PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 58 (72), (1995), 51–70
Slobodan Aljančić memorial volume

REGULAR VARIATION ON HOMOGENEOUS CONES

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To the memory of Professor S. Aljančić

Abstract. The notion of regular variation is extended to functions defined on homogeneous cones in $\mathbf{R}^n$. For these functions we prove the Uniform Convergence Theorem and the Representation Theorem.

Regularly varying functions were defined by Karamata [**10**] in 1930, and the theory of regular variation was developed by him and some of his students (see, for example, [**11, 1, 2, 3**]). Nowadays, there are several books on this theory: [**13**], [**4**] and [**6**] (see also the very extensive bibliography in [**4**]). The following is a standard definition of regular variation.

Definition 1. A function F is said to be *regularly varying at infinity* if it is real-valued, positive and measurable on $[A,\infty)$, for some $A>0$, and if for every $y>0$ the limit

$$\lim_{x\to\infty}\frac{F(xy)}{F(x)}=\Phi(y) \tag{1}$$

exists. □

In this paper we shall extend this definition to functions of several variables. Regular variation in many variables was investigated much less than in one variable. The references [**3, 5, 8, 9, 12, 14**] make a representative, though not an exhaustive list. Most of the papers deal with functions defined on $\mathbf{R}^n_+$ (the cone in $\mathbf{R}^n$ whose elements have all coordinates positive). Jakymiv [**8, 9**] introduces regular variation for functions defined on arbitrary cones, but his notion of convergence is very close to radial convergence. On the other hand, Bajšanski and Karamata [**3**] deal with functions defined on topological groups.

We shall consider functions defined on homogeneous cones (these are cones that have a transitive group of automorphisms), thus choosing a middle way between [**8**] and [**3**] (but relying more on [**3**]). On the one hand, the cones will have

AMS Subject Classification (1991): Primary 26A12; Secondary 43A85

more structure than the arbitrary cones in [**8**], and each will determine its own class of regularly varying functions. For example, by looking just at what is the generalization of the power functions, we see that two homogeneous cones carry two essentially different classes of regularly varying functions. On the other hand, the group of automorphisms of the cone (which is a Lie group) has much more properties than an arbitrary topological group. In particular, it has the exponential mapping, which makes it possible to develop an approach in many respects similar to the one-dimensional case.

Our definition of regular variation in many variables is formally similar to Definition 1 above, and many properties of regular variation will follow along the same lines as in the one-dimensional case. For example, it is well known and easily seen that once the limit $\Phi(u)$ (called the index function) exists in (1), it is necessarily of the form $\Phi(u) = u^\alpha$, for some $\alpha \in \mathbf{R}$ (this is usually called the Characterization Theorem). In other words, Φ is a homomorphism from the group $(\mathbf{R}_+, \cdot)$ into itself. We shall have, similarly, that the index function in many variables is a homomorphism of the group of automorphisms of the cone into $(\mathbf{R}_+, \cdot)$.

There is another well-known fact in the theory of regular variation which is used as a standard trick. Instead of dealing with functions as in Definition 1, we introduce the change of variables

$$f(u) = \log F(e^u) \tag{2}$$

and we pass to the (equivalent) "additive" class of functions $f\colon \mathbf{R} \to \mathbf{R}$, such that the limit

$$\lim_{u\to\infty} (f(u+v) - f(u)) = \phi(v) \tag{3}$$

exists. (We have put $y = e^v$ and $\phi(v) = \log \Phi(e^v)$ in (1).)

There will be an analogous "additive" class in many variables, too. The "change of variables" like the one in (2) will be provided by the exponential mapping of the Lie group G of the automorphisms of the cone

$$\exp\colon \mathfrak{g} \to G \tag{4}$$

where $\mathfrak{g}$ is the Lie algebra of G. For a homogeneous cone the group G has the property that (4) is a diffeomorphism onto, so we shall have again the equivalence of the classes of regularly varying functions and of "additively regularly varying" functions. Since $\mathfrak{g}$ is isomorphic with $\mathbf{R}^n$, we have to deal with functions defined on $\mathbf{R}^n$ satisfying a condition like (3).

We first examine in section 2 these functions, which are regularly varying with respect to an abelian group (the vector space addition). In this more elementary setting we prove the main theorems of the theory of regular variation: the Uniform Convergence Theorem and the Representation Theorem. Everything pertinent to regular variation is contained in this section.

In section 3 we prove some facts about filters with respect to which the limit is taken in the definition of regular variation.

Section 4 deals with the relationship between "multiplicative" and "additive" regular variation. We start with a regularly varying function defined on the group of a homogeneous cone and apply the exponential mapping to obtain an additively regularly varying function. However, we now work with a more restricted definition (Definition 4.1) in which uniformity is assumed with respect to the nilpotent variables. In this setting we have the equivalence of the two classes of functions.

At the end of this introduction let us make two more remarks. The function in Definition 1 is defined in the neighbourhood of infinity. It is easy to have instead a function defined on the whole half-line: just put, for example, $F(x) = 1$ on the complement of $[A, \infty)$. That is why we shall have functions defined on the whole cone, without loss of generality. On the other hand, we shall assume that all our functions are smooth (since they are defined on Lie groups). This can be done without affecting the regular variation property of F, since (1) is a condition on the *oscillation* of F in the neighbourhood of infinity and is not related to the smoothness. Indeed, it is well known that for a regularly varying function as in Definition 1, there is an asymptotically equivalent smooth regularly varying function.

1. Introduction

1.1. Definition of regular variation. In this section we give a definition of regular variation for functions defined on an arbitrary topological group. This is basically the definition of Bajšanski and Karamata [**3**]. Later we shall have two special cases of this definition. First we shall consider regularly varying functions on the group of automorphisms of a homogeneous cone V in $\mathbf{R}^n$ (Definition 1.5). As a second application of the definition we shall have regular variation with respect to the additive group of the vector space $\mathbf{R}^n$ (which carries the Lie algebra of the group of the cone) (Definition 2.2). In Section 4 we show that there is a close relationship between these two classes.

Definition 1.1. Let G be a topological group. Let $\mathcal{U}$ be a filter of open convex subsets of G with a countable basis. Then $\mathcal{U}$ is said to be *G-invariant* if

$$U \in \mathcal{U} \quad \text{and} \quad h \in G \qquad \text{implies} \qquad Uh \in \mathcal{U} \quad \text{and} \quad hU \in \mathcal{U}. \quad \square \tag{1.1}$$

We shall denote by $g \underset{\mathcal{U}}{\rightarrow} \infty$, or simply by $g \to \infty$, the convergence with respect to this filter. Note that (1.1) means

$$g \underset{\mathcal{U}}{\rightarrow} \infty \qquad \text{implies} \qquad gh \underset{\mathcal{U}}{\rightarrow} \infty \quad \text{and} \quad hg \underset{\mathcal{U}}{\rightarrow} \infty \qquad \text{for every } h \in G. \tag{1.1$'$}$$

Definition 1.2. Let G be a topological group and $(\mathbf{R}_+, \cdot)$ the multiplicative group of positive real numbers. Let $F: G \to \mathbf{R}_+$ be a continuous function, and let $\mathcal{U}$ be a G-invariant filter in G. Then F is said to be *regularly varying* with respect to $\mathcal{U}$, if the following limit

$$\lim_{g \underset{\mathcal{U}}{\rightarrow} \infty} \frac{F(gh)}{F(g)} = \Phi(h) \tag{1.2}$$

exists for every $h \in G$. $\square$

Remark. It is possible to consider analogous "right-sided" regular variation, with the quotient in (1.2) replaced by $F(hg)/F(g)$.

