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A FEW REMARKS ON REDUCED IDEAL-PRODUCTS

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Abstract. A nicer shape of the condition $(\Lambda\Psi)$ (which ensures preservation of separation axioms $T_k,\ k=0,1,2,3,3\frac{1}{2}$, in reduced ideal-products) is given. If an reduced-ideal products is T_0, T_1 or T_2 then "almost all" coordinate spaces have this property. This implication holds for T_3-property if the condition $(\Lambda\Psi)$ is satisfied. Some results on mappings and homogenicity of r.i. products are obtained. Finally, it is proved that r.i.p. of topological groups (rings) is a topological group (ring).

1. Preliminaries

The notation and some of the remarks made here were already given in our previous papers on this subject. Thus, they are offered more for the reader's convenience.

Throughout the paper $\{\mathcal{X}_i \mid i \in I\}$ will be a family of topological spaces, where $\mathcal{X}_i = (X_i, \mathcal{O}_i)$. Λ and Ψ will be respectively an ideal and a filter on I and $\pi_j : \prod_{i\in I} X_i \longrightarrow X_j,\ j \in I$, will be the canonical projections.

The topology $\mathcal{O}^\Lambda$ on $\prod_{i\in I} X_i$ is given by its base $\mathcal{B}^\Lambda$ consisting of all sets of the form $\bigcap_{i\in L} \pi_i^{-1}(O_i)$, where $L \in \Lambda$ and $O_i \in \mathcal{O}_i$, for all $i \in L$. The space $(\prod_{i\in I} X_i, \mathcal{O}^\Lambda)$ will be denoted by $\prod^\Lambda \mathcal{X}_i$. The equivalence relation $\sim$ on $\prod_{i\in I} X_i$ defined by: $f \sim g$ iff $\{i \in I \mid f_i = g_i\} \in \Psi$ determines the quotient space $\prod^\Lambda \mathcal{X}_i/\sim$, which will be called the reduced ideal-product (r.i.p.) of the family of spaces $\{\mathcal{X}_i \mid i \in I\}$ and denoted by $\prod_\Psi^\Lambda \mathcal{X}_i$. The natural mapping $q : \prod_{i\in I} X_i \longrightarrow \prod_{i\in I} X_i/\sim$ is given by $q(f) = [f]$ $(= \{g \in \prod_{i\in I} X_i \mid f \sim g\})$. Since q is an open mapping, $\mathcal{B}_\Psi^\Lambda = \{q(B) \mid B \in \mathcal{B}^\Lambda\}$ is a base for the topology $\mathcal{O}_\Psi^\Lambda$ on $\prod_\Psi^\Lambda \mathcal{X}_i$.

To simplify the notation, for an arbitrary $A \subseteq \prod_{i\in I} X_i$ we put $A^* = q^{-1}(q(A))$. The following facts will be used in the sequel (see [**3**]).

Lemma 1.1. *If $A_i, B_i \subseteq X_i$ for all $i \in I,\ L \subseteq I$ and $F \in \Psi$, then:*

(a) $(\prod_{i\in I} A_i)^* \subseteq (\prod_{i\in I} B_i)^*$ *iff* $\{i \in I \mid A_i \subseteq B_i\} \in \Psi$*;*

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(b) $(\prod_{i\in I} A_i)^* = (\prod_{i\in I} B_i)^*$ *iff* $\{i\in I \mid A_i = B_i\} \in \Psi$;

(c) $f \in (\prod_{i\in I} A_i)^*$ *iff* $\{i\in I \mid f_i \in A_i\} \in \Psi$;

(d) $(\prod_{i\in I} A_i)^* \cap (\prod_{i\in I} B_i)^* \neq \emptyset$ *iff* $\{i\in I \mid A_i \cap B_i \neq \emptyset\} \in \Psi$;

(e) $(\bigcap_{i\in L} \pi_i^{-1}(A_i))^* = (\bigcap_{i\in L\cap F} \pi_i^{-1}(A_i))^*$;

(f) $q(\bigcap_{i\in L} \pi_i^{-1}(A_i)) = q(\bigcap_{i\in L\cap F} \pi_i^{-1}(A_i))$;

Lemma 1.2. *If $A \in \Psi$ then:*

(a) $\Psi_A = \{F\cap A \mid F \in \Psi\}$ *is a filter on A and $\Psi_A \subseteq \Psi$;*

(b) $\Lambda_A = \{L\cap A \mid L \in \Lambda\}$ *is an ideal on A and $\Lambda_A \subseteq \Lambda$;*

