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DECOMPOSITIONS OF SEMIGROUPS WITH ZERO

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Abstract. We give a general theory of decompositions of semigroups with zero into an orthogonal, right, left and matrix sum of semigroups. The lattices of such decompositions are characterized by some sublattices of the lattice of equivalence relations on a semigroup with zero, and also by some lattices obtained from the lattices of (left, right) ideals of a semigroup with zero. Using the obtained results we decompose the lattice of (left, right) ideals of a semigroup with zero into a direct product of directly indecomposable lattices.

Introduction and preliminaries

It is well known that many of the "classical" methods of decompositions of semigroups degenerate when a semigroup has zero. For example, any semigroup with zero is matrix indecomposable. This requires some new methods of decompositions specific to semigroups with zero. Some of such methods we meet in a number of papers. The main tools of Dieudonné [**9**], in the theory of rings, and Schwarz [**19**], in the theory of semigroups, were the socle and 0-minimal ideals (two-sided and one-sided). Lallement and Petrich [**12**], studied decompositions of semigroups with zero by congruences whose corresponding factors are 0-rectangular bands. Some types of decompositions of semigroups with zero, similar to matrix decompositions of semigroups without zero, one can find in the book of Steinfeld [**17**].

Decompositions into an orthogonal sum (called also 0-direct union) of semigroups were first defined and studied by Lyapin [**13, 14**], 1950, and Schwarz [**19**], 1951. After that, orthogonal decompositions have been studied by a number of authors, mainly as othogonal sums of (completely) 0-simple semigroups (see [**3, 8, 17**]). A general theory of orthogonal decompositions was developed by Bogdanović and Ćirić [**4**]. They proved that every semigroup with zero has a greatest orthogonal decomposition and that summands in this decomposition are orthogonally indecomposable. Also, they proved that 0-consistent ideals of a semigroup with zero form

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a complete atomic Boolean algebra, whose atoms are summands in the greatest orthogonal decomposition of a semigroup with zero, and that any complete atomic Boolean algebra can be represented by 0-consistent ideals of some semigroup with zero.

In this paper we give a general theory of decompositions of semigroups with zero into an orthogonal, right, left and matrix sum of semigroups. The lattices of such decompositions will be characterized by some sublattices of the lattice of equivalence relations on a semigroup with zero, and also by some lattices obtained from the lattices of (left, right) ideals of a semigroup with zero. Also, using the obtained results, we decompose the lattice of (left, right) ideals of a semigroup with zero into a direct product of directly indecomposable lattices.

A lattice L is *complete for joins* (*complete for meets*) if every nonempty subset of L has a join (meet) and it is *complete* if it is complete both for joins and for meets. A sublattice (Boolean subalgebra) K of a complete lattice (Boolean algebra) L is a *closed sublattice* (*Boolean subalgebra*) of L if K contains the meet and the join of any its nonempty subset. Note that all the closed sublattices (Boolean subalgebras) of a complete lattice (Boolean algebra) form a complete lattice. A lattice L, complete for joins, is *infinitely distributive for meets* if $a \wedge (\bigvee_{\alpha \in Y} x_\alpha) = \bigvee_{\alpha \in Y}(a \wedge x_\alpha)$, for every $a \in S$ and every nonempty subset $\{x_\alpha \mid \alpha \in Y\}$ of L. A nontrivial lattice L is *directly indecomposable* if it has the property: If L is a direct product of lattices L_α, $\alpha \in Y$, then there exists $\alpha \in Y$ such that L_α is isomorphic to L and $|L_\beta| = 1$, for every $\beta \in Y$, $\beta \neq \alpha$. If a lattice L is a direct product of lattices L_α, $\alpha \in Y$, then we will write $L = \prod_{\alpha \in Y} L_\alpha$, and for $\alpha \in Y$, π_α will denote the projection homomorphism of L onto L_α.

An element a of a lattice L with the zero 0 is an *atom* of L, if $a > 0$ and there exists no $x \in L$ such that $a > x > 0$. A complete Boolean algebra B is *atomic* if every element of B is the join of some set of atoms of B. If L is a distributive lattice with zero and unity, then the set $\mathfrak{B}(L)$ of all elements of L having a complement in L is a Boolean algebra and it will be called the *greatest Boolean subalgebra* of L.

Throughout this paper, $\mathbf{Z}^+$ will denote the set of all positive integers, $J(a)$ ($L(a)$, $R(a)$) will denote the principal ideal (left ideal, right ideal) of a semigroup S generated by an element $a \in S$, and $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ will denote the well known Green's relations of S. Further, $S = S^0$ means that S is a semigroup with the zero 0. If $S = S^0$, we will write 0 instead $\{0\}$ and if A is a subset of S, then $A^\bullet = A - 0$, $A^0 = A \cup 0$, $A' = (S - A)^0$.

For a semigroup S, $\mathcal{I}d(S)$ will denote the lattice of all ideals of S. If $S = S^0$, then $\mathcal{I}d(S)$ is a complete lattice, infinitely distributive for meets, with the zero 0 and the unity S. Also, $\mathcal{LI}d(S)$ will denote the lattice of left ideals of a semigroup S defined in the following way: if $S = S^0$, then $\mathcal{LI}d(S)$ consists of all the left ideals of S, and if S is without zero, then $\mathcal{LI}d(S)$ consists of the empty set and all the left ideals of S. In both of these cases $\mathcal{LI}d(S)$ is a complete lattice, infinitely distributive for meets. Clearly, for a semigroup S without zero, the lattice $\mathcal{LI}d(S)$ is isomorphic to $\mathcal{LI}d(S^0)$, where S^0 denotes S with the zero adjoined. Dually we define the lattice of right ideals of S, in notation $\mathcal{RI}d(S)$. In any closed sublattice

K of $\mathcal{I}d(S)$ ($\mathcal{LI}d(S)$, $\mathcal{RI}d(S)$) which contains the unity, the intersection of all the elements of K containing an element a of S is also in K, and it will be called the *principal element* of K generated by a.

A subset A of a semigroup S is *right* (*left*) *consistent* if for $x, y \in S$, $xy \in A$ implies $y \in A$ ($xy \in A$ implies $x \in A$), and A is *consistent* if it is both left and right consistent. If $S = S^0$, then a subset A of S is (*right, left*) *0-consistent* subset of S if $A^\bullet$ is (right, left) consistent subset of S. For a semigroup $S = S^0$, $\mathcal{I}d^{\mathbf{c}}(S)$ ($\mathcal{LI}d^{\mathbf{rc}}(S), \mathcal{RI}d^{\mathbf{lc}}(S)$) will denote the set of all 0-consistent ideals (right 0-consistent left ideals, left 0-consistent right ideals) of S.