LEMMA 1.3. (The Characterization Theorem) *Let $F: G \to \mathbf{R}_+$ be a regularly varying function. Then the function F in* (1.2) *is a homomorphism of G into $\mathbf{R}_+$*.□

Proof. Put $F_g(h) = F(gh)/F(g)$. Then we have

$$F_g(h_1 h_2) = F_{gh_1}(h_2) F_g(h_1). \tag{1.3}$$

Now let $g \underset{\mathcal{U}}{\to} \infty$; then by property (1.1′) we have that $gh_1 \underset{\mathcal{U}}{\to} \infty$ and thus $\Phi(h_1 h_2) = \Phi(h_2)\Phi(h_1)$. □

1.2. Homogeneous cones. The theory of homogeneous cones was founded in [**7**] and [**15**].

Definition 1.4. Let V be an open, convex cone in $\mathbf{R}^n$. The cone V is called *homogeneous* if its group of linear automorphisms is transitive on V. □

In more detail the definition states the following. Consider the subgroup $\mathrm{GL}(V)$ of the general linear group $\mathrm{GL}(\mathbf{R}^n)$ that leaves the cone invariant, i.e. such that $gV = V$, for every $g \in \mathrm{GL}(V)$. The group $\mathrm{GL}(V)$ is said to be *transitive* on V if for every two points x and y of V there is a $g \in \mathrm{GL}(V)$ such that $x = gy$. If we fix an $e \in V$, this is equivalent with: for every $x \in V$ there is an element $g \in \mathrm{GL}(V)$ such that $x = ge$.

When such a correspondence between the elements of V and the elements of a subgroup $G \subseteq \mathrm{GL}(V)$

$$\pi: G \to V, \qquad g \mapsto x \tag{1.4}$$

is one-to-one, the group G is said to be *simply transitive.* Then for every two points x and y of V we have a *unique* $g \in G$ such that $x = gy$.

A most important property of homogeneous cones is that they always have a simply transitive group of automorphisms.

PROPOSITION (see [**15**]). *Let V be a homogeneous cone in $\mathbf{R}^n$. Then there exists a subgroup G in the group of all linear automorphisms* $\mathrm{GL}(V)$ *of the cone V which is simply transitive on V. This group is the maximal triangular subgroup of* $\mathrm{GL}(V)$. □

Here a group is said to be triangular if there is a basis in $\mathbf{R}^n$ in which all the elements of the group are written in the form of (upper) triangular matrices (with real coefficients). In other words, this group is real solvable. This simply transitive group G will be called *the group of the cone.*

Now, on a homogeneous cone one can define a *product* of its elements x and y, which are written as $x = g_x e$ and $y = g_y e$ according to (1.4), by putting

$$xy = g_x g_y e. \tag{1.5}$$

The mapping π from (1.4) makes it also possible to correlate with every function $F: V \to \mathbf{R}_+$ a function $\bar{F}: G \to \mathbf{R}_+$ by putting

$$\bar{F} = F \circ \pi \tag{1.6}$$

Remark. To simplify notation, we shall sometimes omit the bar and use the same letter for both functions, when no confusion is likely.

1.3. Regularly varying functions on V. Let F be a positive function defined on the homogeneous cone. We shall say that F is regularly varying if the corresponding function $\bar{F}$, defined on the group of the cone G by (1.6), is regularly varying in the sense of Definition 1.2. More precisely, given a filter $\mathcal{U}$ of subsets of G define a filter of subsets of V by $\mathcal{W} = \pi(\mathcal{U})$. It is obvious that the filter $\mathcal{W}$ is *translation invariant* in V, in the sense that it has a countable basis and that

$$W \in \mathcal{W} \quad \text{and} \quad x \in V \qquad \text{implies} \qquad Wx \in \mathcal{W} \quad \text{and} \quad xW \in \mathcal{W} \tag{1.7}$$

where the product Wx and xW is defined in (1.5).

Definition 1.5. A smooth function $F: V \to \mathbf{R}_+$ is said to be *regularly varying* with respect to a translation invariant filter $\mathcal{W}$ if for every $y \in V$ the limit

$$\lim_{x \overrightarrow{\mathcal{W}} \infty} \frac{\bar{F}(xy)}{\bar{F}(x)} = \Phi(y) \tag{1.8}$$

exists. □

Obviously, F is regularly varying if and only if the corresponding function $\bar{F}$ defined in (1.6) is regularly varying, that is

$$\lim_{g \overrightarrow{\mathcal{U}} \infty} \frac{\bar{F}(gh)}{\bar{F}(g)} = \bar{\Phi}(h) \tag{1.9}$$

where now the limit is taken with respect to the filter $\mathcal{U} = \pi^{-1}(\mathcal{W})$ in G; (1.9) is obtained from (1.8) by just putting $x = ge$ and $y = he$.

1.4. The exponential mapping. Let, as above, G be the simply transitive group of the cone V and let $\mathfrak{g}$ be the Lie algebra of G. Since G is real solvable, the exponential mapping

$$\exp: \mathfrak{g} \to G \tag{1.10}$$

is a diffeomorphism onto.

This is a very important property for our considerations since it shall enable us to correlate with every function F defined on G a companion function f defined on the Lie algebra $\mathfrak{g}$. By (1.10), we have that

$$\log: G \to \mathfrak{g} \tag{1.11}$$

is defined globally (and gives a coordinate system for G). This means that for every $g \in G$ there is a unique $X \in \mathfrak{g}$ such that $g = \exp X$.

Now to every $F: G \to \mathbf{R}_+$ there is a corresponding function $f: \mathfrak{g} \to R$ defined by

$$f(X) = \log F(\exp X); \tag{1.12}$$

this is in fact the coordinate form of F in the coordinates of (1.11) (compare with (2) in the Introduction).

Next, let $F: G \to \mathbf{R}_+$ be a regularly varying function. By imitating the one-dimensional case, we shall introduce the "change of variables" of (1.12) into the definition (1.9). For $g, h \in G$, let X and Y be the corresponding unique elements of $\mathfrak{g}$

$$g = \exp X, \quad h = \exp Y. \tag{1.13}$$

Let $\mathcal{S}$ be the inverse image under the mapping exp in (1.10) of a G-invariant filter $\mathcal{U}$ in G, i.e.

$$\mathcal{S} = \{S \subseteq \mathfrak{g} : \exp S \in \mathcal{U}\}. \tag{1.14}$$

This shall be written as $\exp \mathcal{S} = \mathcal{U}$. Then $\mathcal{S}$ is obviously a filter of subsets in $\mathfrak{g}$. In section 3 we shall prove that the G-invariance of $\mathcal{U}$ implies the invariance of $\mathcal{S}$ with respect to the additive group in (the vector space) $\mathfrak{g}$.

Now put (1.12), (1.13), (1.14) and

$$\phi(X) = \log \Phi(\exp X) \tag{1.15}$$

into the defining relation (1.9) of regular variation. Although it is obviously not true that $\exp X \exp Y = \exp(X+Y)$, we shall see in section 4 below that under some additional conditions the function f defined in (1.12) still satisfies a relation of the form

$$\lim_{X \xrightarrow[\mathcal{S}]{} \infty} (f(X+Y) - f(X)) = \phi(Y)$$

which means that it is regularly varying with respect to the additive group in the vector space $\mathbf{R}^n$ that carries $\mathfrak{g}$. We shall call such functions *additively regularly varying.* We shall first study these functions.

2. Additive regular variation

2.1. Definition. We now write down the special case of Definition 1.1 and 1.2 for the abelian group $(\mathbf{R}^n, +)$. We shall have a small modification: the codomain of the function is the additive group of reals $(\mathbf{R}, +)$, rather than $(\mathbf{R}_+, \cdot)$.