(c) *the mapping* $\eta : \prod_\Psi^\Lambda X_i \longrightarrow \prod_{\Psi_A}^{\Lambda_A} X_i$, *given by* $\eta([f]) = [f|_A]$, *where* $[f|_A] = \{g \in \prod_{i\in A} X_i \mid g \sim_{\Psi_A} f|_A\}$, *is a homeomorphism.*

Lemma 1.3. *If, for all $i \in I$, $\mathcal{B}_i$ is a base for the topology $\mathcal{O}_i$ then the sets of the form*

$$q\left(\bigcap_{i\in L} \pi_i^{-1}(B_i)\right),$$

where $L \in \Lambda$ and, for all $i \in L$, $B_i \in \mathcal{B}_i$, form a base for the topology $\mathcal{O}_\Psi^\Lambda$.

It is proved in [**3**] that the r.i.p. preserves the separation axioms T_0, T_1, T_2, T_3 and $T_{3\frac{1}{2}}$ if and only if:

$$\forall A \in \Psi\, \forall B \notin \Psi\, \exists L \in \Lambda\, (L \subseteq A \setminus B \text{ and } L^c \notin \Psi) \tag{$\Lambda\Psi$}$$

A part of the proof was the following

Lemma 1.4. *If an ideal Λ and a filter Ψ satisfy the condition $(\Lambda\Psi)$, then:*

(a) *if* $\{i\in I \mid A_i \in \mathcal{F}_i\} \in \Psi$, *then* $(\prod_{i\in I} A_i)^* \in \mathcal{F}^\Lambda$ *and* $q(\prod_{i\in I} A_i) \in \mathcal{F}_\Psi^\Lambda$ *(of course, $\mathcal{F}_i$, $\mathcal{F}^\Lambda$ and $\mathcal{F}_\Psi^\Lambda$ are the families of closed sets of the spaces $\mathcal{X}_i$, $\prod^\Lambda \mathcal{X}_i$, and $\prod_\Psi^\Lambda \mathcal{X}_i$, respectively);*

(b) $\overline{q(\prod_{i\in I} A_i)} = q(\prod_{i\in I} \overline{A_i})$.

R.i. products which satisfy the condition $(\Lambda\Psi)$ were also investigated in [**6**] and [**7**]. Special $(\Lambda\Psi)$-r.i. products are: the Tychonoff product (for $\Lambda = [I]^{<\omega}$ and $\Psi = \{I\}$), the full box product (for $\Lambda = P(I)$ and $\Psi = \{I\}$), the ultraproduct (for $\Lambda = P(I)$ and Ψ an arbitrary ultrafilter on I) and the Knight's box product (for $\Lambda = [I]^{<\kappa}$ and $\Psi = \{A \subseteq I \mid A^c \in [I]^{<\mu}\}$, where κ and μ are cardinals satisfying $|I| \geq \kappa > \mu \geq \omega$).

2. A nicer shape of $(\Lambda\Psi)$

For a filter Ψ on a nonempty set I, let $\equiv$ be the (well-known) congruence relation of the Boolean algebra $\langle P(I), \cup, \cap, {}^c, \emptyset, I\rangle$, defined by: $A \equiv B$ iff for some $F \in \Psi$, $A \cap F = B \cap F$. The equivalence class of an element $A \in P(I)$ will be denoted by $[A]$. Clearly, $\langle P(I)/\equiv, \vee, \wedge, ', \mathbf{0}, \mathbf{1}\rangle$, where: $[A] \vee [B] = [A \cup B]$, $[A] \wedge [B] = [A \cap B]$, $[A]' = [A^c]$, $\mathbf{0} = [\emptyset]$ and $\mathbf{1} = [I]$, is a Boolean algebra.

It is easy to verify that $\Lambda/\equiv\, = \{[L] \mid L \in \Lambda\}$ is an ideal on $P(I)/\equiv$. Also, for each $A \in P(I)$ we have:

$$[A] > \mathbf{0}: \text{ iff } A^c \notin \Psi. \tag{*}$$

Theorem 2.1. *An ideal Λ and a filter Ψ on I satisfy the condition $(\Lambda\Psi)$ iff*

$$\forall x \in P(I)/\equiv\ (x > \mathbf{0} \Rightarrow \exists y \in \Lambda/\equiv\ (\mathbf{0} < y \leq x)).$$

Proof. $(\Rightarrow)$ If $x = [C] \in P(I)/\equiv \setminus \{\mathbf{0}\}$, then (by $(*)$), $B = C^c \notin \Psi$ and $(\Lambda\Psi)$ gives an $L \in \Lambda$ satisfying $L \subseteq B^c = C$ and $L^c \notin \Psi$. Now $y = [L] \in \Lambda/\equiv$ and (again by $(*)$) $y > \mathbf{0}$. Finally, because of $L \subseteq C$, $[L] \leq [C]$, i.e. $y \leq x$.