For a binary relation ξ on a set X and $n \in \mathbf{Z}^+$, ξ^n will denote the n-th power of ξ in the semigroup of binary relations on X, and ξ^∞ will denote the transitive closure of ξ. For an equivalence relation ξ on a set X, $x\xi$ will denote the equivalence class of X containing an element x. For a set X, $\mathcal{E}(X)$ will denote the *lattice of equivalences* (equivalence relations) on X. It is well known that $\mathcal{E}(X)$ is a complete lattice (see Theorem 67 in [**18**]). An equivalence relation ξ on a semigroup $S = S^0$ is *0-restricted* if 0 is a ξ-class. The set $\mathcal{E}^\bullet(S)$ of all 0-restricted equivalences on a semigroup $S = S^0$ is a lattice isomorphic to the lattice $\mathcal{E}(S^\bullet)$, and it is a principal ideal of $\mathcal{E}(S)$ generated by the equivalence χ determine by the partition $\{0, S^\bullet\}$.

For undefined notions and notations we refer to [**1**], [**2**], [**8**], [**16**], [**17**] and [**18**].

2. Lattices of 0-restricted (right, left) 0-consistent equivalences

In analogy with subsets of a semigroup with zero, an equivalence relation ξ on a semigroup $S = S^0$ will be called *right* (*left*) *0-consistent* if for $x, y \in S$, $xy \neq 0$ implies $xy\,\xi\,y$ ($xy \neq 0$ implies $xy\,\xi\,x$), and it will be called *0-consistent* if it is both right 0-consistent and left 0-consistent. For example, for the relations $\overset{\ell}{\sim}$ and $\overset{r}{\sim}$ defined in [**12**] (see also [**10, 11**]) and the relation $\sim$ defined in [**4**] by:

$$
\begin{array}{llll}
x \overset{\ell}{\sim} y & \iff & L(x) \cap L(y) \neq 0, \text{ for } x, y \in S^\bullet, & 0 \overset{\ell}{\sim} 0, \\
x \overset{r}{\sim} y & \iff & R(x) \cap R(y) \neq 0, \text{ for } x, y \in S^\bullet, & 0 \overset{r}{\sim} 0, \\
x \sim y & \iff & J(x) \cap J(y) \neq 0, \text{ for } x, y \in S^\bullet, & 0 \sim 0,
\end{array}
$$

their transitive closures $\kappa = \overset{\ell}{\sim}{}^\infty$, $\upsilon = \overset{r}{\sim}{}^\infty$ and $\delta = \sim^\infty$ are right 0-consistent, left 0-consistent and 0-consistent equivalence, respectively (for an equivalent definition of δ see [**4**]). By the following theorem we characterize all equivalence relations with these properties.

THEOREM 1. *The following conditions for an equivalence ξ of a semigroup $S = S^0$ are equivalent:*

(i) *ξ is* (*right, left*) *0-consistent;*
(ii) $\left(\overset{\ell}{\sim} \subseteq \xi,\ \overset{r}{\sim} \subseteq \xi \right)\ \sim \subseteq \xi;$
(iii) *$(a\xi)^0$ is a* (*right, left*) *0-consistent subset of S, for any $a \in S^\bullet$;*
(iv) *$(a\xi)^0$ is a* (*left, right*) *ideal of S, for any $a \in S^\bullet$.*

Proof. It is sufficient to prove the part of the theorem characterizing right 0-consistent equivalences.

(i) $\Longrightarrow$ (ii). Let ξ be right 0-consistent. If $a, b \in S$ and $a \overset{\ell}{\sim} b$, then $xa = yb \in S^\bullet$, for some $x, y \in S$, whence $a \,\xi\, xa = yb \,\xi\, b$. Thus, $\overset{r}{\sim} \subseteq \xi$.

(ii) $\Longrightarrow$ (i). This follows by the fact that $\overset{\ell}{\sim}$ is right 0-consistent.

(i) $\Longrightarrow$ (iii). Assume $a \in S^\bullet$ and $x, y \in S$ such that $xy \in a\xi$, $xy \neq 0$. Then $xy \,\xi\, y$, whence $y \,\xi\, a$, i.e. $y \in a\xi$. Thus, $(a\xi)^0$ is right 0-consistent.

(iii) $\Longrightarrow$ (iv). Assume $a \in S^\bullet$, $x \in S$, $y \in (a\xi)^0$. If $xy = 0$, then clearly $xy \in (a\xi)^0$. Assume that $xy \neq 0$. Then $y \neq 0$ and $y \,\xi\, a$, and $xy \,\xi\, y$, so $xy \,\xi\, a$, i.e. $xy \in a\xi$. Thus, $(a\xi)^0$ is a left ideal of S.

(iv) $\Longrightarrow$ (i). Assume $x, y \in S$ such that $xy \neq 0$. Then $y \neq 0$ and $(y\xi)^0$ is a left ideal of S, whence $xy \in (y\xi)^0$, i.e. $xy \,\xi\, y$. $\square$

For a semigroup $S = S^0$, $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ and $\mathcal{E}^{\mathbf{lc}}(S)$ will denote the sets of 0-consistent, right 0-consistent and left 0-consistent equivalences on S, respectively. A relation between these sets in $\mathcal{E}(S)$ is given by the following theorem.

Theorem 2. *For a semigroup $S = S^0$, $\mathcal{E}^{\mathbf{c}}(S)$, $\mathcal{E}^{\mathbf{rc}}(S)$ and $\mathcal{E}^{\mathbf{lc}}(S)$ are principal dual ideals of $\mathcal{E}(S)$ generated by δ, κ and υ, respectively, and $\kappa \vee \upsilon = \delta$.*

Proof. Clearly, δ is a 0-consistent equivalence on S and by Theorem 1, $\mathcal{E}^{\mathbf{c}}(S) = \{\xi \in \mathcal{E}(S) \mid \ \sim \subseteq \xi\} = \{\xi \in \mathcal{E}(S) \mid \delta \subseteq \xi\}$, so $\mathcal{E}^{\mathbf{c}}(S)$ is the principal dual ideal of $\mathcal{E}(S)$ generated by δ. Analogously we prove the assertions concerning $\mathcal{E}^{\mathbf{rc}}(S)$ and $\mathcal{E}^{\mathbf{lc}}(S)$.

