PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 57 (71), 1995, 135–142
Đuro Kurepa memorial volume

# SOME REMARKS ON GENERALIZED MARTIN'S AXIOM

**Z. Spasojević**

**Abstract**. Let $GMA$ denote that if $\mathbb{P}$ is well-met, strongly $\omega_1$-closed and $\omega_1$-centered partial order and $\mathcal{D}$ a family of $< 2^{\omega_1}$ dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ which meets every member of $\mathcal{D}$. The consistency of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$ was proved by Baumgartner [**1**] and in [**13**] many of its consequences were considered. In this paper we give a consequence and present an independence result. Namely, we prove that, as a consequence of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$, every $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap and show that the existence of an $\omega_2$-Kurepa tree is consistent with and independent of $2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$.

**1. Introduction.** With the discovery of Martin's Axiom [**8**] and its many consequences a number of set-theorists considered the problem of generalizing Martin's Axiom to higher cardinals. Their aim actually was to generalize the consequences of $MA$ to higher cardinals. One of the first generalizations of Martin's Axiom is due to Baumgartner [**1**] and one of the strongest generalizations is due to Shelah [**9**]. We will return to Shelah's version in the last section but now we state Baumgartner's result. A partial order $\mathbb{P}$ is well-met if any two compatible elements in $\mathbb{P}$ have the greatest lower bound. We denote compatibility of $p, q \in \mathbb{P}$ by $p \not\perp q$ and their incompatibility by $p \perp q$. $\mathbb{P}$ is $\omega_1$-closed if any decreasing $\omega$-sequence in $\mathbb{P}$ has a lower bound and it is strongly $\omega_1$-closed if the greatest lower bound exists for any such sequence. $\mathbb{P}$ is centered if any finite sub-collection of $\mathbb{P}$ has a lower bound and it is $\omega_1$-centered if it is a union of $\omega_1$ many centered partial orders. Baumgartner [**1**] constructed a model for

$$(BA) \qquad 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA$$

and thus obtained the consistency of one of the first versions of Generalized Martin's Axiom. In fact, Baumgartner considered a somewhat bigger class of partial orders, but in this paper we will only consider partial orders which are well-met, strongly $\omega_1$-closed, $\omega_1$-centered and of size $< 2^{\omega_1}$, where $2^{\omega_1}$ is computed in the final model.

Many consequences of $(BA)$ were considered in [**13**]. The object of this paper is to present one more consequence and an independence result. We show that every

---

*AMS Subject Classification* (1991): Primary 03E35

$\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap (see §2 for notation and terminology). As usual, this result will be obtained by applying $GMA$ to suitably chosen partial orders. It will be fairly straight-forward to show that this partial order is well-met and strongly $\omega_1$-closed. Somewhat harder will be to show that it is $\omega_1$-centered. For this we first need to recall the notion of a complete embedding.

*Definition* 1.1. Let $\mathbb{P}$ and $\mathbb{Q}$ be partial orders. An $i: \mathbb{P} \to \mathbb{Q}$ is a complete embedding if

(*a*) $\forall p, p' \in \mathbb{P}(p' \leq p \to i(p') \leq i(p))$,

(*b*) $\forall p, p' \in \mathbb{P}(p' \perp p \leftrightarrow i(p') \perp i(p))$,

(*c*) $\forall q \in \mathbb{Q} \exists p \in \mathbb{P} \forall p' \in \mathbb{P}(p' \leq p \to i(p') \not\perp i(p))$.

We also recall a result from [**13**].

PROPOSITION 1.2. Assume $2^\omega = \omega_1$. Then any countable support iteration of length $\leq 2^{\omega_1}$ with well-met, strongly $\omega_1$-closed and $\omega_1$-centered partial orders yields an $\omega_1$-centered partial order.

At this stage we also point out that $2^\omega = \omega_1$ is assumed throughout this paper. Now, let $\mathbb{P}$ be a partial order which is well-met and strongly $\omega_1$-closed and suppose that all the conditions in $\mathbb{P}$ are countable. To show that $\mathbb{P}$ is $\omega_1$-centered, it suffices to exhibit a sequence $\langle \mathbb{P}_\xi : \xi \leq \alpha \leq 2^{\omega_1} \rangle$ of sub-orders of $\mathbb{P}$ such that $\mathbb{P}_\alpha = \mathbb{P}$ and each $\mathbb{P}_\xi$, for $\xi < \alpha$, is well-met, strongly $\omega_1$-closed and $\omega_1$-centered, as well as a sequence $\langle i_{\xi\eta} : \xi \leq \eta \leq \alpha \rangle$, with $i_{\xi\eta} : \mathbb{P}_\xi \to \mathbb{P}_\eta$, of complete embeddings such that $\forall \xi, \eta, \theta (\xi \leq \eta \leq \theta \leq \alpha \to i_{\xi\theta} = i_{\eta\theta} \circ i_{\xi\eta})$. Then $\mathbb{P}$ can be viewed as a countable support iteration of length $\alpha \leq 2^{\omega_1}$ with well-met, strongly $\omega_1$-closed and $\omega_1$-centered partial orders so that by Proposition 1.2 $\mathbb{P}$ is also $\omega_1$-centered.