$(\Leftarrow)$ Let $A \in \Psi$ and $B \notin \Psi$. By $(*)$ we have $x = [B^c] > \mathbf{0}$ and the observed condition implies the existence of $y = [D] \in \Lambda/\equiv$, where $D \in \Lambda$ and $\mathbf{0} < [D] \leq [B^c]$. Now, $[D] = [D] \wedge [B^c] = [D \cap B^c]$, so there is $F \in \Psi$ such that $D \cap F = D \cap B^c \cap F$. For $L = D \cap F \cap A = D \cap F \cap (A \setminus B)$ we have $L \subseteq A \setminus B$ and from $L \subseteq D$ it follows that $L \in \Lambda$. Suppose $L^c \in \Psi$. Then $D \cap F \cap A \cap L^c = L \cap L^c = \emptyset$ and $F \cap A \cap L^c \in \Psi$ gives $D^c \in \Psi$. But $[D] > \mathbf{0}$ and, by $(*)$, $D^c \notin \Psi$, a contradiction. Thus $L^c \notin \Psi$ and $(\Lambda\Psi)$ is true.

In set theory a subset Λ of a partially ordered set $\mathbf{P}$ is called *dense* iff

$$\forall x \in \mathbf{P}\, \exists y \in \Lambda\, (y \leq x).$$

Hence, by the previous theorem we have: $(\Lambda\Psi)$ *iff* Λ *is dense in* $\mathbf{P}$, where $\mathbf{P} = P(I)/\equiv \setminus \{\mathbf{0}\}$ and $\Lambda = \Lambda/\equiv \setminus \{\mathbf{0}\}$.

3. On separation axioms

By [**3**], for each family $\{\mathcal{X}_i \mid i \in I\}$ of topological spaces and each $k \in \{0, 1, 2, 3, 3\frac{1}{2}\}$ we have:

$$\{i \in I \mid \mathcal{X}_i \text{ is a } T_k\text{-space}\} \in \Psi \Rightarrow \prod\nolimits_{\Psi}^{\Lambda} \mathcal{X}_i \text{ is a } T_k\text{-space}$$

iff the condition $(\Lambda\Psi)$ is satisfied. Naturally, the following question arises: when the following implication holds?

$$\prod\nolimits_{\Psi}^{\Lambda} \mathcal{X}_i \text{ is a } T_k\text{-space} \Rightarrow \{i \in I \mid \mathcal{X}_i \text{ is a } T_k\text{-space}\} \in \Psi \tag{**}$$

Theorem 3.1. *Let Λ be an arbitrary ideal and Ψ an arbitrary filter on a nonempty set I. Then $(**)$ is true for each family of spaces $\{\mathcal{X}_i \mid i \in I\}$ and each $k \in \{0,1,2\}$.*

Proof. We will give the proof for $k = 2$. For $k = 0,1$ the proof is similar. Suppose $H = \{i \in I \mid \mathcal{X}_i \text{ is } T_2\} \notin \Psi$ and for $i \in H$ choose $f_i = g_i \in X_i$. For $i \in I \setminus H$ we pick $f_i, g_i \in X_i$ satisfying $f_i \neq g_i$ and

$$\forall U, V \in \mathcal{O}_i \, (f_i \in U \, \wedge \, g_i \in V \, \Rightarrow \, U \cap V \neq \emptyset). \tag{1}$$

Now $\{i \in I \mid f_i = g_i\} = H \notin \Psi$, so $[f] \neq [g]$. Let $[f] \in q(B_f)$ and $[g] \in q(B_g)$, where $B_f, B_g \in \mathcal{B}^\Lambda$ and $B_f = \prod_{i \in I} S_i$, $B_g = \prod_{i \in I} T_i$. Then, according to Lemma 1.1(c), for $A_f = \{i \in I \mid f_i \in S_i\}$ and $A_g = \{i \in I \mid g_i \in T_i\}$ we have $A_f, A_g \in \Psi$, thus $A = A_f \cap A_g \in \Psi$. If $i \in A \cap H$, then $f_i = g_i \in S_i \cap T_i$. If $i \in A \setminus H$, then $f_i \in S_i$, $g_i \in T_i$ and, by (1), $S_i \cap T_i \neq \emptyset$. So $A \subseteq \{i \in I \mid S_i \cap T_i \neq \emptyset\} = D$ and $D \in \Psi$. Due to lemma 1.1(d), $(\prod_{i \in I} S_i)^* \cap (\prod_{i \in I} T_i)^* \neq \emptyset$ and obviously $q(B_f) \cap q(B_g) \neq \emptyset$. Hence $[f]$ and $[g]$ cannot be separated by basic open sets and $\prod_\Psi^\Lambda \mathcal{X}_i$ is not a Hausdorff space.