Definition 2.1. Let $\mathcal{S}$ be a filter of open convex subsets in $\mathbf{R}^n$ with a countable basis. Then $\mathcal{S}$ is said to be *additively invariant,* or *+-invariant,* if for every $S \in \mathcal{S}$ and for every $x \in \mathbf{R}^n$, the set $x + S$ belongs to $\mathcal{S}$. □

Definition 2.2. Let $\mathcal{S}$ be a +-invariant filter in $\mathbf{R}^n$. A smooth function $f: \mathbf{R}^n \to \mathbf{R}$ is said to be *additively regularly varying* with respect to $\mathcal{S}$ if for every $y \in \mathbf{R}^n$ the following limit exists

$$\lim_{x \xrightarrow[\mathcal{S}]{} \infty} (f(x+y) - f(x)) = \phi(y). \quad \square \tag{2.1}$$

For these functions we shall show that they have the main properties of regular variation, in complete analogy to the classical theory of one variable. This section is independent from the rest of the paper. It presents the theory of regular variation in the abelian case.

Write, for short

$$f_x(y) = f(x+y) - f(x) \tag{2.2}$$

for the expression on the left-hand side of (2.1). Then we have a familly of functions f_x converging to f, for which we prove in the next lemma that it is a linear function. The functions f_x themselves are not linear; still they satisfy the equation

$$f_x(y+u) = f_x(y) + f_{x+y}(u) \tag{2.3}$$

which is an additive version of (1.3).

LEMMA 2.1. *The function ϕ in* (2.1) *is linear; that is $\phi(y) = \alpha_1 y_1 + \cdots + \alpha_n y_n$ for some $\alpha_i \in \mathbf{R}$.* □

Proof. By Lemma 1.3, it follows that ϕ is a homomorphism from $\mathbf{R}^n$ into $\mathbf{R}$, i.e. an additive function $\phi(y+u) = \phi(y) + \phi(u)$. But this is enough to ensure linearity. □

For an (additively) regularly varying function f, we shall call the limit function ϕ, or even the corresponding vector α, the index of regular variation. When $\alpha = 0$, the function f is called *additively slowly varying.* Every additively regularly varying function is the sum of its *index* function and of a slowly varying function $l\colon f = \phi + l$. Thus it is enough to study the properties of additively slowly varying functions.

2.2. The Uniform Convergence Theorem. The most important property of regularly varying functions, from which all other properties follow easily, is the fact that the limit in the defining equation (2.1), which is assumed to exist only pointwise in y, holds in fact uniformly in y on compact sets in $\mathbf{R}^n$. This is Theorem 2.4 below.

The proof, divided into several lemmas, is essentially a generalization of the Banach–Steinhaus Theorem, applied to the functions f_x, which are not linear, but satisfy instead equation (2.3) – a substitute for additivity. Note that we don't need any substitute for homogeneity; thus the theorem holds for any topological group. The proof is essentially a modification of a proof given in [**3**].

LEMMA 2.2. *Let $f_x\colon \mathbf{R}^n \to \mathbf{R}$, with $x \in \mathbf{R}^n$, be a family of continuous functions; let $\mathcal{S}$ be a filter with a countable basis and let $\mathcal{B} = \{B_k : k \in \mathbf{N}\}$ be a basis for $\mathcal{S}$. If for every $u \in \mathbf{R}^n$*

$$\lim_{x \xrightarrow[\mathcal{S}]{} \infty} f_x(u) = 0 \tag{2.4}$$

then for every $\varepsilon > 0$ there is a $k \in \mathbf{N}$ and an open set $U \subseteq \mathbf{R}^n$ such that

$$x \in B_k \implies \sup_{u \in U} |f_x(u)| < \varepsilon. \quad \square \tag{2.5}$$

Remark. A family of functions f_x satisfying condition (2.5) is said to be *asymptotically equicontinuous* at (the points of) the set U (see [**3**]). In the next lemma, when the filter is $+$-invariant and the functions f_x are of a special form, the set U will be moved to the origin. Finally, in Theorem 2.4 a compact set will be covered with translations of this neighbourhood of zero to prove uniform convergence on compacts.

Proof. By the assumption (2.4) we have that

$$(\forall\varepsilon > 0)(\forall u \in \mathbf{R}^n)(\exists k)(\forall x \in B_k)|f_x(u)| < \varepsilon. \tag{2.6}$$

Let us fix an ε. Consider for $x \in \mathbf{R}^n$ the set $E_x := \{u \in \mathbf{R}^n : |f_x(u)| \le \varepsilon/2\}$. The set E_x is closed, since f_x is continuous. For $k \in \mathbf{N}$ put $E_k = \bigcap_{x\in B_k} E_k$. The set E_k is closed, as an intersection of closed sets. Now by the assumption (see (2.6)), for every $u \in \mathbf{R}^n$ there is a $k \in \mathbf{N}$ such that $u \in E_k$. In other words, $\mathbf{R}^n = \bigcup_{k\in\mathbf{N}} E_k$.

Thus $\mathbf{R}^n$ is a countable union of closed sets, and by Baire's Category Theorem, we have that at least one of the sets E_k contains an open set U. That is, there exist a k_0 and a U such that $U \subseteq E_{k_0}$, which, in more detail, means

$$(\exists k_0)(\forall u \in U)(\forall x \in B_{k_0})|f_x(u)| \le \varepsilon/2$$

and this is exactly (2.5). This proves the lemma. □

Lemma 2.3. *Let f be an additively slowly varying function with respect to a $+$-invariant filter $\mathcal{S}$, i.e., let for $f_x(u) = f(x+u) - f(x)$ and for every $u \in \mathbf{R}^n$*

$$\lim_{x \underset{\mathcal{S}}{\to} \infty} f_x(u) = 0.$$

Then for every $\varepsilon > 0$ there is an element B_k of the filter basis and an open neighbourhood W of zero such that

$$x \in B_k \implies \sup_{u\in W} |f_x(u)| < \varepsilon \quad \square \tag{2.7}$$

Proof. Let us fix $\varepsilon > 0$. By the preceding lemma, there is a B_k and an open set U on which (2.5) holds. Let $u_0 \in U$ and put $W = U - u_0$. Thus every $u \in U$ is written as $u = w + u_0$, for some $w \in W$. Now by (2.2)

$$\sup_{w\in W} |f_{x+u_0}(w)| \le \sup_{u\in U} |f_x(u)| + |f_x(u_0)|$$

By (2.5) if $x \in B_k$, then the right-hand side is $< 2\varepsilon$. Thus

$$x \in B_k \implies \sup_{w\in W} |f_{x+u_0}(w)| < 2\varepsilon.$$

But $x + u_0 \in u_0 + B_k$; let $B_{k'}$ be the element of the filter basis contained in $u_0 + B_k$ ($B_{k'}$ exists by the $+$-invariance of the filter). Now by putting this k' into (2.7) we obtain the lemma. □

Theorem 2.4. (The Uniform Convergence Theorem) *Let $f\colon \mathbf{R}^n \to \mathbf{R}$ be an additively slowly varying function with respect to a $+$-invariant filter $\mathcal{S}$. Then* *the limit* (2.1) *is uniform on compact sets in* $\mathbf{R}^n$. *That is, for every compact set* $K \subseteq \mathbf{R}^n$ *we have*

$$\lim_{x \underset{\mathcal{S}}{\to} \infty} \sup_{u\in K} |f(x+u) - f(x)| = \lim_{x \underset{\mathcal{S}}{\to} \infty} \sup_{u\in K} |f_x(u)| = 0. \quad \square$$