It is well known that $2^\omega = \omega_1$ implies the existence of an $\omega_2$-Aronszajn tree (see [**6**]). The results of Laver and Shelah [**7**] and Shelah and Stanley [**10**] show that the existence of an $\omega_2$-Suslin tree is consistent with and independent of $(BA)$. In the final section we consider the influence of $(BA)$ on the existence of $\omega_2$-Kurepa trees. Our result is that the existence of such trees is consistent with and independent of $(BA)$.

**2. Gaps.** Let $\kappa^\kappa$ be the set of all function from $\kappa$ to $\kappa$. If $f, g \in \kappa^\kappa$ then $f \leq^* g$ if and only if $\exists n < \kappa \forall i < \kappa (i \geq n \to f(i) \leq g(i))$ and $f(i) < g(i)$ on a set of size $\kappa$. A $(\kappa^+, \kappa^+)$-pregap in $(\kappa^\kappa, \leq^*)$ is a pair $(a, b)$ where $a = \langle a_\xi : \xi < \kappa^+ \rangle$ and $b = \langle b_\xi : \xi < \kappa^+ \rangle$ are subsets of $\kappa^\kappa$ such that $\forall \xi, \eta < \kappa^+ (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \kappa^+ (a_\xi \leq^* a_\eta \wedge b_\eta \leq^* b_\xi)$. If there is a $c \in \kappa^\kappa$ such that $\forall \xi, \eta < \kappa^+ (a_\xi \leq^* c \leq^* b_\eta)$ then $c$ splits the pregap $(a, b)$. If no such $c$ exists then $(a, b)$ is a $(\kappa^+, \kappa^+)$-gap.

Hausdorff [**4**] showed (in $ZFC$) that $(\omega^\omega, \leq^*)$ contain an $(\omega_1, \omega_1)$-gap. Herink [**5**] and independently Blaszczyk and Szymanski [**2**] generalized Hausdorff's result to higher cardinals by proving that if $\kappa$ is a regular cardinal then $(\kappa^\kappa, \leq^*)$ contains a $(\kappa^+, \kappa^+)$-gap. Hausdorff's result was refined in [**11**] by showing that $MA$ implies that every $\leq^*$-increasing $\omega_1$-sequence in $(\omega^\omega, \leq^*)$ is a lower half of some $(\omega_1, \omega_1)$-gap. And this last result was further improved in [**12**] by establishing that $\mathfrak{t} > \omega_1$ is in fact equivalent to the statement that every $\leq^*$-increasing $\omega_1$-sequence in $(\omega^\omega, \leq^*)$ is a lower half of some $(\omega_1, \omega_1)$-gap. The goal of this section is to show that $(BA)$ implies that every $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$ is a lower half of some $(\omega_2, \omega_2)$-gap and thus refine the results of Herink [**5**] and Blaszczyk and Szymanski [**2**].

Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$. A $\leq^*$-decreasing $\omega_2$-sequence $b = \langle b_\xi : \xi < \omega_2 \rangle$ on top of $a$, such that $(a, b)$ is an $(\omega_2, \omega_2)$-gap, will be obtained from an application of $GMA$ to a suitably defined partial order $\mathbb{P}_a$. In order to guarantee that $(a, b)$ is in fact a gap, the elements of the sequences $a$ and $b$ have to satisfy the following condition:

$$(\star) \quad \forall \xi < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\xi(i)) \wedge \forall \xi, \eta < \omega_2 (\xi < \eta \to \exists i < \omega_1 (b_\xi(i) < a_\eta(i))).$$

This condition is a refinement of the following condition due to Kunen for $(\omega_1, \omega_1)$-gaps in $(\omega^\omega, \leq^*)$ (unpublished work):

$$\forall \xi < \omega_1 \forall i < \omega (a_\xi(i) \leq b_\xi(i)) \qquad \text{and}$$
$$\forall \xi, \eta < \omega_1 (\xi \neq \eta \to \exists i < \omega (a_\xi(i) \not\leq b_\eta(i) \vee a_\eta(i) \not\leq b_\xi(i))).$$

Now we show that if $2^\omega = \omega_1$ then every $(\omega_2, \omega_2)$-pregap in $(\omega_1^{\omega_1}, \leq^*)$ satisfying $(\star)$ is in fact a gap.