Let us now consider the case $k = 3$. Clearly, a space $\mathcal{X} = (X, \mathcal{O})$ is regular iff for each $x \in X$ and each basic open set B containing x there is a basic open set B_1 containing x and satisfying $\overline{B_1} \subseteq B$.

Theorem 3.2. *Let Λ and Ψ satisfy the condition $(\Lambda\Psi)$. Then:*

(i) $\prod_\Psi^\Lambda \mathcal{X}_i$ *is regular* $\Rightarrow$ $\{i \in I \mid \mathcal{X}_i$ *is regular*$\} \in \Psi$*;*

(ii) $\prod_\Psi^\Lambda \mathcal{X}_i$ *is a* T_3*-space* $\Rightarrow$ $\{i \in I \mid \mathcal{X}_i$ *is a* T_3*-space*$\} \in \Psi$.

Proof. (i) If $H = \{i \in I \mid \mathcal{X}_i \text{ is regular}\} \notin \Psi$, then $(\Lambda\Psi)$ gives $L \in \Lambda$ where $L \subseteq H^c$ and $L^c \notin \Psi$. For $i \in L$, the space $\mathcal{X}_i$ is not regular, so we can pick $O_i \in \mathcal{O}_i$ and $f_i \in O_i$ such that

$$\forall U \in \mathcal{O}_i \, (f_i \in U \, \Rightarrow \, \overline{U} \not\subseteq O_i). \tag{2}$$

For $i \in L^c$ we choose an arbitrary $f_i \in X_i$. Let $B = \bigcap_{i \in L} \pi_i^{-1}(O_i) = \prod_{i \in I} T_i$. Then $f \in B$ and $[f] \in q(B) \in \mathcal{B}_\Psi^\Lambda$. Assume $W = \bigcap_{i \in L_1} \pi_i^{-1}(U_i) = \prod_{i \in I} G_i \in \mathcal{B}^\Lambda$ and $[f] \in q(W)$. Then $f \in (\prod_{i \in I} G_i)^*$ and $E = \{i \in I \mid f_i \in G_i\} \in \Psi$. By lemma 1.1(f), for $L_2 = L_1 \cap E$ and $B_1 = \bigcap_{i \in L_2} \pi_i^{-1}(U_i)$ we have $q(W) = q(B_1)$. According to lemma 1.4 $\overline{q(B_1)} = q(\bigcap_{i \in L_2} \pi_i^{-1}(\overline{U_i})) = q(\prod_{i \in I} S_i)$. If $i \in L \cap L_2$, then $T_i = O_i$, $S_i = \overline{U_i}$ and $f_i \in G_i = U_i$. So, by (2), $\overline{U_i} \not\subseteq O_i$, that is $S_i \not\subseteq T_i$. If $i \in L \setminus L_2$, then $T_i = O_i \neq X_i = S_i$ and again $S_i \not\subseteq T_i$. Now $\{i \in I \mid S_i \subseteq T_i\} \subseteq L^c \notin \Psi$, thus $\{i \in I \mid S_i \subseteq T_i\} \notin \Psi$ and by 1.1(a), $(\prod_{i \in I} S_i)^* \not\subseteq (\prod_{i \in I} T_i)^*$, i.e. $\overline{q(W)} \not\subseteq q(B)$. It follows that: $[f] \in q(B) \in \mathcal{B}_\Psi^\Lambda$ and for all $q(W) \in \mathcal{B}_\Psi^\Lambda$ if $[f] \in q(W)$ then $\overline{q(W)} \not\subseteq q(B)$, whence $\prod_\Psi^\Lambda \mathcal{X}_i$ is not a regular space.

(ii) is a consequence of (i) and the previous theorem.

Note. The previous theorem does not hold for $T_{3\frac{1}{2}}$-spaces. One counterexample is given in [**6**, Remark 3.1].