Proof. Let $K \subseteq \mathbf{R}^n$ be a compact set and, given an $\varepsilon > 0$, let W be the neighbourhood of zero from Lemma 2.3, i.e. such that (2.7) holds. By compactness, there are points $t_i \in K$, $i = 1, \dots, m$, such that $K \subseteq \bigcup_{i=1}^m (t_i + W)$. Denote by W_i the translations $t_i + W$; then

$$\sup_{u\in K} |f(u)| \le \max_{1\le i\le m} \sup_{u\in W_i} |f_x(u)| \tag{2.8}$$

and as before, by (2.2)

$$\sup_{u\in W_i} |f_x(u)| = \sup_{w\in W} |f_x(t_i + w)| \le |f_x(t_i)| + \sup_{w\in W} |f_{x+t_i}(w)| \tag{2.8$'$}$$

Now if B_k is as in Lemma 2.3 and $x \in -t_i + B_k$, then by (2.7) the last supremum is $< \varepsilon$. Thus if $B_{k'}$ is the element of the filter basis contained in $\bigcap_{i=1}^m (-t_i + B_k)$, then

$$x \in B_{k'} \implies \sup_{w\in W} |f_{x+t_i}(w)| < \varepsilon \qquad \text{for } i = 1, \dots, m$$

On the other hand, since $f_x(t_i)$ tends to 0, for each t_i, we can find a $B_{k''}$ such that

$$x \in B_{k''} \implies |f_x(t_i)| < \varepsilon \qquad \text{for } i = 1, \dots, m.$$

Finally, let $B_{k_0} \subseteq B_{k'} \cap B_{k''}$; then for $x \in B_{k_0}$ we have that (2.8′) is $< 2\varepsilon$ and then also (2.8) is less than 2ε. This proves the theorem. $\square$

Corollary 2.5. *Let* f *be an additively regularly varying function with index function* ϕ, *and* K *a compact set in* $\mathbf{R}^n$. *Then*

$$\lim_{x\to\infty} \sup_{u\in K} |f_x(u) - \phi(u)| = 0. \quad \square$$

2.3. The Representation Theorem. The Uniform Convergence Theorem will be used to obtain the following "representation".

Theorem 2.6. *Let* $f\colon \mathbf{R}^n \to \mathbf{R}$ *be an additively slowly varying function, with respect to a* $+$*-invariant filter* $\mathcal{S}$. *Then there are two smooth functions* h *and* $\delta\colon \mathbf{R}^n \to \mathbf{R}$ *such that* $f(x) = h(x) + \delta(x)$ *with*

$$\lim_{x\to\infty} \frac{\partial h}{\partial x_i} = 0, \qquad i = 1, \dots, n \tag{2.9}$$

and

$$\lim_{x\to\infty} \delta(x) = 0. \tag{2.10}$$

Proof. Let $I_n = [0,1] \times \cdots \times [0,1]$ be the unit interval in $\mathbf{R}^n$. Put

$$h(x) = \int_{I_n} f(x+t)\,dt \quad \text{and} \quad \delta(x) = f(x) - h(x) = \int_{I_n} (f(x) - f(x+t))\,dt$$

(where $dt = dt_1 \cdots dt_n$). We shall prove that these two functions satisfy the conditions of the theorem. First $|\delta(x)| \le \sup_{t \in I_n} |f(x) - f(x+t)|$ and (2.10) follows by the Uniform Convergence Theorem applied to the interval I_n.

Next for h we have

$$\frac{\partial h}{\partial x_1}(x) = \int_{I_{n-1}} (f(x_1+1, x_2+t_2, \dots, x_n+t_n) - f(x_1, x_2+t_2, \dots, x_n+t_n))\,dt_2 \dots dt_n$$

and by adding and subtracting $f(x)$ we obtain two terms to which we can apply the Uniform Convergence Theorem: for the first term with respect to $t \in \{1\} \times I_{n-1}$ and for the second with respect to I_{n-1}. (Here I_{n-1} is the $(n-1)$-dimensional interval in the variables $t_2, \dots, t_n$.) This proves that the first partial derivative of h tends to zero. The same reasoning applied to the other partial derivatives proves (2.9). This proves the theorem. $\square$

COROLLARY 2.7. *A function $f\colon \mathbf{R}^n \to \mathbf{R}$ is additively slowly varying if and only if there are two smooth functions h and $\delta\colon \mathbf{R}^n \to R$ such that $f(x) = h(x) + \delta(x)$ with*

$$\lim_{x \overrightarrow{\mathcal{S}} \infty} \|h'(x)\| = 0 \tag{2.11}$$

and $\lim_{x \overrightarrow{\mathcal{S}} \infty} \delta(x) = 0$. (Here $h'(x)$ is the derivative of h at x.) $\square$

Proof. The necessity follows from the theorem. To prove the sufficiency, we only need to prove that h (satisfying (2.11)) is additively slowly varying. For every $x, t \in \mathbf{R}^n$ we have $h(x+t) - h(x) = h'(x+\theta t)(t)$ with $0 < \theta < 1$. (This is Lagrange's theorem applied to the function $h \circ \eta$, where $\eta\colon \mathbf{R} \to \mathbf{R}^n$, $\eta(\tau) = (1-\tau)x + \tau(x+t)$, maps the interval $[0,1]$ to the line segment with endpoints x and $x+t$.) Thus, by (2.11), h is additively slowly varying. $\square$

3. Invariant filters

Next we would like to transfer the properties of the function f of the previous section to the function F related as in (1.12). In doing this we encounter two problems: we need to know that the filter $\mathcal{S}$ is $+$-invariant, and we need a substitute for $\exp(X+Y) = \exp X \exp Y$. This will be solved in section 3.5 and section 3.2, respectively.

3.1. The Lie algebra $\mathfrak{g}$. We start with some facts about the Lie algebra of the group of the cone, which were established in [**15**]. The Lie algebra $\mathfrak{g}$ decomposes into

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$$

where $\mathfrak{a}$ is the maximal abelian (diagonal) subalgebra, and $\mathfrak{n}$ is the nilpotent subalgebra. We have

$$n = [\mathfrak{g}, \mathfrak{g}] \qquad \text{and} \qquad [\mathfrak{a}, \mathfrak{n}] \subseteq n. \tag{3.1}$$

Thus $\mathfrak{n}$ is the derived subalgebra (ideal) of $\mathfrak{g}$, and $\mathfrak{a}$ normalizes $\mathfrak{n}$.

Let $A = \exp \mathfrak{a}$ and $N = \exp \mathfrak{n}$ be the connected Lie subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$. The Lie group G is decomposed accordingly: $G = AN$, which means that every element $g \in G$ is written in a unique way as

$$g = an \tag{3.2}$$

with $a \in A$ and $n \in N$. If g_1 and g_2 are as in (3.2), then their product is equal to

$$g_1 g_2 = an \tag{3.3}$$

with $a = a_1 a_2$ and $n = a_2 n_1 a_2^{-1} n_2 \in N$ (by (3.1)).

Let $E_1, \dots, E_m$ be a basis for the abelian subalgebra $\mathfrak{a}$. Then an element $H \in \mathfrak{a}$ is written as

$$H = H(t) = t_1 E_1 + \cdots + t_m E_m \tag{3.4}$$

with $t_i \in \mathbf{R}$. The dimension m of $\mathfrak{a}$ is called the *rank* of $\mathfrak{g}$ (and also the rank of the cone).

The nilpotent subalgebra $\mathfrak{n}$ is decomposed into a direct sum of subspaces

$$\mathfrak{n} = \sum_{i<j} \mathfrak{n}_{ij} \qquad i, j = 1, \dots, m \tag{3.5}$$

where $\mathfrak{n}_{ij}$ are the root spaces of the adjoint representation, i.e., for $Z_{ij} \in \mathfrak{n}_{ij}$

$$[H(t), Z_{ij}] = (t_i - t_j) Z_{ij} \tag{3.6}$$

with the corresponding roots $\lambda_{ij}(H) = (t_i - t_j)$. The dimensions d_{ij} of the spaces $\mathfrak{n}_{ij}$ can be arbitrary—they may be different from one another and possibly zero.