Lemma 2.1. *Assume $2^\omega = \omega_1$ and let $(a, b) = \langle a_\xi, b_\xi : \xi < \omega_2 \rangle$ be an $(\omega_2, \omega_2)$-pregap in $(\omega_1^{\omega_1}, \leq^*)$ whose elements satisfy $(\star)$. Then $(a, b)$ is a gap.*

*Proof.* By way of contradiction, assume $(a, b)$ is split by $c : \omega_1 \to \omega_1$. Then

$$(\circ) \qquad \forall \xi < \omega_2 \exists n_\xi < \omega_1 \forall n \geq n_\xi (a_\xi(n) \leq c(n) \leq b_\xi(n)).$$

By a first thinning process we may assume that $\forall \xi < \omega_2 (n_\xi = m)$, for some fixed $m < \omega_1$. Since $2^\omega = \omega_1$ and $m$ is a countable ordinal, we have $| \omega_1^m | = \omega_1$. Hence, by another thinning process we may assume that

$$(\bullet) \qquad \forall \xi, \eta < \omega_2 (a_\xi \restriction m = a_\eta \restriction m \wedge b_\xi \restriction m = b_\eta \restriction m).$$

But then $(\circ)$,$(\bullet)$ and the first clause of $(\star)$ imply that $\forall \xi, \eta < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\eta(i))$, which contradicts the second clause of $(\star)$. Hence, $(a, b)$ is a gap and the Lemma is proved. $\square$

Therefore, the definition of $\mathbb{P}_a$ has to incorporate the requirements in $(\star)$.

*Definition* 2.2. Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$.

$$\begin{aligned} \mathbb{P}_a = \{ \langle x, y, n, s \rangle : f x, y \in [\omega_2]^{<\omega_1} \wedge n < \omega_1 \wedge s : y \to \omega_1^n \wedge \\ f \forall \xi \in y ((\xi \in x \to \forall i < n (a_\xi(i) \leq s(\xi)(i))) \wedge \\ f \forall \eta \in x (\eta > \xi \to \exists i < n (s(\xi)(i) < a_\eta(i)))) \} \end{aligned}$$

where $\langle x_2, y_2, n_2, s_2 \rangle \leq \langle x_1, y_1, n_1, s_1 \rangle$ if and only if

(1) $x_1 \subseteq x_2, y_1 \subseteq y_2, n_1 \leq n_2$,

(2) $\forall \xi \in y_1 (s_2(\xi) \restriction n_1 = s_1(\xi))$,

(3) $\forall \xi, \eta \in y_1 \forall i < \omega_1 (\xi \leq \eta \wedge n_1 \leq i < n_2 \rightarrow s_2(\eta)(i) \leq s_2(\xi)(i))$,

(4) $\forall \xi \in x_1 \forall \eta \in y_1 \forall i < \omega_1 (n_1 \leq i < n_2 \rightarrow a_\xi(i) \leq s_2(\eta)(i))$.

Clearly $\mathbb{P}_a$ is a partial order and the next step is to show that $\mathbb{P}_a$ is well-met, strongly $\omega_1$-closed and $\omega_1$-centered so that $GMA$ can be applied to it.

So let $\langle x_1, y_1, n_1, s_1 \rangle, \langle x_2, y_2, n_2, s_2 \rangle \in \mathbb{P}_a$ and suppose $\langle u, v, k, t \rangle \in \mathbb{P}_a$ is their lower bound. We may assume that $u = x_1 \cup x_2$ and $v = y_1 \cup y_2$. Then there is the least $m$ such that $\max(n_1, n_2) \leq m \leq k$ and $\langle u, v, m, t \restriction m \rangle \in \mathbb{P}_a$, where $t \restriction m$ is a function with domain $v$ such that $\forall \xi \in v ((t \restriction m)(\xi) = t(\xi) \restriction m)$. Then it is easily seen that $\langle u, v, m, t \restriction m \rangle$ is the greatest lower bound of $\langle x_1, y_1, n_1, s_1 \rangle$ and $\langle x_2, y_2, n_2, s_2 \rangle$ so that $\mathbb{P}_a$ is well-met.

Now let $\langle x_0, y_0, n_0, s_0 \rangle \geq \langle x_1, y_1, n_1, s_1 \rangle \geq \cdots$ be a decreasing $\omega$-sequence in $\mathbb{P}_a$. Let $u = \bigcup_{i<\omega} x_i$, $v = \bigcup_{i<\omega} y_i$, $m = \sup_{i<\omega}(n_i)$ and let $t$ be a function with domain $v$ such that $\forall \xi \in v (t(\xi) = \bigcup \{ s_i(\xi) : \xi \in y_i \})$. Then $\langle u, v, m, t \rangle$ is the greatest lower bound in $\mathbb{P}_a$ of the above sequence so that $\mathbb{P}_a$ is strongly $\omega_1$-closed.