Further, by Theorem 1, any equivalence relation on S containing a right and a left 0-consistent equivalence is both right and left 0-consistent, i.e. it is 0-consistent. Thus, $\kappa \vee \upsilon$ is 0-consistent, so $\delta \subseteq \kappa \vee \upsilon$. On the other hand, by $\kappa, \upsilon \subseteq \delta$ it follows $\kappa \vee \upsilon \subseteq \delta$. Hence, $\kappa \vee \upsilon = \delta$. $\square$

Theorem 3. *For any semigroup $S = S^0$, $\mathcal{I}d^{\mathbf{c}}(S)$, $\mathcal{LI}d^{\mathbf{rc}}(S)$ and $\mathcal{RI}d^{\mathbf{lc}}(S)$ are complete atomic Boolean algebras and $\mathcal{I}d^{\mathbf{c}}(S) = \mathfrak{B}(\mathcal{I}d(S))$, $\mathcal{LI}d^{\mathbf{rc}}(S) = \mathfrak{B}(\mathcal{LI}d(S))$ and $\mathcal{RI}d^{\mathbf{lc}}(S) = \mathfrak{B}(\mathcal{RI}d(S))$.*

Proof. The assertions for $\mathcal{I}d^{\mathbf{c}}(S)$ were proved in [**4**]. Analogously one can prove the remaining cases. $\square$

For a semigroup $S = S^0$, let $\mathcal{E}^{\bullet\mathbf{c}}(S)$ ($\mathcal{E}^{\bullet\mathbf{rc}}(S)$, $\mathcal{E}^{\bullet\mathbf{lc}}(S)$) denote the set of all 0-restricted (right, left) 0-consistent equivalences on S.

Theorem 4. *For any semigroup $S = S^0$, $\mathcal{E}^{\bullet\mathbf{c}}(S)$ ($\mathcal{E}^{\bullet\mathbf{rc}}(S)$, $\mathcal{E}^{\bullet\mathbf{lc}}(S)$) is a closed sublattice of $\mathcal{E}(S)$ and it is dually isomorphic to the lattice of closed Boolean subalgebras of $\mathcal{I}d^{\mathbf{c}}(S)$ ($\mathcal{LI}d^{\mathbf{rc}}(S)$, $\mathcal{RI}d^{\mathbf{lc}}(S)$).*

Proof. We will prove the assertion concerning $\mathcal{E}^{\bullet\mathbf{c}}(S)$. Analogously we prove the remaining assertions. By Theorem 2 and Lemma 1, $\mathcal{E}^{\bullet\mathbf{c}}(S)$ is equal to the (closed) interval $[\delta,\chi]$ of $\mathcal{E}(S)$, so it is a complete sublattice of $\mathcal{E}(S)$. For $\xi \in \mathcal{E}^{\bullet\mathbf{c}}(S)$, let B_ξ be a subset of $\mathcal{I}d^{\mathbf{c}}(S)$ defined by:

$$B_\xi = \Big\{ A \in \mathcal{I}d^{\mathbf{c}}(S) \;\Big|\; A = \bigcup_{a\in A} (a\xi)^0 \Big\}. \tag{1}$$

Clearly, $0, S \in B_\xi$. Let $A \in B_\xi$ and $x \in A \cap \big(\bigcup_{a\in A'}(a\xi)^0\big)$. Then $x \in a\xi$, for some $a \in A'$, $a \neq 0$, whence $a \in x\xi \subseteq A$, which is in contradiction with $a \in A'$, $a \neq 0$. Thus, $\bigcup_{a\in A'}(a\xi)^0 \subseteq A'$, whence $A' \in B_\xi$.

Assume $\{A_\alpha \mid \alpha \in Y\} \subseteq B_\xi$. Clearly, $\bigcup_{\alpha\in Y} A_\alpha \in B_\xi$. Let $A = \bigcap_{\alpha\in Y} A_\alpha$ and $x \in \bigcup_{a\in A}(a\xi)^0$. If $x \neq 0$, then clearly $x \in A$. Assume that $x \neq 0$. Then $x \in a\xi$, for some $a \in A^\bullet$, and since $a \in A_\alpha$, for each $\alpha \in Y$, then $x \in \bigcup_{a\in A_\alpha}(a\xi)^0 = A_\alpha$, for each $\alpha \in Y$, whence $x \in A$, Thus, $\bigcup_{a\in A}(a\xi)^0 \subseteq A$, whence $A = \bigcap_{\alpha\in Y} A_\alpha \in B_\xi$. Therefore, B_ξ is a closed Boolean subalgebra of $\mathcal{I}d^{\mathbf{c}}(S)$, i.e. the mapping $\xi \mapsto B_\xi$ maps $\mathcal{E}^{\bullet\mathbf{c}}(S)$ into the set of closed Boolean subalgebras of $\mathcal{I}d^{\mathbf{c}}(S)$.

Let B be a closed Boolean subalgebra of $\mathcal{I}d^{\mathbf{c}}(S)$. By Theorem 56 of [**18**], a complete Boolean algebra is atomic if and only if it is completely distributive, whence any closed Boolean subalgebra of a complete atomic Boolean algebra is also atomic. Thus, B is atomic and S is the union of all its atoms. Thus, for any $a \in S^\bullet$ there exists a unique atom of B containing it, which will be denoted by $B(a)$. Now, define a relation ξ on S by:

$$a\,\xi\,b \iff B(a) = B(b), \quad \text{for } a, b \in S^\bullet, \qquad 0\,\xi\,0. \tag{2}$$

Clearly, ξ is a 0-restricted equivalence. It can be checked easily that $(a\xi)^0 = B(a)$, for any $a \in S^\bullet$, so by Theorem 1, ξ is 0-consistent, and for $A \in \mathcal{I}d^{\mathbf{c}}(S)$

$$A \in B \iff A = \bigcup_{a\in A} B(a) \iff A = \bigcup_{a\in A} (a\xi)^0 \iff A \in B_\xi,$$

so $B = B_\xi$. Hence, the mapping $\xi \mapsto B_\xi$ is onto.