(*Remark.* When all d_{ij} are equal to the same number d, we obtain the classical symmetric (self-adjoint) cones, and the only possible choices for d (when $m \geq 3$) are 1, 2 and 4 (and 8 only in the case $m = 3$). These are the cones of positive definite matrices with real, complex or quaternionic coefficients, respectively. There is one more type of symmetric cone: when the rank is 2, there is only one root space $\mathfrak{n}_{12}$ and its dimension may be arbitrary. This is the circular cone in $\mathbf{R}^n$.)

3.2. Some exponential formulas. In this section we look for formulas for exp of the sum.

Lemma 3.1. *Let $g = \exp X = \exp(H + Z)$, with $H \in \mathfrak{a}$ and $Z \in \mathfrak{n}$, be decomposed according to* (3.2) *as $g = an = \exp H' \exp Z'$ with some $H' \in \mathfrak{a}$ and $Z' \in \mathfrak{n}$. Then $H' = H$.* □

Proof. Indeed, we have

$$\exp(H+Z) = \exp H' \exp Z'. \tag{3.7}$$

Now, by the Campbell–Hausdorff formula (which holds globally in an exponential group), $\exp(-H')\exp(H+Z)$ equals the exponential of

$$(-H'+H+Z) + \frac{1}{2!}[-H', H+Z] + \frac{1}{3!}[-H',[-H',H+Z]] + \cdots \tag{3.8}$$

and in order to have (3.7) we must have that (3.8) is an element of $\mathfrak{n}$. By (3.1), all the terms in (3.8) starting from the second belong to $\mathfrak{n}$. Thus to have $-H'+H+Z \in \mathfrak{n}$ we must have $H' = H$, which proves the lemma. □

LEMMA 3.2. *Let* $g = \exp X = \exp(H_1 + Z_1)$, $h = \exp Y = \exp(H_2 + Z_2)$, *with* $H_i \in \mathfrak{a}$, $Z_i \in \mathfrak{n}$. *Then*

$$gh = \exp(H+Z) \tag{3.9}$$

with $H = H_1 + H_2$ *and some* $Z \in \mathfrak{n}$. □

Remark. We can also write (3.9) as

$$gh = \exp X \exp Y = \exp(X + Y') = \exp(X' + Y) \tag{3.9$'$}$$

where X' is an element of $\mathfrak{g}$ with the same $\mathfrak{a}$ part as X, i.e. such that $X' - X \in \mathfrak{n}$ (and Y' is an element of $\mathfrak{g}$ with the same $\mathfrak{a}$ part as Y), or also

$$\exp(X+Y) = \exp X \exp Y'' = \exp X'' \exp Y \tag{3.9$''$}$$

for some $X'' \in \mathfrak{g}$ such that $X'' - X \in \mathfrak{n}$ and some $Y'' \in \mathfrak{g}$ such that $Y'' - Y \in \mathfrak{n}$.

Proof. By Lemma 3.1 we can write

$$g = \exp(H_1 + Z_1) = \exp H_1 \exp Z_1'$$
$$h = \exp(H_2 + Z_2) = \exp H_2 \exp Z_2'$$

for some Z_1' and $Z_2' \in \mathfrak{n}$. Then, by (3.3), we have $gh = \exp(H_1 + H_2)\exp Z_3$ for some $Z_3 \in \mathfrak{n}$, and now, again by Lemma 3.1, we have $gh = \exp(H_1 + H_2 + Z_4)$ for some $Z_4 \in \mathfrak{n}$. This proves the lemma. □

3.3. The projection of the filter to $\mathfrak{a}$. Let $\mathcal{U}$ be a G-invariant filter in G and let $\mathcal{S}$ be the inverse image of $\mathcal{U}$ under exp, as in (1.14). We shall write $X \in \mathfrak{g}$ as $X = H + Z$, with some $H \in \mathfrak{a}$ and $Z \in \mathfrak{n}$.

Consider the projections to the subgroup A of the elements of $\mathcal{U}$: $\mathrm{pr}_A U = \{a = \exp H \colon g = \exp(H+Z) \in U\}$. Let $\mathcal{U}_A$ be the family of all the projections

$$\mathcal{U}_A = \{D : D = \mathrm{pr}_A U,\ \ U \in \mathcal{U}\}. \tag{3.10}$$

Consider also the projections to $\mathfrak{a}$ of the elements of $\mathcal{S}$: $\mathrm{pr}_{\mathfrak{a}} S = \{H : X = H + Z \in S\}$ and let $\mathcal{S}_{\mathfrak{a}}$ be the familly of all such projections

$$\mathcal{S}_{\mathfrak{a}} = \{C : C = \mathrm{pr}_{\mathfrak{a}} S,\ \ S \in \mathcal{S}\}. \tag{3.11}$$

Lemma 3.3. *The family* $\mathcal{U}_A$ *defined in* (3.10) *is an* A*-invariant filter in* A*. For* $\mathcal{S}_\mathfrak{a}$ *as in* (3.11) *we have* $\exp \mathcal{S}_\mathfrak{a} = \mathcal{U}_A$ *and* $\mathcal{S}_\mathfrak{a}$ *is a* $+$*-invariant filter in* $\mathfrak{a}$*.* □

Proof. Let $D \in \mathcal{U}_A$ and $b \in A$; we have to prove that $bD \in \mathcal{U}_A$. Now by (3.10) $D \in \mathcal{U}_A$ iff $D = \mathrm{pr}_A U$, with $U \in \mathcal{U}$. By the G-invariance of $\mathcal{U}$ we have $bU = U_1 \in \mathcal{U}$. We shall prove that

$$bD = \mathrm{pr}_A U_1 \tag{3.12}$$

thus proving that $bD \in \mathcal{U}_A$.

Now $g \in U_1$ iff $g = bu$, with $u \in U_1$, thus $g = \exp H_b \exp(H_u + Z_u) = \exp(H_b + H_u + Z')$, by Lemma 3.2. Thus $a \in \mathrm{pr}_A U_1$ iff $a = \exp(H_b + H_u) = \exp H_b \exp H_u = bd$, where $d \in \mathrm{pr}_A U = D$. This proves (3.12).

Next if $C = \mathrm{pr}_\mathfrak{a} S$, with some $S \in \mathcal{S}$, then there is a $U \in \mathcal{U}$ such that $\exp \mathcal{S} = U$, and then $\exp C = \mathrm{pr}_A U \in \mathcal{U}_A$.

Finally, to prove that $\mathcal{S}_\mathfrak{a}$ is $+$-invariant in $\mathfrak{a}$, let $C \in \mathcal{S}_\mathfrak{a}$ and $H \in \mathfrak{a}$. We have to prove that $H + C \in \mathcal{S}_\mathfrak{a}$ which is equivalent with

$$\exp(H + C) \in \mathcal{U}_A. \tag{3.13}$$

Since everything commutes, we have $\exp(H+C) = \exp H \exp C$. But $C \in \mathcal{S}_\mathfrak{a}$, thus $\exp C = D \in \mathcal{U}_A$ and by the A-invariance of $\mathcal{U}_A$ we have that $\exp H \exp C \in \mathcal{U}_A$. This proves (3.13) and the lemma. □

Examples. (1) We shall give two examples of $+$-invariant filters in $\mathfrak{a}$ by defining the elements of their filter basis. Let a typical element of $\mathfrak{a}$ be written as in (3.4). Let

$$B_N^1 = \{H(t) \in \mathfrak{a} : t_1 > N_1, \dots, t_m > N_m\}$$

with $N = (N_1, \dots, N_m) \in \mathbf{N}^m$, and

$$B_N^2 = \{H(t) \in \mathfrak{a} : t_1 \cdots t_m > N\}$$

with $N \in \mathbf{N}$, be the elements of the filter basis for the filters $\mathcal{S}_\mathfrak{a}^1$ and $\mathcal{S}_\mathfrak{a}^2$.