As indicated in §1, to show that $\mathbb{P}_a$ is $\omega_1$-centered it suffices to show that there is a sequence $\langle \mathbb{P}_\alpha : \alpha \leq \omega_2 \rangle$ of sub-orders of $\mathbb{P}_a$ such that $\mathbb{P}_{\omega_2} = \mathbb{P}_a$ and a sequence $\langle i_{\alpha\beta} : \alpha \leq \beta \leq \omega_2 \rangle$, with $i_{\alpha\beta} : \mathbb{P}_\alpha \rightarrow \mathbb{P}_\beta$, of complete embeddings such that $\forall \alpha, \beta, \gamma (\alpha \leq \beta \leq \gamma \leq \omega_2 \rightarrow i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$ and such that each $\mathbb{P}_\alpha$, for $\alpha < \omega_2$, is well-met, strongly $\omega_1$-closed and $\omega_1$-centered. Then $\mathbb{P}_a$ can be viewed as a countable support iteration of length $\omega_2$ with well-met, strongly $\omega_1$-closed and $\omega_1$-centered partial orders, since $\mathbb{P}_a$ consists of countable conditions. Then, by Proposition 1.2, $\mathbb{P}_a$ is also $\omega_1$-centered.

For each $\alpha \leq \omega_2$ let $\mathbb{P}_\alpha = \{ \langle x, y, n, s \rangle \in \mathbb{P}_a : y \subseteq \alpha \}$ and for each $\alpha \leq \beta \leq \omega_2$ let $i_{\alpha\beta} : \mathbb{P}_\alpha \rightarrow \mathbb{P}_\beta$ be the inclusion map $i(p) = p$. Then $\mathbb{P}_a = \mathbb{P}_{\omega_2}$, each $\mathbb{P}_\alpha$ is a sub-order of $\mathbb{P}_a$ with the ordering relation inherited from $\mathbb{P}_a$, and $\forall \alpha \leq \beta \leq \gamma \leq \omega_2 (i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta})$. Analogous proof can be used to show that each $\mathbb{P}_\alpha$, for $\alpha < \omega_2$, is well-met and strongly $\omega_1$-closed as the one used to show that $\mathbb{P}_a$ has these properties.

Lemma 2.3. *For each $\alpha \leq \beta \leq \omega_2$, $i_{\alpha\beta}$ is a complete embedding.*

*Proof.* Properties $(a)$ and $(b)$ of Definition 1.1 are satisfied in a trivial way. For $(c)$, let $q = \langle x_q, y_q, n_q, s_q \rangle \in \mathbb{P}_\beta$. Then $p = \langle x_q, y_q \cap \alpha, n_q, s_q \restriction (y_q \cap \alpha) \rangle$ has the required property. $\square$

Lemma 2.4. *Assume $2^\omega = \omega_1$. Then for each $\alpha < \omega_2$, $\mathbb{P}_\alpha$ is $\omega_1$-centered.*

*Proof.* Let $\alpha < \omega_2$ and for each $y \in [\alpha]^{<\omega_1}, n < \omega_1, s \in (\omega_1^n)^y$ let $\mathbb{P}_{\alpha yns} = \{ \langle x, z, m, t \rangle \in \mathbb{P}_\alpha : z = y \wedge m = n \wedge t = s \}$. Then $\mathbb{P}_\alpha = \bigcup \{ \mathbb{P}_{\alpha yns} : y \in [\alpha]^{<\omega_1} \wedge n < \omega_1 \wedge s \in (\omega_1^n)^y \}$ and since $2^\omega = \omega_1$, hence $\omega_1^\omega = \omega_1$, we have that $\mathbb{P}_\alpha$ is a union of $\omega_1$ many sub-orders. Furthermore, if $\langle x_1, y, n, s \rangle, ..., \langle x_k, y, n, s \rangle \in \mathbb{P}_{\alpha yns}$ then $\langle x_1 \cup ... \cup x_k, y, n, s \rangle \in \mathbb{P}_{\alpha yns}$ and $\langle x_1 \cup ... \cup x_k, y, n, s \rangle \leq \langle x_1, y, n, s \rangle, ..., \langle x_k, y, n, s \rangle$. Thus, each $\mathbb{P}_{\alpha yns}$ is centered so that $\mathbb{P}_\alpha$ is $\omega_1$-centered. $\square$

Therefore, Lemmas 2.3 and 2.4 imply that $\mathbb{P}_a$ can be viewed as a countable support iteration of length $\omega_2$ with partial orders which are well-met, strongly $\omega_1$-closed and $\omega_1$-centered. Thus, by Proposition 1.2, $\mathbb{P}_a$ is also $\omega_1$-centered.