Further, assume $\xi, \eta \in \mathcal{E}^{\bullet\mathbf{c}}(S)$. If $\xi \subseteq \eta$, then $(a\xi)^0 \subseteq (a\eta)^0$, for each $a \in S^\bullet$, whence $A \in B_\eta$ implies $\bigcup_{a\in A}(a\xi)^0 \subseteq \bigcup_{a\in A}(a\eta)^0 = A$, i.e. $A \in B_\xi$, so $B_\eta \subseteq B_\xi$. Conversely, if $B_\eta \subseteq B_\xi$, then $a\eta \in B_\eta \subseteq B_\xi$ and $a\xi \subseteq \bigcup_{x\in a\eta}(x\xi)^0 = a\eta$, for any $a \in S^\bullet$, whence $\xi \subseteq \eta$. Thus, $\xi \subseteq \eta$ if and only if $B_\eta \subseteq B_\xi$. Hence, the mapping $\xi \mapsto B_\xi$ is a dual order isomorphism, so it is a dual lattice isomorphism. $\square$

An equivalence relation ξ on a semigroup $S = S^0$ will be called *quasi-0-consistent* if it is an intersection of a right 0-consistent and a left 0-consistent equivalence on S. An example of such an equivalence is the equivalence $\mu = \kappa \cap \upsilon$. For a semigroup $S = S^0$, $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ will denote the set of all 0-restricted quasi-0-consistent equivalences on S.

Theorem 5. *For any semigroup $S = S^0$, $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ is a complete lattice.*

Proof. Consider $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ as a subset of the complete lattice $\mathcal{E}^\bullet(S)$. Clearly, $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ contains the unity of $\mathcal{E}^\bullet(S)$. Moreover, it is easy to check that the intersection of any family of elements of $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ is also in $\mathcal{E}^{\bullet\mathbf{qc}}(S)$. By these facts it follows that $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ is a complete lattice. $\square$

Note that the zero and the unity of $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ are μ and χ, respectively.

For a quasi-0-consistent equivalence ξ on a semogroup $S = S^0$, by Theorem 1 we have that $(a\xi)^0$ is an intersection of a right 0-consistent left ieal and a left 0-consistent right ideal of S, for each $a \in S$, and hence, it is a quasi-ideal. By this fact we derive the following

Problem 1. Find necessary and sufficient conditions for a quasi-ideal Q of a semigroup $S = S^0$ to be an intersection of a right 0-consistent left ideal and a left 0-consistent right ideal of S and give an answer to the question: Can $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ be embedded into the lattice of quasi-ideals of S?

3. Orthogonal, right, left and matrix sums of semigroups

Recall that a semigroup $S = S^0$ is an *orthogonal sum* of semigroups S_α, $\alpha \in Y$, in notation $S = \Sigma_{\alpha\in Y} S_\alpha$, if $S_\alpha \neq 0$, for all $\alpha \in Y$, $S = \bigcup_{\alpha\in Y} S_\alpha$ and $S_\alpha \cap S_\beta = S_\alpha S_\beta = 0$, for all $\alpha, \beta \in Y$, $\alpha \neq \beta$. In this case, the family $\mathcal{D} = \{S_\alpha \mid \alpha \in Y\}$ is a *decomposition into an orthogonal sum* or an *orthogonal decomposition* of S and S_α are *orthogonal summands* of S or *summands* in $\mathcal{D}$. On a set of all orthogonal decompositions of $S = S^0$ we define a partial order $\leq$ by: $\mathcal{D} \leq \mathcal{D}'$ if each member of $\mathcal{D}'$ is a subset of some member of $\mathcal{D}$. A semigroup $S = S^0$ is orthogonally indecomposable if $\mathcal{D} = \{S\}$ is the unique orthogonal decomposition of S [**4**].

A semigroup $S = S^0$ is a *right sum* of semigroups S_α, $\alpha \in Y$, in notation $S = R\Sigma_{\alpha\in Y} S_\alpha$, if $S_\alpha \neq 0$, for all $\alpha \in Y$, $S = \bigcup_{\alpha\in Y} S_\alpha$ and $S_\alpha \cap S_\beta = 0$ and $S_\alpha S_\beta \subseteq S_\beta$, for all $\alpha, \beta \in Y$, $\alpha \neq \beta$. Dually we define a *left sum* of semigroups S_α, $\alpha \in Y$, in notation $L\Sigma_{\alpha\in Y} S_\alpha$.

A semigroup $S = S^0$ will be called a *matrix sum* of semigroups S_u, $u \in M$, in notation $S = M\Sigma_{u\in M} S_u$, if $\varnothing \neq M \subseteq I \times \Lambda$, where I and Λ are nonempty sets, $S = \bigcup_{u\in M} S_u$, $S_u \cap S_v \neq 0$, for $u \neq v$, $u, v \in M$, and for any $(i,\lambda), (j,\nu) \in M$ the following condition holds

$$S_{(i,\lambda)} S_{(j,\nu)} \begin{cases} \subseteq S_{(i,\nu)} & \text{if } (i,\nu) \in M \\ = 0 & \text{otherwise.} \end{cases}$$

If $M = I \times \Lambda$, then we will say that S is a *complete matrix sum* of semigroups $S_{(i,\lambda)}$, $i \in I$, $\lambda \in \Lambda$.

Previously defined notions concerning orthogonal sums we naturally translate to the related notions concerning right, left and matrix sums of semigroups.

Theorem 6. *Decompositions of a semigroup $S = S^0$ into an orthogonal (right, left, matrix) sum of semigroups form a complete lattice which is dually isomorphic to the lattice $\mathcal{E}^{\bullet\mathbf{c}}(S)$ ($\mathcal{E}^{\bullet\mathbf{rc}}(S)$, $\mathcal{E}^{\bullet\mathbf{lc}}(S)$, $\mathcal{E}^{\bullet\mathbf{qc}}(S)$).*

Proof. We will prove the assertions concerning orthogonal and matrix sums. The assertions concerning right and left sums can be proved similarly as the assertion for orthogonal sums.

