/x) = \text{O}(1)g(x)$,

(iii) $f(x) = o(1)g(x)$ *iff* $F(1/x) = o(1)g(x)$,

(iv) $f(x) \in$ ORV *iff* $F(1/x) \in$ ORV *and both statements imply that* $F(1/x)/f(x)$ *is bounded away from 0 and* ∞.

(v) $f(x) \in \Pi(L)$ *iff* $F(1/x) \in \Pi(L)$.

(vi) $f(x) \in \text{O}\Pi(L)$ iff $F(1/x) \in \text{O}\Pi(L)$.

As an example of this method, we prove part of the results of Hawkes and Jenkins. The full result is quoted in Theorem B below. Suppose that $\alpha > 0$. Then it follows from Theorem A(i) that

$$f_a(n) \sim \alpha n \text{ iff } A(e^{-1/x}) \sim \alpha x. \tag{2.1}$$

By replacing $\exp(-1/x)$ by $1 - 1/y$, it follows that (2.1) holds iff

$$A(1 - 1/y) \sim \alpha y. \tag{2.2}$$

Now let $C(x) := B(1 - 1/x)$. Note that $x^2 C'(x) = B'(1 - 1/x)$. From (1.2) we have $xC'(x)/C(x) = A(1 - 1/x)/(x - 1)$. It easily follows that (2.2) holds iff $xC'(x)/C(x) \to \alpha$. Since $\alpha > 0$ and $C(x)$ is monotone, this is equivalent to

$$C(x) \in \mathrm{RV}_\alpha. \tag{2.3}$$

Again using Theorem A (i), we obtain that (2.3) holds iff

$$f_b(n) \in \mathrm{RV}_\alpha. \tag{2.4}$$

The result ((2.1) iff (2.4)) we proved is part (ii) of the result of Hawkes and Jenkins. The full statement of their result is the following Theorem B.

THEOREM B. (i) *If $a(n) \in \mathrm{RV}_\alpha, \alpha < 0$, then $b(n) \in \mathrm{RV}_{\alpha-1}$ and $nb(n) \sim B(1)a(n)$.*

(ii) *If $\alpha > 0$, then $f_a(n) \sim \alpha n$ if and only if $f_b(n) \in \mathrm{RV}_\alpha$.*

(iii) *If $\alpha > 0$, then $a(n) \to \alpha$ implies that $b(n) \in \mathrm{RV}_{\alpha-1}$.*

In Theorems C and D below we complete the result of Hawkes and Jenkins by considering the case $\alpha = 0$ and by considering O-analogues of the results.

THEOREM C. (i) *The following two statements are equivalent:*

$$f_a(n) = o(1)n \tag{2.5}$$

and

$$f_b(n) \in \mathrm{SV}. \tag{2.6}$$

Moreover, if $a(n) \to 0$, then $nb(n) = o(1)f_b(n)$.

(ii) *The following two statements are equivalent:*

$$f_a(n) = \mathrm{O}(1)n \tag{2.7}$$

and

$$f_b(n) \in \mathrm{ORV}. \tag{2.8}$$

Moreover, if $a(n) = \mathrm{O}(1)$, then $nb(n) = \mathrm{O}(1)f_b(n)$.

THEOREM D. (i) *The following two statements are equivalent:*

$$f_a(n)/n \subset \mathrm{SV} \text{ and (2.5) holds} \tag{2.9}$$

and

$$f_b(n) \in \Pi(L) \text{ for some } L(x) \in \mathrm{SV}. \tag{2.10}$$

Moreover, if $a(n) \to 0$ and if $a(n) \in \mathrm{SV}$, then $nb(n) = (1 + o(1))a(n)f_b(n)$.

(ii) *The following two statements are equivalent:*

$$f_a(n) = \mathrm{O}(1)L(n)n \text{ where } L(x) \in \mathrm{SV} \text{ and } L(x) = o(1). \tag{2.11}$$

and

$$f_b(n) \in \mathrm{O}\Pi(K) \text{ where } K(x)\mathrm{SV} \text{ and } K(n) = o(1)f_b(n). \tag{2.12}$$

Moreover, if $a(n) = \mathrm{O}(1)nL(n)$ *where* $L(x)\mathrm{SV}$ *and* $L(x) = o(1)$, *then* $nb(n) = \mathrm{O}(1)L(n)f_b(n)$.