Now we come to the main result of this section.

THEOREM 2.5 *Assume* $(BA)$. *Then every* $\leq^*$*-increasing* $\omega_2$*-sequence in* $(\omega_1^{\omega_1}, \leq^*)$ *is a lower half of some* $(\omega_2, \omega_2)$*-gap.*

*Proof.* Let $a = \langle a_\xi : \xi < \omega_2 \rangle$ be an $\leq^*$-increasing $\omega_2$-sequence in $(\omega_1^{\omega_1}, \leq^*)$. Then by the previous results, $\mathbb{P}_a$ is well-met, strongly $\omega_1$-closed and $\omega_1$-centered. Let $G$ be a filter in $\mathbb{P}_a$ and for each $\eta < \omega_2$ let

$$b_\eta = \bigcup \{ s(\eta) : \exists p \in G(p = \langle x_p, y_p, n_p, s_p \rangle \wedge s = s_p)\}.$$

Condition (4) of Definition 2.2 together with the requirement that for each $\xi, \eta < \omega_2$ and each $m < \omega_1$ the filter $G$ has a nonempty intersection with the following dense sets

$$D_{\xi\eta m} = \{\langle x, y, n, s \rangle \in \mathbb{P}_a : \xi \in x \wedge \eta \in y \wedge n \geq m\}$$

will guarantee that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$. In addition, condition (3) of Definition 2.2 together with the requirement that for each $\xi < \eta < \omega_2$ and each $m < \omega_1$ the filter $G$ has a nonempty intersection with the following dense sets

$$E_{\xi\eta m} = \{\langle x, y, n, s \rangle \in \mathbb{P}_a : \xi, \eta \in y \wedge \mid \{ i : s(\eta)(i) < s(\xi)(i)\} \mid \geq m\}$$

will guarantee that $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$. Then the total number of these dense sets $D_{\xi\eta m}$ and $E_{\xi\eta m}$ is $\omega_2$. Therefore, to satisfy the requirements that $\forall \xi, \eta < \omega_2 (a_\xi \leq^* b_\eta)$ and $\forall \xi < \eta < \omega_2 (b_\eta \leq^* b_\xi)$ the filter $G$ needs to intersect $\omega_2$ dense subsets of $\mathbb{P}_a$ and by $(BA)$ there is one such filter. In addition, the definition of $\mathbb{P}_a$ implies that $\forall \xi < \omega_2 \forall i < \omega_1 (a_\xi(i) \leq b_\xi(i))$ and $\forall \xi, \eta < \omega_2 (\xi < \eta \rightarrow \exists i < \omega_1 (b_\xi(i) < a_\eta(i)))$ so that $(a, b)$ is in fact an $(\omega_2, \omega_2)$-gap in $(\omega_1^{\omega_1}, \leq^*)$. $\square$

**3. Trees.** It is well known that $2^\omega = \omega_1$ implies the existence of an $\omega_2$-Aronszajn tree (see [**6**]). Since $2^\omega = \omega_1$ is a part of $(BA)$, it follows that $(BA)$ sattles the existance of an $\omega_2$-Aronszajn tree. By the results of Laver and Shelah [**7**] and Shelah and Stanley [**10**] the existence of an $\omega_2$-Suslin tree is consistent with and independent of $(BA)$. In this section we consider the influence of $(BA)$ on the existence of $\omega_2$-Kurepa trees. We show that the existence of such trees is consistent with and independent of a stronger version of Generalized Martin's Axiom, due to Shelah [**9**], than the one we have considered so far. Recall that an $\omega_2$-Kurepa tree is a tree $\mathbb{T} = (T, \leq_T)$ of hight $\omega_2$ such that any level of $\mathbb{T}$ is of size $< \omega_2$. If $x \in T$ let $\hat{x} = \{y \in T : y <_T x\}$. We also assume that $T = \omega_2$ and that all our trees have the following properties:

1) $\mid \mathrm{Lev}_0(\mathbb{T}) \mid = 1$,
2) $\forall \alpha < \beta < \mathrm{hight}(\mathbb{T}) \forall x \in \mathrm{Lev}_\alpha(\mathbb{T}) \exists y_1, y_2 \in \mathrm{Lev}_\beta(\mathbb{T})(y_1 \neq y_2 \wedge x <_T y_1, y_2)$,
3) $\forall \alpha < \mathrm{hight}(\mathbb{T}) \forall x, y \in \mathrm{Lev}_\alpha(\mathbb{T})(\text{limit } \alpha \rightarrow (x = y \leftrightarrow \hat{x} = \hat{y}))$.