Let $\xi \in \mathcal{E}^{\bullet\mathbf{c}}(S)$ and let

$$\mathcal{D}_\xi = \{(a\xi)^0 \mid a \in S^\bullet\}. \tag{3}$$

By Theorem 1, the members of $\mathcal{D}_\xi$ are nonzero 0-consistent ideals of S whose union is S, so $\mathcal{D}_\xi$ is an orthogonal decomposition of S. Consider a mapping $\xi \mapsto \mathcal{D}_\xi$, of $\mathcal{E}^{\bullet\mathbf{c}}(S)$ into the poset of orthogonal decompositions of S. If $\mathcal{D} = \{S_\alpha \mid \alpha \in Y\}$ is an orthogonal decomposition of S, then the relation ξ on S defined by:

$$a\,\xi\,b \iff (\exists \alpha \in Y)\ a, b \in S_\alpha^\bullet, \quad \text{for } a, b \in S^\bullet, \qquad 0\,\xi\,0, \tag{4}$$

is a 0-restricted 0-consistent equivalence on S and $\mathcal{D} = \mathcal{D}_\xi$. Thus, the mapping $\xi \mapsto \mathcal{D}_\xi$ is onto.

Assume $\xi, \eta \in \mathcal{E}^{\bullet\mathbf{c}}(S)$. If $\xi \subseteq \eta$, then for each $A \in \mathcal{D}_\xi$, $A = (a\xi)^0$, for some $a \in S^\bullet$, whence $A = (a\xi)^0 \subseteq (a\eta)^0 \in \mathcal{D}_\eta$, so $\mathcal{D}_\eta \leq \mathcal{D}_\xi$. Conversely, let $\mathcal{D}_\eta \leq \mathcal{D}_\xi$. Assume $a, b \in S^\bullet$ such that $a\,\xi\,b$. By the hypothesis, $(a\xi)^0 = (b\xi)^0 = (c\eta)^0$, for some $c \in S^\bullet$, whence $a, b \in c\eta$, so $a\,\eta\,b$. Thus, $\mathcal{D}_\eta \leq \mathcal{D}_\xi$ implies $\xi \subseteq \eta$. Hence, the mapping $\xi \mapsto \mathcal{D}_\xi$ is a dual order isomorphism.

Further, let $\theta \in \mathcal{E}^{\bullet\mathbf{qc}}(S)$, i.e. $\theta = \xi \cap \eta$, where $\xi \in \mathcal{E}^{\bullet\mathbf{rc}}(S)$ and $\eta \in \mathcal{E}^{\bullet\mathbf{lc}}(S)$, and let $\mathcal{D}_\xi = \{L_\lambda \mid \lambda \in \Lambda\}$ and $\mathcal{D}_\eta = \{R_i \mid i \in I\}$ be decompositions into a right sum and a left sum of semigroups, determined by ξ and η as in (3), i.e. for any $\lambda \in \Lambda$, $i \in I$, $L_\lambda = (a\xi)^0$, $R_i = (b\eta)^0$, for some $a, b \in S^\bullet$. Let $M = \{(i, \lambda) \in I \times \Lambda \mid R_i \cap L_\lambda \neq 0\}$, and for $(i, \lambda) \in M$, let $S_{(i,\lambda)} = R_i \cap L_\lambda$ and $\mathcal{D}_\theta = \{S_u \mid u \in M\}$. It is easy to check that $\mathcal{D}_\theta$ is a decomposition of S into a matrix sum. Also, for any $u \in M$, $S_u = (a\theta)^0$, for some $a \in S^\bullet$.

Consider a mapping $\theta \mapsto \mathcal{D}_\theta$ of $\mathcal{E}^{\bullet\mathbf{qc}}(S)$ into the poset of decompositions of S into a matrix sum of semigroups. Let $\mathcal{D} = \{S_u \mid u \in M\}$ be a decomposition of S into a matrix sum of semigroups S_u, $u \in M$, where $M \subseteq I \times \Lambda$ and I and Λ are nonempty sets. Define relations ξ and η on S by:

$$\begin{aligned} a\,\xi\,b &\iff (\exists \lambda \in \Lambda)(\exists i, j \in I)\ a \in S_{(i,\lambda)},\ b \in S_{(j,\lambda)},\ \text{for } a, b \in S^\bullet, \quad 0\,\xi\,0, \\ a\,\eta\,b &\iff (\exists i \in I)(\exists \lambda, \nu \in \Lambda)\ a \in S_{(i,\lambda)},\ b \in S_{(i,\nu)},\ \text{for } a, b \in S^\bullet, \quad 0\,\eta\,0. \end{aligned}$$

Then $\xi \in \mathcal{E}^{\bullet\mathbf{rc}}(S)$, $\eta \in \mathcal{E}^{\bullet\mathbf{qc}}(S)$, $\theta = \xi \cap \eta$ and $\mathcal{D} = \mathcal{D}_\theta$. Therefore, the mapping $\theta \mapsto \mathcal{D}_\theta$ is onto. The rest of the proof is similar to the related part of the proof for the case of orthogonal decompositions. $\square$

By Theorems 4 and 6 we obtain

THEOREM 7. *The lattice of decompositions of a semigroup $S = S^0$ into an orthogonal (right, left) sum of semigroups is isomorphic to the lattice of closed Boolean subalgebras of $\mathcal{I}d^{\mathbf{c}}(S)$ ($\mathcal{LI}d^{\mathbf{rc}}(S)$, $\mathcal{RI}d^{\mathbf{lc}}(S)$).*

Example 1. Let $\mathcal{M}^\bullet$ denote the set of all matrices over a ring R and let $\mathcal{M} = \mathcal{M}^\bullet \cup \{0\}$, where $0 \notin \mathcal{M}^\bullet$. Define a multiplication $\circ$ on $\mathcal{M}$ by:

$$A \circ B = \begin{cases} AB & \text{if } A, B \in \mathcal{M}^\bullet \text{ and } AB \text{ is defined,} \\ 0 & \text{otherwise,} \end{cases}$$

where on the right-hand side is the usual multiplication of matrices A and B. With this multiplication, $\mathcal{M}$ is a semigroup with the zero 0 and it will be called the *semigroup of matrices* over R.

For $m \in \mathbf{Z}^+$, let $\mathcal{M}_{*\times m}$ ($\mathcal{M}_{m\times *}$) denote the set of all the matrices over R of the type $k \times m$ ($m \times k$), $k \in \mathbf{Z}^+$, with 0 adjoined, and for $m, n \in \mathbf{Z}^+$, let $\mathcal{M}_{m\times n} = \mathcal{M}_{m\times *} \cap \mathcal{M}_{*\times n}$. It is easy to verify that $\{\mathcal{M}_{*\times m} \mid m \in \mathbf{Z}^+\}$ ($\{\mathcal{M}_{m\times *} \mid m \in \mathbf{Z}^+\}$, $\{\mathcal{M}_{m\times n} \mid m, n \in \mathbf{Z}^+\}$) is the greatest decomposition of $\mathcal{M}$ into a right (left, matrix) sum of semigroups, whose summands are indecomposable into a right (left, matrix) sum of semigroups. Moreover, $\mathcal{M}$ is a complete matrix sum of semigroups $\mathcal{M}_{m\times n}$, $m, n \in \mathbf{Z}^+$, and $\mathcal{M}$ is orthogonally indecomposable.