There are many possible choices for filters in $\mathfrak{a}$ each yielding a different class of regularly varying functions. In section 3.5 we shall see that the filter in $\mathfrak{g}$ is determined by its $\mathfrak{a}$ part (it does not depend on $\mathfrak{n}$).

In these two examples we have moreover that all the elements of the filter basis are translations of a single set: $B_N = B + N$.

(2) Let B be as in (1) – a "generator" of a filter $\mathcal{S}_\mathfrak{a}$ in $\mathfrak{a}$. If B is a cone, or contains an open convex cone, then it is easily seen that the filter $\mathcal{S}_\mathfrak{a}$ is $+$-invariant. If, on the other hand, B is a paraboloid (doesn't contain a cone), then $\mathcal{S}_\mathfrak{a}$ is not $+$-invariant.

Motivated by these examples we shall impose the following conditions on our filter $\mathcal{S}$ in $\mathfrak{g}$:

(3.14) every element B_N of the filter basis contains a set C_N which is (a translation of) an open convex cone.

(3.15) There is an open covex cone C_0 such that every element S of the filter contains a cone C which is obtained by a translation and/or rotation of C_0.

(This means that the filter elements are not "smaller" than a given cone C_0). If, under the assumption (1), condition (2) is not satisfied, then we shall have a sequence of filter elements whose inscribed cones become smaller than any given cone – they tend to a half-line, and the convergence with respect to this filter degenerates to the convergence along a half-line. We want to exclude such *degenerate* filters, and this is why we assume (3.14) and (3.15) for the rest of the paper.

3.4. The $\mathrm{Ad}(g)$**-invariance of** $\mathcal{S}$**.** The filter $\mathcal{S}$ has the following property.

LEMMA 3.4. *Let $\mathcal{U}$ be a G-invariant filter in G and let $\mathcal{S}$ be the inverse image of $\mathcal{U}$ under* exp, *as in* (1.14). *Then for every $S \in \mathcal{S}$ and every $g \in G$ we have* $\mathrm{Ad}(g)S \in \mathcal{S}$. □

Remark. We shall say that $\mathcal{S}$ is $\mathrm{Ad}(g)$-invariant. Thus the inverse image under exp of a G-invariant filter is $\mathrm{Ad}(g)$-invariant.

Proof. By the definition of $\mathcal{S}$, we have $\exp S = U$ for some $U \in \mathcal{U}$. By the G-invariance of $\mathcal{U}$, we have $gUg^{-1} \in \mathcal{U}$. But, if $gUg^{-1} = g(\exp S)g^{-1} = \exp(\mathrm{Ad}(g)S)$ is an element of $\mathcal{U}$, then $\mathrm{Ad}(g)S$ is an element of $\mathcal{S}$; which proves the lemma. □

LEMMA 3.5. *Let Z_{ij} be an element of $\mathfrak{n}$ (belonging to the subspace $\mathfrak{n}_{ij}$) and let $n(s) = \exp sZ_{ij}$ $(s \in R)$. Let $X = H(t) + \nu Z_{ij}$, with $H(t) = t_1E_1 + \cdots + t_mE_m \in \mathfrak{a}$. Then*

$$\mathrm{Ad}(n(s))X = H(t) + (\nu + s(t_j - t_i))Z_{ij}. \quad \square \tag{3.16}$$

Proof. By (3.6) we have $\mathrm{ad}(sZ_{ij})H(t) = (t_j - t_i)sZ_{ij}$. Then $\mathrm{ad}^2(sZ_{ij})H(t) = 0$ and

$$\mathrm{Ad}(n(s))H(t) = e^{\mathrm{ad}(sZ_{ij})}H(t) + \mathrm{ad}(sZ_{ij})H(t) = H(t) + (t_j - t_i)sZ_{ij}$$

and this proves the lemma, since $\mathrm{Ad}(n(s))\nu Z_{ij} = \nu Z_{ij}$. □

In particular, if we consider a two-dimensional plane π spanned by an $H(t) \in \mathfrak{a}$ and a $Z_{ij} \in \mathfrak{n}_{ij}$, the lemma says that π is invariant under $\mathrm{Ad}(n(s))$. For later use we rewrite the lemma in the following form.

COROLLARY 3.6. *Let $H(t) \in \mathfrak{a}$ and $Z_{ij} \in \mathfrak{n}$ and let $n(s) = \exp sZ_{ij}$ $(s \in \mathbf{R})$. Let π be the two-dimensional subspace spanned by $H(t)$ and Z_{ij}. Then π is invariant under $\mathrm{Ad}(n(s))$ and if we write the elements of π as $X = \lambda H(t) + \nu Z_{ij} = (\lambda, \nu)$, then the restriction of $\mathrm{Ad}(n(s))$ to this subspace has the matrix*

$$\mathrm{Ad}(n(s)) = \begin{bmatrix} 1 & 0 \\ s(t_j - t_i) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ s\tau & 1 \end{bmatrix}$$

where we have put for short $\tau = (t_j - t_i)$; this is a constant in π. □

Proof. Indeed, since $\lambda H(t) = H(\lambda t)$, from (3.16) it follows that

$$\mathrm{Ad}(n(s))(\lambda H(t) + \nu Z_{ij}) = \lambda H(t) + (\nu + (t_j - t_i)s)Z_{ij}. \quad \square$$

3.5. The form of the filter. In this section we prove that $\mathcal{S}$ is $+$-invariant. The following proposition provides the main step in the proof.

PROPOSITION 3.7. *Let $\mathcal{U}$ be a G-invariant filter in G and let $\mathcal{S}$ be the inverse image of $\mathcal{U}$ under exp, as in* (1.14). *Then every $S \in \mathcal{S}$ is equal to*

$$S = S_{\mathfrak{a}} \times \mathfrak{n} \tag{3.17}$$

(*where $S_{\mathfrak{a}}$ is the projection of S to $\mathfrak{a}$, see* (3.11)). $\square$

Proof. Let $S \in \mathcal{S}$. Take an $H \in \mathfrak{a}$ such that the line $\{\lambda H : \lambda \in \mathbf{R}\}$ has a nonempty intersection with $S_{\mathfrak{a}}$. (Note that since, by assumption (3.14), $S_{\mathfrak{a}}$ contains a cone, the intersection of λH and $S_{\mathfrak{a}}$ is necessarily a half-line, when it is nonempty.)

Now let π be the two-dimensional subspace spanned by such an $H \in \mathfrak{a}$ and a $Z_{ij} \in \mathfrak{n}$. For this π all the intersections $S^\pi = S \cap \pi$, where $S \in \mathcal{S}$, constitute a filter in π, which will be denoted by $\mathcal{S}^\pi$.

We know by Lemma 3.4 that $\mathcal{S}$ is $\mathrm{Ad}(g)$ invariant, for every $g \in G$, and in particular for $n(s) = \exp sZ_{ij}$. But the plane π is invariant under $\mathrm{Ad}(n(s))$, by Corollary 3.6, and thus

$$\mathcal{S}^\pi \text{ is } \mathrm{Ad}(n(s))\text{-invariant.} \tag{3.18}$$

Now consider the form of the sets S^π. They are certainly convex and if the assertion of the proposition is true, then S^π is equal to a "vertical" half-plane $\lambda > c$.