We begin by formulating Shelah's version of Generalized Martin's Axiom. A partial order $\mathbb{P}$ is $\omega_2$-normal if $\{p_\alpha : \alpha < \omega_2\} \subseteq \mathbb{P}$ then there is a club $C \subseteq \omega_2$ and a regressive function $f : \omega_2 \to \omega_2$ such that for $\alpha, \beta \in C$ if $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta) = \omega_1$ and $f(\alpha) = f(\beta)$ then $p_\alpha$ and $p_\beta$ are compatible. Note that $\omega_2$-normality is a strengthening of $\omega_2$-Knaster condition, which states that if $\{p_\alpha : \alpha < \omega_2\} \subseteq \mathbb{P}$ then there is an $A \in [\omega_2]^{\omega_2}$ such that any two elements in $\{p_\alpha : \alpha \in A\}$ are compatible. Let $GMA^*$ denote the statement that if $\mathbb{P}$ is a partial order such that $|\mathbb{P}| < 2^{\omega_1}$, it is well-met, strongly $\omega_1$-closed and $\omega_2$-normal and $\mathcal{D}$ is a family of $< 2^{\omega_1}$ dense subsets of $\mathbb{P}$ then there is a filter $G \subseteq \mathbb{P}$ meeting all the elements of $\mathcal{D}$. The following Lemma is due to Shelah [**9**].

LEMMA 3.1. *Suppose $2^\omega = \omega_1$, $2^{<\kappa} = \kappa$ and $\kappa$ is a regular cardinal. Let $\langle\langle \mathbb{P}_\alpha : \alpha \le \kappa\rangle, \langle \mathbb{Q}_\alpha : \alpha < \kappa\rangle\rangle$ be a countable support iteration such that*

$$\mathbf{1} \Vdash_{\mathbb{P}_\alpha} \text{“ } \mathbb{Q}_\alpha \textit{ is well-met, strongly } \omega_1\textit{-closed and } \omega_2\textit{-normal ”.}$$

*Then $\mathbb{P}_\kappa$ is strongly $\omega_1$-closed and $\omega_2$-normal.*

This Lemma is essentially all that is needed in Shelah's proof [**9**] of the consistency of

$$(SA) \qquad 2^\omega = \omega_1 + 2^{\omega_1} > \omega_2 + GMA^*.$$

This Lemma will also play a role in the analysis below.

To obtain a model for $(SA)$ in which there is an $\omega_2$-Kurepa tree, we start with a ground model $V$ for $ZFC + GCH$ in which there is an $\omega_2$-Kurepa tree. For example, the constructible universe $L$ has this property. Then iterate, as in [**9**], to obtain a model for $(SA)$. By Lemma 3.1 cofinalities and hence cardinals are preserved by the iteration so that any $\omega_2$-Kurepa tree in the ground model remains an $\omega_2$-Kurepa tree in the extension. Thus, the existence of an $\omega_2$-Kurepa tree is consistent with $(SA)$.

The construction of a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ in which there are no $\omega_2$-Kurepa trees requires the existence of a strongly inaccessible cardinal and it is analogous to Devlin's construction [**3**] of a model for $2^\omega = \omega_2 + MA$ in which there are no $\omega_1$-Kurepa trees. Therefore we only present an outline of our construction.

The construction will proceed as follows. Start with a model for $ZFC+GCH$ in which $\kappa$ is a strongly inaccessible cardinal. Then collapse $\kappa$ to $\omega_3$ by the Levy collapse $\mathbb{L}_\kappa$ (see below). In the extension, there are no $\omega_2$-Kurepa trees. Then iterate, as in [**9**], to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. We use Lemma 3.1 to show that in the final model there are no $\omega_2$-Kurepa trees.

Now, we define the Levy collapse $\mathbb{L}_\kappa$ and present some of its properties whose proofs are standard.

*Definition* 3.2. $\mathbb{L}_\kappa = \{p : |p| \le \omega_1 \wedge p$ is a function $\wedge\, \mathrm{dom}(p) \subseteq \kappa \times \omega_2 \wedge \forall(\alpha, \xi) \in \mathrm{dom}(p)(p(\alpha, \xi) \in \alpha)\}$, where $p \le q$ if and only if $p \supseteq q$.

For $\lambda < \kappa$ let $\mathbb{L}_\lambda = \{p \in \mathbb{L}_\kappa : \mathrm{dom}(p) \subseteq \lambda \times \omega_2\}$ and $\mathbb{L}^\lambda = \{p \in \mathbb{L}_\kappa : \mathrm{dom}(p) \subseteq (\kappa \setminus \lambda) \times \omega_2\}$. then $\mathbb{L}_\lambda \times \mathbb{L}^\lambda$ is isomorphic to $\mathbb{L}_\kappa$.