In order to prove the results, we introduce some notation. As before let $A(x), B(x)$ and $D(x)$ denote the generating functions of the sequences $\{a(n)\}$, $\{b(n)\}$ and $\{nb(n)\}$. From (1.1) it follows that

$$D(x) = xB'(x) = A(x)B(x). \tag{2.13}$$

Now let $C(x) = B(1-1/x)$. It follows that $x^2C(x) = B'(1-1/x)$ and from (2.13) that

$$\frac{xC'(x)}{C(x)} = \frac{A(1-1/x)}{(x-1)} \tag{2.14}$$

and

$$x^2C'(x) \sim D(1-1/x) \tag{2.15}$$

Now define $g(x) := f_b(x), f(x) := f_{nb(n)}(x)$ and $h(x) = f(x)/x$. Obviously $G(s) = B(e^{-s})$ and $F(s) = D(e^{-s})$. Moreover straightforward calculations show that $f(x)$ and $g(x)$ are related as follows:

$$f(x) = xg(x) - \int_0^x g(s)ds \tag{2.16}$$

$$g(x) = h(x) + \int_0^x \frac{h(s)}{s}ds \tag{2.17}$$

In the sequel we use the notation [A(i)], [A(ii)] etc. when we refer to Theorem A(i) etc.

Proof of Theorem C(i). Using [A(i)], (2.5) is equivalent to $A(1-1/x) = o(1)x$; using (2.14) this in turn is equivalent to $xC'(x) = o(1)C(x)$. This implies that $C(x)$ is SV and using [A(i)], (2.6) follows. Conversely, if (2.6) holds, by [A(i)], $C(x)$ is SV and $C(x) \sim g(x)$. Using (2.16) it follows that $f(x) = o(1)xg(x)$ and an application of [A(iii)] yields $D(1-1/x) = o(1)xg(x)$. Using (2.15) we obtain $xC'(x) = o(1)C(x)$ and from (2.14) we obtain $A(1-1/x) = o(1)x$. Hence (2.5) follows.

To prove the final statement of part (i), choose $\varepsilon > 0$ and N so that $a(n) < \varepsilon$ for all $n \geq N$. Using (1.1) we write $nb(n) = \left(\sum_{k=0}^{N} + \sum_{k=N+1}^{n}\right) a(k)b(n-k) :=$I+II Since $a(n)$ is bounded by, say, M, we have I$\leq M(g(n) - g(n-N-1))$. Since $g(x)$ is SV, it follows that I$= o(1)g(n)$. As to II we have II$\leq g(n-N-1)$. Using $g(x) \in$ SV this yields $\limsup \mathrm{II}/g(n) \leq \varepsilon$. Now combine the two estimates and let $\varepsilon \to 0$ to obtain that $nb(n) = o(1)g(n)$. This proves the result.

Proof of Theorem C(ii). Using (2.14) and [A(ii)] we have (2.7) iff

$$xC'(x) = \mathrm{O}(1)C(x) \tag{2.18}$$

Hence $C(x) \in \mathrm{ORV}$ and by [A(iv)] we obtain (2.8). Conversely, in (2.8) we state that $g(x) \in \mathrm{ORV}$ and from [A(iv)] it follows $C(x) \in \mathrm{ORV}$ and $g(x) = \mathrm{O}(1)C(x)$. To prove (2.18) note that $x^2C'(x)$ is nondecreasing. For fixed $t > 0$ we obtain that

$$xC'(x)\left(1-\frac{1}{t}\right) \le \int_x^{tx} C'(s)ds = C(xt) - C(x) \tag{2.19}$$

Using $C(x) \in \mathrm{ORV}$, it follows that (2.18) holds and hence also (2.7). The final statement of part (ii) follows as in part (i).

Proof of Theorem D(i). Using [A(i)] and [A(ii)] (2.9) holds iff $A(1-1/x)/x \in \mathrm{SV}$ and $A(1-1/x) = o(1)x$. Using (2.14) this is equivalent to $xC'(x)/C(x) \in \mathrm{SV}$ and $xC'(x) = o(1)C(x)$. This is equivalent to $C'(x) \in \mathrm{RV}_{-1}$. Using (2.15) it follows that (2.9) holds iff $D(1-1/x) \in \mathrm{RV}_1$ which in turn holds iff $f(x) \in \mathrm{RV}_1$. Using (2.17) this is equivalent to $g(x) \in \Pi(h(x))$.