We write the coordinates in π as (λ, ν) and consider the following cone containing S^π: $C = C(S^\pi) = \{t(\lambda, \nu) : (\lambda, \nu) \in S^\pi, \quad t > 0\}$ (we may assume that $0 \notin S^\pi$, since the filter "tends to infinity"). Obviously, S^π is equal to a vertical half-plane if and only if $C(S^\pi) = C_0 = \{(\lambda, \nu) : \lambda > 0\}$.

Suppose that the proposition is not true. Then we would have a set $S \in \mathcal{S}$ and a plane π such that S^π would not be equal to a vertical half-plane. Thus $C = C(S^\pi)$ is not equal to C_0. The cone C is an angle between two half-lines $\nu = a\lambda$ and $\nu = b\lambda$ which are not both vertical. We have the following two types of C.

$$C = \{(\lambda, \nu) : b\lambda < \nu < a\lambda, \lambda > 0\} \tag{3.19}$$

for some $-\infty < b < a < +\infty$ and

$$C = \{(\lambda, \nu) : \nu > \max(a\lambda, b\lambda)\} \tag{3.20}$$

with $-\infty \leq b < a < +\infty$. (All other possibilities are analogous to these two.)

We see that C belongs to the filter $\mathcal{S}^\pi$ as a superset of a filter element and by (3.18) we must have that

$$\mathrm{Ad}(n(s))C \in \mathcal{S}^\pi \tag{3.21}$$

By Corollary 3.6: $\mathrm{Ad}(n(s))$ transforms the line $\nu = a\lambda$ into $\nu = (a + \tau s)\lambda$. (If $a = \infty$, the line $\lambda = 0$ is invariant.) Now if C is of the form (3.19) we can chose s_1 and s_2 so that $b + \tau s_1 > 0$ and $a + \tau s_2 < 0$. Then, on one hand, by (3.21) both sets

$$\mathrm{Ad}(n(s_1))C = \{(\lambda, \nu) : 0 < (b + \tau s_1)\lambda < \nu < (a + \tau s_1)\lambda, \quad \lambda > 0\}$$
$$\mathrm{Ad}(n(s_2))C = \{(\lambda, \nu) : (b + \tau s_2)\lambda < \nu < (a + \tau s_2)\lambda < 0, \quad \lambda > 0\}$$

should belong to $\mathcal{S}^\pi$, but on the other hand, they are obviously disjoint. This yields the desired contradiction and proves the proposition in this case.

In the case C is of the form (3.20),

$$\mathrm{Ad}(n(s))C = \{(\lambda, \nu) : \nu > \max((a + \tau s)\lambda, (b + \tau s)\lambda)\}.$$

For every $M > 0$ chose an s_1 so that $b + \tau s_1 < -M$ (when $b = -\infty$, this holds for every s_1) and chose an s_2 so that $a + \tau s_2 > M$ and put $C_1 = \mathrm{Ad}(n(s_1))C$ and $C_2 = \mathrm{Ad}(n(s_2))C$. Then $C_1 \cap C_2$ should contain a filter element, for every M. But

$$C_1 \cap C_2 \subseteq \{(\lambda, \nu) : \nu > \max(M\lambda, -M\lambda)\}$$

and when M is large enough, this set doesn't contain a filter element by (3.15). In other words the filter with C as in (3.20) is degenerate; this proves the proposition in the second case. □

PROPOSITION 3.8. *$\mathcal{S}$ is a $+$-invariant filter in $\mathfrak{g}$.* □

Proof. Let $S \in \mathcal{S}$ and $X \in \mathfrak{g}$. Then $\exp S = U \in \mathcal{U}$, and we want to prove $\exp(X + S) \in \mathcal{U}$. Let $Y \in S$; then by (3.9″) $\exp(X + Y) = \exp X \exp Y''$, with $Y'' = Y \pmod{\mathfrak{n}}$. Now, by Proposition 3.7, if $Y \in S$, then $Y'' \in S$. Thus $\exp(X + S) = \exp X \exp S$ and it belongs to $\mathcal{U}$, by the G-invariance of $\mathcal{U}$. This proves the proposition. □

Definition 3.9. Let $\mathcal{S}$ be a filter of open convex subsets in $\mathfrak{g}$ with a countable basis. Then $\mathcal{S}$ is said to be $\mathfrak{g}$-invariant if $\mathcal{S}$ is $+$-invariant and its elements are of the form (3.17). □

COROLLARY 3.10. *Let $\mathcal{S}$ be a $\mathfrak{g}$-invariant filter in $\mathfrak{g}$. Let $\mathcal{U}$ be a filter of subsets in G defined by $U \in \mathcal{U}$ if and only if $U = \exp S$, for some $S \in \mathcal{S}$. Then $\mathcal{U}$ is G-invariant.* □

4. Multiplicative and additive regular variation

4.1. The equivalence of the two classes. In this section we examine the relationship between a regularly varying function F on G (the group of a homogeneous cone) and the function f, defined on the Lie algebra $\mathfrak{g}$ of G by the formula (1.12) $f(X) = \log F(\exp X)$. Let, as in section 3.1, A and N be the subgroups of G and $\mathfrak{g}$, $\mathfrak{a}$ and $\mathfrak{n}$ the corresponding Lie algebras.

We have seen in section 3 that if $\mathcal{U}$ is a G-invariant filter on G, then $\mathcal{S}$ (its inverse image under exp) is $\mathfrak{g}$-invariant (that is: $+$-invariant and its elements are of the form (3.17); in other words, the convergence $X \to \infty$ is uniform with respect to the $\mathfrak{n}$-part of X).

Definition 4.1. Let G be the group of a homogeneous cone and $\mathcal{U}$ a G-invariant filter on G. A smooth function $F\colon G \to \mathbf{R}_+$ is said to be $\mathfrak{n}$-*uniformly regularly varying* with respect to $\mathcal{U}$ if the limit

$$\lim_{g \xrightarrow{\mathcal{U}} \infty} \frac{F(gh)}{F(g)} = \phi(h) \tag{4.1}$$

exists for every $h \in G$ and

$$\lim_{g \xrightarrow{\mathcal{U}} \infty} \sup_{n \in N} \left| \frac{F(gn)}{F(g)} - 1 \right| = 0. \quad \square \tag{4.2}$$

Definition 4.2. Let $\mathfrak{g}$ be the Lie algebra of a homogeneous cone and $\mathcal{S}$ a $\mathfrak{g}$-invariant filter on $\mathfrak{g}$. A smooth function $f\colon \mathfrak{g} \to \mathbf{R}$ is said to be $\mathfrak{n}$-*uniformly additively regularly varying* with respect to $\mathcal{S}$ if the limit

$$\lim_{X \xrightarrow{\mathcal{S}} \infty} (f(X+Y) - f(X)) = \phi(Y) \tag{4.3}$$

exists for every $Y \in \mathfrak{g}$ and

$$\lim_{X \xrightarrow{\mathcal{S}} \infty} \sup_{Z \in \mathfrak{n}} |f(X+Y) - f(X)| = 0. \tag{4.4}$$

THEOREM 4.3. *Let G be the group of a homogeneous cone V and let $\mathfrak{g}$ be its Lie algebra. Let $\mathcal{U}$ be a G-invariant filter in G and let $\mathcal{S}$ be the inverse image of $\mathcal{U}$ under exp, as in* (1.14). *Let $F\colon G \to \mathbf{R}_+$ be a smooth function and let $f\colon \mathfrak{g} \to \mathbf{R}$ be defined by $f(X) = \log F(\exp X)$. Then F is $\mathfrak{n}$-uniformly regularly varying with respect to $\mathcal{U}$ if and only if f is $\mathfrak{n}$-uniformly additively regularly varying with respect to $\mathcal{S}$, and*

$$\phi(X) = \log F(\exp X). \quad \square \tag{4.5}$$

The proof of the theorem will be given in Section 4.3.