4. Mappings of reduced ideal-products

Let, for $i \in I$, $\mathcal{X}_i = (X_i, \mathcal{O}_i^X)$ and $\mathcal{Y}_i = (Y_i, \mathcal{O}_i^Y)$ be topological spaces and φ_i a mapping of X_i into Y_i. The usual product of the mappings φ_i, $i \in I$, denoted by φ^Λ, is the mapping of the topological space $\prod^\Lambda \mathcal{X}_i$ into the topological space $\prod^\Lambda \mathcal{Y}_i$; we recall: $\varphi^\Lambda(\langle f_i \mid i \in I\rangle) = \langle \varphi_i(f_i) \mid i \in I\rangle$.

LEMMA 4.1. *If each φ_i, $i \in I$, is onto ($1-1$, continuous, open and onto) then φ^Λ has the same property.*

Proof. Supposing the mappings φ_i, $i \in I$, are continuous, the continuity of φ^Λ follows from the equality:

$$(\varphi^\Lambda)^{-1}\left(\bigcap_{i\in L} p_i^{-1}(O_i)\right) = \bigcap_{i\in L} \pi_i^{-1}(\varphi_i^{-1}(O_i)),$$

where $\pi_j : \prod_{i\in I} X_i \longrightarrow X_j$ and $p_j : \prod_{i\in I} Y_i \longrightarrow Y_j$ are the canonical projections and $O_i \in \mathcal{O}_i^Y$. When the openness is in question we use the relation

$$\varphi^\Lambda(\bigcap_{i\in L} \pi_i^{-1}(O_i)) = \bigcap_{i\in L} p_i^{-1}(\varphi_i(O_i)),$$

where $O_i \in \mathcal{O}_i^X$.

It is easy to check that the mapping $\varphi^\Lambda_\Psi : \prod^\Lambda_\Psi X_i \longrightarrow \prod^\Lambda_\Psi Y_i$, given by $\varphi^\Lambda_\Psi([f]) = [\varphi^\Lambda(f)]$ is well-defined. Moreover:

THEOREM 4.2. *Let $\mathcal{P}$ be one of the properties: to be onto, "$1-1$", continuous, open and onto, homeomorphism. If $A = \{i \in I \mid \varphi_i$ has the property $\mathcal{P}\} \in \Psi$ then φ^Λ_Ψ has the property $\mathcal{P}$.*

Proof. By Lemma 1.2, there are homeomorphisms η_X and η_Y described by the following diagram

$$\begin{array}{ccc}
\prod\limits_{i\in I}{}^{\Lambda}_{\Psi}\mathcal{X}_i & \xrightarrow{\varphi^\Lambda_\Psi} & \prod\limits_{i\in I}{}^{\Lambda}_{\Psi}\mathcal{Y}_i \\
{\scriptstyle\eta_X}\big\downarrow & & \big\downarrow{\scriptstyle\eta_Y} \\
\prod\limits_{i\in A}{}^{\Lambda_A}_{\Psi_A}\mathcal{X}_i & \xrightarrow{\varphi^{\Lambda_A}_{\Psi_A}} & \prod\limits_{i\in A}{}^{\Lambda_A}_{\Psi_A}\mathcal{Y}_i
\end{array}$$

where $\eta_X([f]) = [f|_A]$ and $\eta_Y([g]) = [g|_A]$. Obviously, $\varphi^\Lambda_\Psi = \eta_Y^{-1} \circ \varphi^{\Lambda_A}_{\Psi_A} \circ \eta_X$, so it is enough to prove that $\varphi^{\Lambda_A}_{\Psi_A}$ has the desired property. In other words we can assume $A = I$. Firstly, we note that the diagram

$$\begin{array}{ccc} \prod^{\Lambda} \mathcal{X}_i & \xrightarrow{\varphi^{\Lambda}} & \prod^{\Lambda} \mathcal{Y}_i \\ {\scriptstyle q_X}\downarrow & & \downarrow{\scriptstyle q_Y} \\ \prod^{\Lambda}_{\Psi} \mathcal{X}_i & \xrightarrow{\varphi^{\Lambda}_{\Psi}} & \prod^{\Lambda}_{\Psi} \mathcal{Y}_i \end{array}$$

commutes, i.e. $\varphi^{\Lambda}_{\Psi} \circ q_X = q_Y \circ \varphi^{\Lambda}$. Now, the verification of the first two properties is elementary.

The continuity of φ^{Λ}_{Ψ} follows from the relation

$$(\varphi^{\Lambda}_{\Psi})^{-1}(O) = q_X((\varphi^{\Lambda})^{-1}(q_Y^{-1}(O)))$$

(where O is open in $\prod^{\Lambda}_{\Psi} \mathcal{Y}_i$), the continuity of q_Y, the openness of q_X and the continuity of φ^{Λ} (proved in the previous lemma).