Let I be an ideal of $\mathcal{M}$ consisting of 0 and of zero matrices of an arbitrary type. The factor semigroup $\mathcal{M}/I$ will be called the *reduced semigroup of matrices* over R. We state the following

Problem 2. Can the previous assertions for the semigroup of matrices be proved for the reduced semigroup of matrices over R?

Example 2. Recall that summands in the greatest orthogonal decomposition of a semigroup with zero are orthogonally indecomposable [**4**]. But, summands in the greatest decomposition into a right (left, matrix) sum of semigroups need not be indecomposable into a right (left, matrix) sum of semigroups. For example, consider a semigroup given by the following table:

	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	d	d
b	0	a	b	b	d
c	0	a	b	b	d
d	0	a	b	b	d

The greatest decomposition of this semigroup into a right sum of semigroups is $\{\{0, a\}, \{0, b, c, d\}\}$, and $\{0, b, c, d\}$ is a right sum of $\{0, b, c\}$ and $\{0, d\}$.

Finishing this section, note that analogues of decompositions of a semigroup with zero into a right, left and complete matrix sum of semigroups are decompositions of a semigroup without zero into a right, left and rectangular band (matrix) of semigroups, respectively, investigated by Petrich in [**15**] (see also [**16**]). Orthogonal decompositions have not such an analogue, but we refer to [**2**, **3**, **5**, **6**, **7**] to make some interesting comparisons between the methodologies of orthogonal decompositions and semilattice decompositions of semigroups.

4. Some special decompositions

For an element a of a semigroup $S = S^0$ we define a sequence $\Delta_n(a)$, $n \in \mathbf{Z}^+$, of subsets of S, and a sequence δ_n, $n \in \mathbf{Z}^+$, of equivalence relations on S by: $\Delta_n(a) = \{x \in S \mid x \sim^n a\}^0$ and

$$a\,\delta_n\,b \iff \Delta_n(a) = \Delta_n(b) \qquad (a, b \in S).$$

Similarly, using $\overset{\ell}{\sim}$ instead of $\sim$, we define a sequence $\mathrm{K}_n(a)$, $n \in \mathbf{Z}^+$, of subsets of S, and a sequence κ_n, $n \in \mathbf{Z}^+$, of equivalences on S. Also, using $\overset{r}{\sim}$ instead of $\sim$, we define a sequence υ_n, $n \in \mathbf{Z}^+$, of equivalences on S; then a sequence μ_n, $n \in \mathbf{Z}^+$, of equivalences on S is defined by: $\mu_n = \kappa_n \cap \upsilon_n$. Clearly, any of these sequences is increasing. Bogdanović and Ćirić in [**4**] proved that for any $a \in S$, $\Delta(a) = \bigcup_{n\in\mathbf{Z}^+} \Delta_n(a)$ is the principal 0-consistent ideal of S generated by a, and for all $a, b \in S$, $a\,\delta\,b$ if and only if $\Delta(a) = \Delta(b)$. It can be proved similarly that $\mathrm{K}(a) = \bigcup_{n\in\mathbf{Z}^+} \mathrm{K}_n(a)$ is the principal right 0-consistent ideal of S generated by a, and that for all $a, b \in S$, $a\,\kappa\,b$ if and only if $\mathrm{K}(a) = \mathrm{K}(b)$.

By the following lemma, some relations between the equivalences introduced above are described:

Lemma 1. *If $S = S^0$, then*

$$\begin{array}{ccccccccccc}
\mathcal{H} & \subseteq & \mu_1 & \subseteq & \mu_2 & \subseteq\cdots\subseteq & \mu_n & \subseteq\cdots\subseteq & \mu \\
\cap\!\!\!\!| & & \cap\!\!\!\!| & & \cap\!\!\!\!| & & \cap\!\!\!\!| & & \cap\!\!\!\!| \\
\mathcal{L} & \subseteq & \kappa_1 & \subseteq & \kappa_2 & \subseteq\cdots\subseteq & \kappa_n & \subseteq\cdots\subseteq & \kappa \\
\cap\!\!\!\!| & & & & & & & & \cap\!\!\!\!| \\
\mathcal{J} & \subseteq & \delta_1 & \subseteq & \delta_2 & \subseteq\cdots\subseteq & \delta_n & \subseteq\cdots\subseteq & \delta \\
\cup\!\!\!\!| & & & & & & & & \cup\!\!\!\!| \\
\mathcal{R} & \subseteq & \upsilon_1 & \subseteq & \upsilon_2 & \subseteq\cdots\subseteq & \upsilon_n & \subseteq\cdots\subseteq & \upsilon
\end{array}.$$

Lemma 2. *Let $S = S^0$ and let $n \in \mathbf{Z}^+$. Then*

(1) *for each $a \in S$, $\Delta_n(a)$ is a 0-consistent subset of S;*
(2) *for all $x, y \in S$, $\Delta_n(xy) \subseteq \Delta_n(x) \cap \Delta_n(y)$.*

Proof. (1) Let $a \in S^\bullet$ and $xy \in (\Delta_1(a))^\bullet$. Since $J(xy) \subseteq J(x)$ and $J(xy) \cap J(a) \neq 0$, then $x \sim a$, i.e. $x \in \Delta_1(a)$, and similarly $y \in \Delta_1(a)$, so $\Delta_1(a)$ is 0-consistent. Using this, by induction we obtain (1).

(2) For $x, y \in S$, $xy \neq 0$ and $a \in \Delta_n(xy)$ implies $xy \in (\Delta_n(a))^\bullet$, whence $x, y \in \Delta_n(a)$, by (1), so $a \in \Delta_n(x) \cap \Delta_n(y)$. Thus, (2) holds. $\square$

Lemma 3. *Let $S = S^0$, $S = \Sigma_{\alpha\in Y} S_\alpha$ and let $n \in \mathbf{Z}^+$.*

(1) *Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$. If $x \sim^n y$ in S, then $\alpha = \beta$.*
(2) *Let $\alpha \in Y$, $x, y \in S_\alpha$. Then $x \sim^n y$ in S if and only if $x \sim^n y$ in S_α.*

Proof. (1) Since $J(x) \subseteq S_\alpha$ and $J(y) \subseteq S_\beta$, then $x \sim y$ implies $\alpha = \beta$, so by induction we obtain (1).