4.2. The index function. We have seen in Lemma 1.3 that the index function F of a regularly varying function is a Lie group homomorphism of G into $\mathbf{R}_+$. Then its derivative $d\Phi$ (at the group idenity) is a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathbf{R}$, and it satisfies the formula $\Phi(\exp X) = \exp d\Phi(X)$. By compairing this with (4.5), we see that the limit function ϕ in (4.3) equals $d\Phi$.

In the next lemma we are going to find the general form of ϕ and Φ (compare this with Lemma 2.1). Here ϕ is not only linear, but it doesn't depend on the $\mathfrak{n}$ coordinates, but only on the $\mathfrak{a}$ coordinates; (a typical element of $\mathfrak{a}$ is written as in (3.4)).

LEMMA 4.4. *Let $\phi : \mathfrak{g} \to \mathbf{R}$ be a Lie algebra homomorphism. Then:*
(a) *$\phi(Z) = 0$, for $Z \in \mathfrak{n}$,* (b) *for $H \in \mathfrak{a}$, $H = t_1E_1 + \cdots + t_mE_m$ we have*

$$\phi(H) = \alpha_1 t_1 + \cdots + \alpha_m t_m \tag{4.6}$$

for some $\alpha_i \in \mathbf{R}$. $\square$

Proof. (a) By (3.1) we have $Z = [X_1, X_2]$ for some $X_1, X_2 \in \mathfrak{g}$. Then $\phi(Z) = [\phi(X_1), \phi(X_2)] = 0$, since the brackets are in R.

(b) Every linear function on the abelian Lie algebra $\mathfrak{a}$ is a Lie algebra homomorphism from $\mathfrak{a}$ into $\mathbf{R}$. $\square$

It follows that Φ is trivial on N, that is $\Phi(g) = \Phi(a)$, if $g = an$ (as in (3.2)). Also for $a = \exp H$ we have by (4.6)

$$\Phi(g) = \Phi(a) = e^{\alpha_1 t_1 + \cdots + \alpha_m t_m}.$$

This is the general form of the index function.

4.3. Proof of the Theorem. In this section we prove Theorem 4.3. Note first that by Propositions 3.7 and 3.8 the G-invariance of $\mathcal{U}$ implies the $\mathfrak{g}$-invariance of $\mathcal{S}$.

Suppose F is $\mathfrak{n}$-uniformly regularly varying with respect to the filter $\mathcal{U}$. We shall prove first that (4.2) implies (4.4). Put $g = \exp X$, $n = \exp Z$ in (4.2). Then by Lemma 3.2 we have

$$gn = \exp(X + Z') \tag{4.7}$$

with some $Z' \in \mathfrak{n}$. Now when n runs through N, Z' will run through all of $\mathfrak{n}$. If we put (4.7) into (4.2), we obtain that the function f satisfies (4.4).

Now to prove (4.3) we put $g = \exp X = \exp(H(X) + Z(X))$, $h = \exp Y = \exp(H(Y) + Z(Y))$ in (4.1). Then, as above, Lemma 3.2 yields $gh = \exp(X + Y')$ with $Y' = H(Y) + Z'$ (with some $Z' \in \mathfrak{n}$). Then (4.1) will give for f

$$\lim_{X \underset{\mathcal{S}}{\to} \infty} (f(X + Y') - f(X)) = \phi(Y) \tag{4.8}$$

Now we shall write

$$f(X + Y') - f(X) = (f(X + Y) - f(X)) + (f(X + Y') - f(X + Y)) \tag{4.9}$$

and consider the second summand on the right-hand side

$$\begin{aligned} f(X + Y') - f(X + Y) &= f(X + H(Y) + Z') - f(X + Y) \\ &= f(X + Y + Z' - Z(Y)) - f(X + Y). \end{aligned}$$

Now, since $\mathcal{S}$ is $+$-invariant, we have that when $X \to \infty$ then also $X + Y \to \infty$, and thus by (4.4), proved above, it follows that

$$\lim_{X \underset{\mathcal{S}}{\to} \infty} (f(X + Y + Z' - Z(Y)) - f(X + Y)) = \lim_{U \underset{\mathcal{S}}{\to} \infty} (f(U + Z'') - f(U)) = 0$$

Thus from (4.9) we see that $\lim(f(X + Y') - f(X)) = \lim(f(X + Y) - f(X))$ and this with (4.8) gives (4.3). This proves the first half of the theorem.

To prove the converse suppose we have (4.4). We put $X = \log g$ and $Y = \log h$, and claim that there is a $h' \in G$ such that $h' = \exp Y'$ with $Y - Y' \in \mathfrak{n}$ so that

$$X + Y = \log(gh'). \tag{4.10}$$

Indeed, by Lemma 3.2, $\exp(X+Y) = \exp X \exp Y' = gh'$, which is exactly (4.10). The proof now follows along the same lines as the proof of the direct part of the theorem. □

5. Going back to the group

A simple combination of Theorem 4.3 with Section 2 will yield the corresponding theorems for regularly varying functions on the group G.

THEOREM 5.1 (The Uniform Convergence Theorem). *Let G be the group of a homogeneous cone and $\mathcal{U}$ a G-invariant filter on G. If $F\colon G \to \mathbf{R}_+$ is regularly varying with respect to $\mathcal{U}$, then the limit*

$$\lim_{g \underset{\mathcal{U}}{\to} \infty} \frac{F(gh)}{F(g)} = \Phi(h)$$

is uniform in $h \in K$, for any compact set K in G. □

Remark. Actually, by (4.2) we have a stronger statement: the convergence is uniform for K of the form $K_A \times N$, where K_A is a compact set in A.

THEOREM 5.2 (The Representation Theorem). *Let F be a $\mathfrak{n}$-uniformly regularly varying function on G, with respect to a G-invariant filter $\mathcal{U}$. Then there exist two smooth functions $H, \Delta\colon G \to \mathbf{R}_+$ such that $F(g) = H(g)\Delta(g)$ with*

$$\lim_{g \underset{\mathcal{U}}{\to} \infty} \|H'(g)\| = 0 \quad \textit{and} \quad \lim_{g \underset{\mathcal{U}}{\to} \infty} \Delta(g) = 1.$$

It is enough to put $H(g) = e^{h(\log g)}$ and $\Delta(g) = e^{\delta(\log g)}$ in Corollary 2.7. Here $H'(g)$ denotes the derivative of the function H in the normal coordinates $g \mapsto \log g$. (Recall that $\log\colon G \to \mathfrak{g}$ is a global coordinate system for G, since the group is exponential.)

In the classical case when $F\colon \mathbf{R}_+ \to \mathbf{R}_+$, we have $F(t) = \dfrac{t dF(t)/dt}{F(t)}$ (with d/dt the usual derivative in R) and this gives a well-known form of the Representation Theorem for differentiable functions.

Acknowledgements. I acknowledge gratefully the support of the Mathematical Institute in Belgrade through Grant 0401A of the Science Fund of Serbia.

Part of the work on this paper was done during my visit to the University of Maine. I would like to thank William O. Bray for thorough discussions and helpful suggestions and the Department of Mathematics for its warm hospitality.

Some of the results of this paper were presented at the Department of Mathematics of the University of Montpellier III, which I would like to thank for its warm hospitality.

I would like to thank D. Aranđelović for reading carefully the manuscript and pointing out a mistake in the proof.

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(Received 05 07 1994)
(Revised 22 05 1995)