If O is an arbitrary open set in $\prod^{\Lambda}_{\Psi} \mathcal{X}_i$, then $\varphi^{\Lambda}_{\Psi}(O) = q_Y(\varphi^{\Lambda}(q_X^{-1}(O)))$, and by the continuity of q_X, the openness of φ^{Λ} (proved in the previous lemma) and the openness of q_Y we have the openness of φ^{Λ}_{Ψ}.

COROLLARY 4.3. *If $\{i \in I \mid \mathcal{X}_i$ is a homogeneous space$\} \in \Psi$, then $\prod^{\Lambda}_{\Psi} \mathcal{X}_i$ is a homogeneous space.*

5. Iterated reduced ideal-products

Let I be a nonempty set, $\{J_i \mid i \in I\}$ a family of nonempty, disjoint sets, Λ an ideal and Ψ a filter on I and for each $i \in I$, Λ_i an ideal and Ψ_i a filter on J_i. Then $\Lambda_0 = \{\bigcup_{i\in L} L_i \mid L \in \Lambda, L_i \in \Lambda_i \text{ for } i \in L\}$ and $\Psi_0 = \{A \subseteq J \mid \{i \in I \mid A \cap J_i \in \Psi_i\} \in \Psi\}$ are respectively an ideal and a filter on the set $J = \bigcup_{i\in I} J_i$. Moreover:

THEOREM 5.1. *If the pairs Λ, Ψ and Λ_i, Ψ_i, $i \in I$, satisfy the condition $(\Lambda\Psi)$, then the pair Λ_0, Ψ_0 satisfies this condition as well.*

Proof. For $A \subseteq J$ let us define $I_A = \{i \in I \mid A \cap J_i \in \Psi_i\}$. Let $A \in \Psi_0$ and $B \notin \Psi_0$. Then $I_A \in \Psi$ and $I_B \notin \Psi$ and, since the pair Λ, Ψ satisfies the condition $(\Lambda\Psi)$, there is an $L \in \Lambda$ such that $L \subseteq I_A \setminus I_B$ and $L^c \notin \Psi$. For each $i \in L$ we have $A \cap J_i \in \Psi_i$ and $B \cap J_i \notin \Psi_i$. Thus, since the pair Λ_i, Ψ_i satisfies the condition $(\Lambda\Psi)$, we can pick $L_i \in \Lambda_i$ such that $L_i \subseteq (A \setminus B) \cap J_i$ and $J_i \setminus L_i \notin \Psi_i$. Now, $L_0 = \bigcup_{i\in L} L_i$ is an element of Λ_0 and $L_0 \subseteq A \setminus B$. It remains to prove that $J \setminus L_0 \notin \Psi_0$, i.e. $I' = \{i \in I \mid J_i \setminus L_0 \in \Psi_i\} \notin \Psi$. For $i \in L$ we have $J_i \setminus L_0 \subseteq J_i \setminus L_i \notin \Psi$, so $J_i \setminus L_0 \notin \Psi_i$ proved $L \subseteq I \setminus I'$. Thus $I' \subseteq I \setminus L \notin \Psi$ and $I' \notin \Psi$.

Let the preceeding assumptions, without the condition $(\Lambda\Psi)$, hold and let $\{(X_j, O_j) \mid j \in J\}$ be a family of topological spaces.

THEOREM 5.2. *The mapping* $F : \prod^{\Lambda_0}_{\Psi_0}{}_{j\in J} X_j \longrightarrow \prod^{\Lambda}_{\Psi}{}_{i\in I}\prod^{\Lambda_i}_{\Psi_i}{}_{j\in J_i} X_j$, *given by*

$$F([f]_{\Psi_0}) = [\langle [f|_{J_i}]_{\Psi_i} \mid i \in I\rangle]_{\Psi}$$

is a homeomorphism.

Proof. F is a well-defined injection since for arbitrary $f, g \in \prod_{j\in J} X_j$, $F([f]_{\Psi_0}) = F([g])_{\Psi_0})$ iff $[f]_{\Psi_0} = [g]_{\Psi_0}$.

For $y = [\langle \varphi_i \mid i \in I\rangle]_{\Psi}$ from $\prod^{\Lambda}_{\Psi}\prod^{\Lambda_i}_{\Psi_i} \mathcal{X}_j$, where $\varphi_i = [f^i]_{\Psi_i}$ and $f^i \in \prod_{j\in J_i} X_j$, define $f \in \prod_{j\in J} X_j$ by $f = \bigcup_{i\in I} f^i$. Then $y = F([f]_{\Psi_0})$ and F is onto.