LEMMA 3.3. *$\mathbb{L}_\kappa$ is $\omega_2$-closed. If $\kappa$ is strongly inaccessible, then $\mathbb{L}_\kappa$ has the $\kappa$-cc.*

LEMMA 3.4. *Let $M$ be a countable transitive model (c.t.m.) for $ZFC+GCH$ and suppose $\kappa$ is strongly inaccessible in $M$ and $G$ is $\mathbb{L}_\kappa$-generic over $M$. Then $\omega_1^{M[G]} = \omega_1^M$, $\omega_2^{M[G]} = \omega_2^M$, $\omega_3^{M[G]} = \kappa$ and, in $M[G]$, there are no $\omega_2$-Kurepa trees.*

So, by extending with $\mathbb{L}_\kappa$, $\omega_1$ and $\omega_2$ remain unchanged and $\kappa$ gets collapsed to $\omega_3$ and if $GCH$ holds in $M$ it also holds in $M[G]$.

The idea now is to start with a model $M[G]$, as above, and iterate, as in [**9**], $\omega_3$ times to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. But we need to know that the iteration does not introduce any new $\omega_2$-Kurepa trees. The next two Lemmas are toward this end. We omit the proofs as the Lemmas and their proofs are the analogues of the corresponding Lemmas for $\omega_1$-trees. The first one is the analogue of Lemma 3.6 in [**3**] and the second one is the analogue of Theorem 8.5 in [**1**].

LEMMA 3.5. *Let $M$ be a c.t.m. for $ZFC$ and suppose that, in $M$, $\mathbb{P}$ and $\mathbb{Q}$ are partial orders such that $\mathbb{P}$ is strongly $\omega_1$-closed and $\omega_2$-normal and $\mathbb{Q}$ is $\omega_2$-closed. Let $G$ be $\mathbb{P}\times\mathbb{Q}$-generic over $M$. Let $G_\mathbb{P} = \{p \in \mathbb{P}: (p,1) \in G\}$ and $G_\mathbb{Q} = \{q \in \mathbb{Q}: (1,q) \in G\}$. Then if $\mathbb{T}$ is an $\omega_2$-tree in $M[G_\mathbb{P}]$ and $b$ is an $\omega_2$-branch of $\mathbb{T}$ in $M[G]$, then $b \in M[G_\mathbb{P}]$. In addition $\omega_1^{M[G]} = \omega_1^M$ and $\omega_2^{M[G]} = \omega_2^M$.*

LEMMA 3.6. *Suppose $\mathbb{T}$ is an $\omega_2$-tree and $\mathbb{P}$ is strongly $\omega_1$-closed and $\omega_2$-normal partial order. Then forcing with $\mathbb{P}$ adds no new $\omega_2$-branches through $\mathbb{T}$.*

Now we state and prove the main result of this section.

THEOREM 3.7. *Let $M$ be a c.t.m. for $ZFC + GCH$ and $\kappa$ strongly inaccessible in $M$. Then there is an extension of $M$ which is a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ in which there are no $\omega_2$-Kurepa trees.*

*Proof.* Let $M$ be as above and $G$ $\mathbb{L}_\kappa$-generic over $M$. Then, by Lemma 3.4, in $N = M[G]$ there are no $\omega_2$-Kurepa trees and $GCH$ still holds. In $N$, we perform a countable support iteration of length $\omega_3$, as in [**9**], to obtain a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$. Let $\langle\langle \mathbb{P}_\alpha : \alpha \le \omega_3\rangle, \langle \mathbb{Q}_\alpha : \alpha < \omega_3\rangle\rangle$ be such iteration and $H$ $\mathbb{P}_{\omega_3}$-generic over $N$. Then $N[H]$ is a model for $2^\omega = \omega_1 + 2^{\omega_1} = \omega_3 + GMA^*$ and we now show that there are no $\omega_2$-Kurepa trees in $N[H]$. In $N$, let $\tau$ be a nice $\mathbb{P}_{\omega_3}$-name for an $\omega_2$-tree in $N[H]$ (see [**6**] for the definition of a nice name). Since, by Lemma 3.1, $\mathbb{P}_{\omega_3}$ has $\omega_2$-cc there is an $\alpha < \omega_3$ such that $\tau$ is in fact a $\mathbb{P}_\alpha$-name. Then $H_\alpha$, the restriction of $H$ to $\mathbb{P}_\alpha$, is $\mathbb{P}_\alpha$ generic over $N$. Since $\alpha < \omega_3$, the iteration is with countable supports, we are considering only partial orders of size $< \omega_3$ (i.e. $\mathbf{1} \Vdash_{\mathbb{P}_\alpha}$ " $\mid \mathbb{Q}_\alpha \mid < \check{\omega}_3 \mid$ "), $GCH$ holds in $M[G]$, hence the density of $\mathbb{P}_\alpha$ is $< \omega_3$, we may assume that $\mid \mathbb{P}_\alpha \mid < \omega_3$. Now, in $M$, $\mathbb{L}_\kappa$ has the $\kappa$-cc (by Lemma 3.3), so there is some $\lambda < \kappa$ such that if $G_\lambda$ is the restriction of $G$ to $\mathbb{L}_\lambda$ then $\mathbb{P}_\alpha \in M[G_\lambda]$ and $H_\alpha$ is $\mathbb{P}_\alpha$-generic over $M[G_\lambda]$. Now $\mathbb{T} = \tau[G] \in M[G_\lambda][H_\alpha]$ and, by Lemma 3.5, any $\omega_2$-branch of $\mathbb{T}$ which is in $M[G_\lambda][H_\alpha][G^\lambda]$ is already in