(2) Since the principal ideal of S generated by a is equal to the principal ideal of S_α generated by a, then $x \sim y$ in S if and only if $x \sim y$ in S_α, so by induction we obtain (2). $\square$

A semigroup $S = S^0$ is *0-δ_n-simple* if it has exactly two δ_n-classes, i.e. if $x \sim^n y$, for all $x, y \in S^\bullet$. By the following theorem we describe orthogonal sums of 0-δ_n-simple semigroups.

Theorem 8. *Let $n \in \mathbf{Z}^+$. Then the following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is on orthogonal sum of 0-δ_n-simple semigroups;*

(ii) $(\forall x,y\in S)\; xy\neq 0 \implies [(x \sim^n a \text{ or } y \sim^n a) \implies xy \sim^n a]$;
(iii) *for each* $a\in S$, $\Delta_n(a)$ *is an ideal of* S;
(iv) $\sim^n$ *is an equivalence relation on* S;
(v) δ_n *is a 0-consistent equivalence on* S.

Proof. (i) $\implies$ (ii). Let $S=\Sigma_{\alpha\in Y}S_\alpha$, where S_α, $\alpha\in Y$, are 0-δ_n-simple semigroups. Assume $x,y\in S$ such that $xy\neq 0$. Then $x,y\in S_\alpha$, for some $\alpha\in Y$. Let $x\sim^n a$, for some $a\in S$. By Lemma 3, $a\in S_\alpha$ and $a\neq 0$. Since S_α is 0-δ_n-simple, then by $a,xy\in S_\alpha^\bullet$ it follows that $xy\sim^n a$ in S_α, and also in S. Similarly we prove that $y\sim^n a$ implies $xy\sim^n a$.

(ii) $\implies$ (iii). Assume $a\in S$, $x\in\Delta_n(a)$, $y\in S$. If $xy=0$, then clearly $xy\in\Delta_n(a)$. If $xy\neq 0$, then by $x\sim^n a$, by (ii), we obtain $xy\sim^n a$, so $xy\in\Delta_n(a)$. Thus, $\Delta_n(a)$ is a right ideal. Similarly we prove that $\Delta_n(a)$ is a left ideal.

(iii) $\implies$ (iv). By Lemma 2, for each $a\in S$, $\Delta_n(a)$ is a 0-consistent ideal of S, whence $\Delta_n(a)=\Delta(a)$, for each $a\in S$, so $\sim^n=\sim^\infty=\delta$.

(iv) $\implies$ (v). Clearly, $\sim^n=\sim^\infty$, i.e. $\delta_n=\delta$, so (v) holds.

(v) $\implies$ (i). By Theorem 4, $S=\Sigma_{\alpha\in Y}S_\alpha$, where for each $\alpha\in Y$, $S_\alpha^\bullet$ is a δ_n-class of S. Assume $\alpha\in Y$, $x,y\in S_\alpha^\bullet$. Then $x\sim^n y$ in S, so by Lemma 3, $x\sim^n y$ in S_α. Hence, S_α is 0-δ_n-simple. □

Corollary 1. *Let $S=S^0$ be a finite semigroup. Then there exists $n\in\mathbf{Z}^+$, $n\leq |S|$, such that S is an orthogonal sum of 0-δ_n-simple semigroups.*

Similarly we prove the following

Theorem 9. *Let $n\in\mathbf{Z}^+$. Then the following conditions on a semigroup $S=S^0$ are equivalent:*

(i) $(\forall x,y\in S)\; xy\neq 0 \implies (y \overset{\ell}{\sim}{}^n a \implies xy \overset{\ell}{\sim}{}^n a)$;
(ii) *for each* $a\in S$, $\mathrm{K}_n(a)$ *is a left ideal of* S;
(iii) $\overset{\ell}{\sim}{}^n$ *is an equivalence relation on* S;
(iv) κ_n *is a right 0-consistent equivalence on* S.

Example 3. The semigroup $\mathcal{M}$ from Example 1 is 0-δ_1-simple. If we analogously define 0-κ_n-simple, 0-υ_n-simple and 0-μ_n-simple semigroups, then $\mathcal{M}_{*\times m}$ is 0-κ_1-simple, $\mathcal{M}_{m\times *}$ is 0-υ_1-simple and $\mathcal{M}_{m\times n}$ is 0-μ_1-simple.

Example 4. Any semigroup with zero and unity is 0-δ_2-simple, 0-κ_2-simple, 0-υ_2-simple and 0-μ_2-simple.

5. Lattices of ideals of semigroups with zero

Here we investigate direct decompositions of lattices of ideals and lattices of left ideals of a semigroup with zero. In the proof of the main theorems of this section we use a more general result concerning direct decompositions with directly indecomposable components of complete lattices, infinitely distributive for meets, or equivalently, of complete Brouwerian lattices. In the sources that are available to the authors, there is no information that this result is proved before, so it will be proved in Theorem 9. First we prove the following

LEMMA 4. *Let L be a complete lattice, infinitely distributive for meets. Then L is directly indecomposable if and only if $\mathfrak{B}(L) = \{0,1\}$.*

Proof. If $\{a_\alpha \mid \alpha \in Y\}$ is a subset of L whose join is 1 and $a_\alpha \wedge a_\beta = 0$ whenever $\alpha \neq \beta$, then the mapping $\phi : L \to \prod_{\alpha\in Y}[0, a_\alpha]$ defined by:

$$x\phi = (x \wedge a_\alpha)_{\alpha\in Y} \qquad (x \in L)$$

is an isomorphism of L onto $\prod_{\alpha\in Y}[0, a_\alpha]$. By this it follows that $\mathfrak{B}(L) = \{0,1\}$ whenever L is directly indecomposable.