We introduce the notation for the projections and quotient mappings considered below by the following diagram

$$\begin{tikzcd}
\prod^{\Lambda_i}_{\Psi_i}{}_{j\in J_i}\mathcal{X}_j & \prod^{\Lambda}_{i\in I}(\prod^{\Lambda_i}_{\Psi_i}{}_{j\in J_i}\mathcal{X}_j) \arrow[l, "\pi_i"'] \arrow[r, "q"] & \prod^{\Lambda}_{\Psi}{}_{i\in I}(\prod^{\Lambda_i}_{\Psi_i}{}_{j\in J_i}\mathcal{X}_j) \\
\prod^{\Lambda_i}_{j\in J_i}\mathcal{X}_j \arrow[u, "q_i"] \arrow[d, "p_j"'] & & \prod^{\Lambda_0}_{\Psi_0}{}_{j\in J}\mathcal{X}_j \arrow[u, "F"'] \\
\mathcal{X}_j & & \prod^{\Lambda_0}_{j\in J}\mathcal{X}_j \arrow[ll, "r_j"'] \arrow[u, "q_{\Psi_0}"']
\end{tikzcd}$$

According to Lemma 1.3 the sets of form

$$q\left(\bigcap_{i\in L}\pi_i^{-1}\left(q_i\left(\bigcap_{j\in L_i}p_j^{-1}(B^i_j)\right)\right)\right),$$

where $L \in \Lambda,\ L_i \in \Lambda_i$ for $i \in L$ and $B^i_j \in \mathcal{B}_j$ for $i \in L$ and $j \in L_i$, form a topology base of the space $\prod^{\Lambda}_{\Psi}(\prod^{\Lambda_i}_{\Psi_i} \mathcal{X}_j)$ and the sets

$$q_{\Psi_0}\left(\bigcap_{j\in\cup_{i\in L}L_i} r_j^{-1}(B_j)\right),$$

where $L \in \Lambda,\ L_i \in \Lambda_i$ for $i \in L$ and $B_j \in \mathcal{B}_j$ for $j \in \bigcup_{i\in L} L_i$, form a base for the topology on $\prod^{\Lambda_0}_{\Psi_0} \mathcal{X}_j$. It is easy to check that

$$q_{\Psi_0}\left(\bigcap_{j\in\cup_{i\in L}L_i} r_j^{-1}(B_j)\right) = F^{-1}\left(q\left(\bigcap_{i\in L}\pi_i^{-1}\left(q_i\left(\bigcap_{j\in L_i}p_j^{-1}(B_j)\right)\right)\right)\right),$$

which proves that F is a continuous mapping. Using direct images of both sides of the last equality we get that F is open.

6. Reduced ideal-product of topological algebras

The aim of this section is (roughly speaking) to prove that the r.i.p. of topological groups (rings, ...) is a topological group (ring, ...). We will consider a more general situation when the "coordinate spaces" are equipped with an arbitrary continuous n-ary operation.

Lemma 6.1. *Let $\{(X_i^k, \mathcal{O}_i^k) \mid i \in I\}$ be a family of topological spaces for $k \in \{1, \dots, n\}$. Then the mapping*

$$\eta : \prod_{k=1}^{n} \left(\prod\nolimits^{\Lambda} X_i^k \right) \longrightarrow \prod\nolimits^{\Lambda} \left(\prod_{k=1}^{n} X_i \right),$$

given by $\eta(\langle f_i^1 \rangle, \dots, \langle f_i^n \rangle) = \langle (f_i^1, \dots, f_i^n) \rangle$ is a homeomorphism.

Proof. η is obviously a bijection. Let us prove the continuity of η. A base for the topology on $\prod_{k=1}^{n} X_i$ consists of sets of the form $\prod_{k=1}^{n} U_i^k$, where $U_i^k \in \mathcal{O}_i^k$ for each $k \in \{1, \dots, n\}$. By Lemma 1.3, the sets of the form

$$\bigcap_{i \in L} \pi_i^{-1} \left(\prod_{k=1}^{n} U_i^k \right),$$

where $L \in \Lambda$, form a topology base on $\prod^{\Lambda} (\prod_{k=1}^{n} X_i^k)$. The proof of the equality

$$\eta^{-1} \left(\bigcap_{i \in L} \pi_i^{-1} \left(\prod_{k=1}^{n} U_i^k \right) \right) = \prod_{k=1}^{n} \left(\bigcap_{i \in L} \pi_i^{-1} (U_i^k) \right) \tag{1}$$

is direct. Since the sets $\bigcap_{i \in L} \pi_i^{-1}(U_i^k)$, $k = 1, \dots, n$, are open in $\prod^{\Lambda} X_i^k$, η is continuous.