To prove the final statement of part (i) we first obtain the exact asymptotic form of the function $h(x)$. Since $a(n) \in \mathrm{SV}$ it follows from [A(i)] and (2.14) that $A(1-1/x) \sim xa([x])$ and $xC'(x)/C(x) \sim a([x])$. Since also $g(x) \sim C(x) \in \mathrm{SV}$ it follows that $xC'(x) \sim a([x])g(x)$. Using (2.15) we obtain $D(1-1/x) \sim xa([x])g(x) \in \mathrm{RV}_1$ and consequently $f(x) \sim xa([x])g(x)$. This shows that $h(x) \sim a([x])g(x)$. Now we rewrite nb(n) as follows, cf. (1.1): for $r > 0$ we have $nb(n) = \left(\sum_{k=0}^{[nr]} + \sum_{k=[nr]+1}^{n}\right) a(k)b(n-k) = \mathrm{I} + \mathrm{II}$. Since $a(k)$ is bounded, by, say M, we have $\mathrm{I} \le M(g(n) - g(n-[nr]))$. Since $g(x) \in \Pi(h(x))$ it follows that $\limsup \mathrm{I}/h(n) \le M\log(1/(1-r))$. As to the second term II, since $a(n)$ is SV we have $|a(k)/a(n) - 1| \to 0$ uniformly in k, $[nr]+1 \le k \le n$. It follows that II $\sim a(n)g(n-[nr]-1)$. Since g is SV we obtain that II $\sim a(n)g(n) \sim h(n)$. Combining the two estimates, it follows that $\limsup |nb(n)/h(n) - 1| \le M\log(1/(1-r)$. Now let $r \to 0$ to obtain the desired result.

Proof of Theorem D(ii). As before we have (2.11) iff $xC'(x) = \mathrm{O}(1)L(x)C(x)$ and $L(x) \in \mathrm{SV}$, $L(x) = o(1)$. This implies $C(x) \sim g(x)$ SV and $D(x) = \mathrm{O}(1)xL(x)C(x)$. Hence also $f(x) = \mathrm{O}(1)xL(x)C(x)$. Using (2.17) we obtain $g(x) \in \mathrm{O}\Pi(K(x))$ where $K(x) = C(x)L(x) \sim g(x)L(x) \in \mathrm{SV}$ and $K(x) = o(1)g(x)$. This is (2.12).

Conversely, from (2.12) we obtain $C(x) \sim g(x) \in \mathrm{SV}$, $C(x) \in \mathrm{O}\Pi(K(x))$ and $K(x) = o(1)C(x)$. Using (2.19) we have $xC'(x) = \mathrm{O}(1)K(x)$. If we set $L(x) = K(x)/C(x)$ we have $L(x) \in \mathrm{SV}$, $L(x) = o(1)$ and $xC'(x)/C(x) = \mathrm{O}(1)L(x)$. Hence $A(1-1/x) = \mathrm{O}(1)xL(x)$ and (2.11) follows.

The final statement of part (ii) follows as in the proof of the final statment in part (i).

3. Concluding remarks 1) For nondecreasing functions $f(x)$ and $g(x)$ with $f(0) = g(0) = 0$, consider the convolution equation $xf(x) = f \# g(x)$ where $\#$ denotes the Stieltjes convolution (i.e.). $f \# g(x) = \int_0^x f(x-y)dg(y)$). Using Laplace transforms one easily obtains that $x(xF(1/x))' = F(1/x)G(1/x)$ and with

$C(x) = xF(1/x)$ we have $xC'(x)/C(x) = G(1/x)/x$. If for example $g(x) \sim \alpha x$ wtih $\alpha \geq 1$, then [A(i)] shows that $G(1/x) \sim \alpha x$ and hence $xC'(x)/C(x) \to \alpha$. This implies that $C(x) \in \mathrm{RV}_\alpha$ and $F(1/x) \in \mathrm{RV}_{\alpha-1}$. It follows that $f(x) \subset \mathrm{RV}_{\alpha-1}$.

2) Sundt (1982) studied discrete probability distributions $\{b(n)\}$ for which there exist sequences $\{c(n)\}$ and $\{a(n)\}$ so that $nb(n) = nc * b(n) + a * b(n)$ The case where $c(n) = 0$ corresponds to (1.1). It may be of interest to study the relationship between the asymptotic behaviour of the sequences involved.

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