Conversely, if $\mathfrak{B}(L) = \{0,1\}$, then it is easy to verify that L is directly indecomposable. □

Now we are ready to consider direct decompositions of complete Brouwerian lattices:

THEOREM 10. *The following conditions for a complete lattice L, infinitely distributive for meets, are equivalent:*

(i) *L can be decomposed into a direct product of directly indecomposable lattices;*
(ii) *$\mathfrak{B}(L)$ is a closed sublattice of L;*
(iii) *$\mathfrak{B}(L)$ is a complete atomic Boolean algebra.*

Proof. (i) $\Longrightarrow$ (ii). Let $L = \prod_{\alpha\in Y} L_\alpha$, where L_α, $\alpha \in Y$, are directly indecomposable lattices. For any $\alpha \in Y$, L_α is a homomorphic image of L, so L_α has a zero 0_α and a unity 1_α. Also, if $a_\alpha \in L$ is an element for which

$$a_\alpha\pi_\beta = \begin{cases} 1_\alpha & \text{for } \beta = \alpha, \\ 0_\beta & \text{for } \beta \neq \alpha, \end{cases} \qquad (\beta \in Y),$$

then L_α is isomorphic to the interval $[0, a_\alpha]$ of L, so L_α is complete and infinitely distributive for meets, and by Lemma 4, $\mathfrak{B}(L_\alpha) = \{0_\alpha, 1_\alpha\}$. Now it is easy to check that $\mathfrak{B}(L) = \prod_{\alpha\in Y}\mathfrak{B}(L_\alpha)$, whence $\mathfrak{B}(L)$ is a closed sublattice of L.

(ii) $\Longrightarrow$ (iii). This follows immediately by Theorem 56 of [**18**].

(iii) $\Longrightarrow$ (i). Let $A = \{a_\alpha \mid \alpha \in Y\}$ be the set of all atoms of $\mathfrak{B}(L)$. By the proof of Lemma 4, L is isomorphic to the lattice $\prod_{\alpha\in Y}[0, a_\alpha]$. Assume $\alpha \in Y$. If x is an element of $[0, a_\alpha]$ with the complement y in $[0, a_\alpha]$, then for $W = Y - \{\alpha\}$, $z = y \vee (\bigvee_{\beta\in W} a_\beta)$ is a complement of x in L. Thus, every element of $[0, a_\alpha]$ having a complement in $[0, a_\alpha]$, also has a complement in L. Since a_α is an atom of L, then $\mathfrak{B}([0, a_\alpha]) = \{0, a_\alpha\}$. Therefore, by Lemma 4, $[0, a_\alpha]$, $\alpha \in Y$, are directly indecomposable lattices. □

Now we begin a study of lattices of ideals of a semigroup with zero. First we give the following

THEOREM 11. *The lattice of ideals of a semigroup $S = S^0$ is directly indecomposable if and only if S is orthogonally indecomposable.*

Proof. This follows immediately by Lemma 4 of [**4**], Theorem 1 of [**4**] (or Theorems 3) and Lemma 4. $\square$

Further, it is not hard to prove the following

LEMMA 5. *Let A be a 0-consistent ideal of a semigroup $S = S^0$. Then $\mathcal{LI}d(A) \subseteq \mathcal{LI}d(S)$, $\mathcal{I}d(A) \subseteq \mathcal{I}d(S)$, $\mathcal{LI}d^{\mathbf{rc}}(A) \subseteq \mathcal{LI}d^{\mathbf{rc}}(S)$ and $\mathcal{I}d^{\mathbf{c}}(A) \subseteq \mathcal{I}d^{\mathbf{c}}(S)$.*

Now we are going to the main theorem of this section:

THEOREM 12. *Let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest orthogonal decomposition of a semigroup $S = S^0$. Then the lattice $\mathcal{I}d(S)$ is isomorphic to the direct product of lattices $\mathcal{I}d(S_\alpha)$, $\alpha \in Y$, which are directly indecomposable.*

Proof. By the proof of Theorem 10, $\mathcal{I}d(S)$ is isomorphic to the direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, that are directly indecomposable lattices. By Lemma 5, the interval $[0, S_\alpha]$ of $\mathcal{I}d(S)$ is isomorphic to the lattice $\mathcal{I}d(S_\alpha)$, for each $\alpha \in Y$. $\square$

A similar result we obtain for lattices of left ideals of a semigroup with zero:

THEOREM 13. *Let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest decomposition of a semigroup $S = S^0$ into a right sum. Then the lattice $\mathcal{LI}d(S)$ is isomorphic to the direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable lattices.*

Proof. This follows by Theorems 3, 7 and 9. $\square$

Let us emphasize that in the previous theorem, some of the intervals $[0, S_\alpha]$ of $\mathcal{LI}d(S)$ can be different from $\mathcal{LI}d(S)$, since Lemma 5 cannot be proved for right 0-consistent left ideals (see Example 2). A connection between the lattice $\mathcal{LI}d(S)$ and lattices $\mathcal{LI}d(S_\alpha)$, $\alpha \in Y$, is given by:

COROLLARY 2. *Let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest decomposition of a semigroup $S = S^0$ into a right sum. Then the lattice $\mathcal{LI}d(S)$ can be embedded into the direct product of lattices $\mathcal{LI}d(S_\alpha)$, $\alpha \in Y$.*

Another approach to the study of the lattice $\mathcal{LI}d(S)$ is given by the following

THEOREM 14. *Let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest orthogonal decomposition of a semigroup $S = S^0$. Then the lattice $\mathcal{LI}d(S)$ is isomorphic to the direct product of lattices $\mathcal{LI}d(S_\alpha)$, $\alpha \in Y$.*

Proof. Since $\{S_\alpha \mid \alpha \in Y\} \subseteq \mathcal{I}d^{\mathbf{c}}(S) \subseteq \mathcal{LI}d^{\mathbf{rc}}(S)$, then by the proof of Lemma 4, $\mathcal{LI}d(S)$ is isomorphic to the direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$ (generally, they can be further directly decomposed). By Lemma 5, the interval $[0, S_\alpha]$ of $\mathcal{LI}d(S)$ is isomorphic to the lattice $\mathcal{LI}d(S_\alpha)$, for each $\alpha \in Y$. $\square$

Note that the results obtained above for semigroups with zero can be naturally translated to semigroups with kernel. Also, the results concerning lattices of left ideals of a semigroup with zero can be translated to lattices of left ideals of a semigroup without zero:

THEOREM 15. *Let S be a semigroup without zero and let $\{S_\alpha \mid \alpha \in Y\}$ be the greatest decomposition of S into a right zero band of semigroups. Then the lattice $\mathcal{LI}d(S)$ is isomorphic to the direct product of its intervals $[0, S_\alpha]$, $\alpha \in Y$, which are directly indecomposable lattices.*

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