Since η is a bijection, from (1) we have:

$$\eta \left(\prod_{k=1}^{n} \bigcap_{i \in L} \pi_i^{-1} (U_i^k) \right) = \bigcap_{i \in L} \pi_i^{-1} \left(\prod_{k=1}^{n} U_i^k \right)$$

which gives the openness of η.

Lemma 6.2. *Let $\{(X_i, \mathcal{O}_i) \mid i \in I\}$ be a family of topological spaces, Λ an ideal on I and let $\oplus_i : X_i^n \longrightarrow X_i$, $i \in I$, be continuous n-ary operations. Then the operation $\oplus : (\prod^{\Lambda} X_i)^n \longrightarrow \prod^{\Lambda} X_i$, given by $\oplus(f^1, \dots, f^n) = \langle \oplus_i (f_i^1, \dots, f_i^n) \rangle$ is continuous.*

Proof. By Lemma 4.1, the direct product $\oplus^{\Lambda} = \prod_{i \in I} \oplus_i$ of the family of mappings $\{\oplus_i \mid i \in I\}$ is a continuous mapping. It is easy to prove that the diagram

$$\begin{tikzcd}[column sep=small] \prod^{\Lambda} X_i^n \arrow[rr, "\oplus^{\Lambda}"] & & \prod^{\Lambda} X_i \\ & \left(\prod^{\Lambda} X_i\right)^n \arrow[ul, "\eta"] \arrow[ur, "\oplus"'] & \end{tikzcd}$$

where η is the homeomorphism from the previous lemma, commutes. Thus $\oplus^{\Lambda} \circ \eta = \oplus$, and the continuity of $\oplus^{\Lambda}$ and η implies the continuity of $\oplus$.

Theorem 6.3. *Suppose that the conditions of the previous lemma are satisfied. Then the operation* $\oplus_{\Psi} : \left(\prod_{\Psi}^{\Lambda} X_i\right)^n \longrightarrow \prod_{\Psi}^{\Lambda} X_i$*, defined by*

$$\oplus_{\Psi}([f^1], \dots, [f^n]) = [\oplus(f^1, \dots, f^n)],$$

(where $\oplus$ *is the operation from the previous lemma) is continuous.*

Proof. It is easy to check that $\oplus_{\Psi}$ is well-defined. Let us prove that the diagram

$$\begin{tikzcd} \left(\prod^{\Lambda} X_i\right)^n \arrow[r, "q^n"] \arrow[d, "\oplus"'] & \left(\prod_{\Psi}^{\Lambda} X_i\right)^n \arrow[d, "\oplus_{\Psi}"] \\ \prod^{\Lambda} X_i \arrow[r, "q"] & \prod_{\Psi}^{\Lambda} X_i \end{tikzcd}$$

commutes, i.e. that $q \circ \oplus = \oplus_{\Psi} \circ q^n$. For an arbitrary $(f^1, \dots, f^n) \in (\prod^{\Lambda} X_i)^n$ we have:

$$\begin{aligned} (\oplus_{\Psi} \circ q^n)(f^1, \dots, f^n) &= \oplus_{\Psi}(([f^1], \dots, [f^n])) = [\oplus(f^1, \dots, f^n)] \\ &= q(\oplus(f^1, \dots, f^n)) = (q \circ \oplus)(f^1, \dots, f^n). \end{aligned}$$

By Lemma 6.2, $\oplus$ is a continuous mapping and since q is also continuous we obtain the continuity of $q \circ \oplus$. Hence, $\oplus_{\Psi} \circ q^n$ is a continuous mapping. The mapping q is continuous, open and onto, thus the direct product q^n has the same properties. The continuity of $\oplus_{\Psi}$ follows from the following lemma, the proof of which is rather obviuos.

Lemma 6.4. *Let* $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ *and* $(Z, \mathcal{O}_Z)$ *be topological spaces and let the mapping* $q : X \longrightarrow Y$ *be continuous, open and onto. Then for each mapping* $\varphi : Y \longrightarrow Z$ *we have:*

φ *is continuous iff* $\varphi \circ q$ *is continuous.*

Now, since the reduced product of groups (rings) is a group (ring), we are able to state

Corollary 6.5. *The reduced ideal-product of an arbitrary family of topological groups (rings) is a topological group (ring), where the corresponding operations are defined as in the previous theorem.*

Remark. Sometimes the definition of a topological group includes the condition that the topology is T_1. In this case the previous statement is valid if the condition $(\Lambda\Psi)$ is satisfied.

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