$M[G_\lambda][H_\alpha]$. So, in $M[G_\lambda][H_\alpha]$, $\mathbb{T}$ has at most $2^{\omega_2} = \theta$ such branches and since $\kappa$ is still strongly inaccessible we have that $\theta < \kappa$. But, in $M[G_\lambda][H_\alpha][G^\lambda]$, $\kappa$ is collapsed to $\omega_3$. So $\mathbb{T}$ can have at most $\aleph_2$ many $\omega_2$-branches in $M[G_\lambda][H_\alpha][G^\lambda]$. But $M[G_\lambda][H_\alpha][G^\lambda] = M[G_\lambda][G^\lambda][H_\alpha] = M[G][H_\alpha] = N[H_\alpha]$. So $\mathbb{T}$ has at most $\aleph_2$ many $\omega_2$-branches in $N[H_\alpha]$. However, by Lemma 3.1, $\mathbb{P}^\alpha$ is $\omega_2$-normal so that, by Lemma 3.6, $\mathbb{T}$ does not obtain any new $\omega_2$-branches in the extension $N[H_\alpha][H^\alpha]$. But $N[H_\alpha][H^\alpha] = N[H]$. So $\mathbb{T}$ can not be an $\omega_2$-Kurepa tree in $N[H]$ which proves that in the model $N[H]$ there are no $\omega_2$-Kurepa trees. This finishes the proof of the Theorem. $\square$

Therefore, the existence of an $\omega_2$-Kurepa tree is consistent with and independent of $(SA)$ and hence consistent with and independent of $(BA)$.

## REFERENCES


[1] J. Baumgartner, *Iterated Forcing, in: A.R.D. Mathias, ed., Surveys in Set Theory,* London Mathematical Society Lecture Note Series **87**, 1983, 1–60.

[2] A. Blaszczyk and A. Szymanski, *Hausdorff's gaps versus normality*, Bull. Acad. Polon. Sci. **30** (1982), 371–378.

[3] K. Devlin, $\aleph_1$-*Trees*, Ann. Math. Logic **13** (1978), 267–330.

[4] F. Hausdorff, Die Graduierung nach dem Endverlauf, Abh. König. Sächs. Ges. Wiss., Math.-Physis. Kl. **31** (1909), 296–334.

[5] C. Herink, *Some Applications of Iterated Forcing*, Ph.D. Thesis, University of Wisconsin-Madison, 1977.

[6] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, North-Holland, Amsterdam, 1980.

[7] R. Laver and S. Shelah, *The* $\aleph_2$-*Suslin hypothesis*, Trans. Amer. Math. Soc. **264** (1981), 411–418.

[8] D. Martin and R. Solovay, *Internal Cohen Extensions*, Ann. Math. Logic **2** (1970), 143–178.

[9] S. Shelah, *A weak generalization of MA to higher cardinals*, Israel J. Math. **36** (1978), 297–304.

[10] S. Shelah and L. Stanley, *Generalized Martin's axiom and Suslin's hypothesis for higher cardinals*, Israel J. Math. **43** (1982), 225–236.

[11] Z. Spasojević, *Some results on gaps*, Topology Appl. **56** (1994), 129–139.

[12] Z. Spasojević, *Gaps in* $(\mathcal{P}(\omega), \subset^*)$ *and* $(\omega^\omega, \leq^*)$, Proc. American Math. Soc., to appear.

[13] F. Tall, *Some applications of generalized Martin's axiom*, Topology Appl. **57** (1994), 215–248.

[14] S. Todorčević, *Trees, Subtrees and Order Types*, Ann. Math. Logic **20** (1980), 233–268.


Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel

(Received 01 07 1995)
(Revised 26 08 1995)
(Revised 28 11 1995)