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ON THE FOURTH MOMENT OF THE RIEMANN ZETA-FUNCTION

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Dedicated to the memory of Professor Đuro Kurepa

Abstract. Atkinson proved in 1941 that $\int_0^\infty e^{-t/T}|\zeta(1/2+it)|^4\,dt = TQ_4(\log T) + O(T^c)$ with $c = 8/9+\varepsilon$, where $Q_4(y)$ is a suitable polynomial in y of degree four. We improve Atkinson's result by showing that $c = 1/2$ is possible, and we provide explicit expressions for all the coefficients of $Q_4(y)$ and the closely related polynomial $P_4(y)$.

1. Introduction

In recent years there has been much progress with problems involving the function $E_2(T)$ (see [**7**], [**8**], [**9**], [**10**], [**12**], [**13**], [**15**]). This important function, which represents the error term in the asymptotic formula for the fourth moment of the Riemann zeta-function $\zeta(s)$ on the so-called "critical line" $\mathrm{Re}\,s = \frac{1}{2}$, is defined by the relation

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt = T\sum_{j=0}^4 a_j \log^j T + E_2(T). \tag{1.1}$$

In 1926 Ingham [**4**] proved that $a_4 = 1/(2\pi^2)$. Much later in 1979 Heath-Brown [**3**] proved that $E_2(T) \ll T^{7/8+\varepsilon}$ ($f \ll g$ and $f = O(g)$ both mean that $|f(x)| \le Cg(x)$ for $x \ge x_0$, $C > 0$ and $g(x) > 0$), and calculated

$$a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2}, \tag{1.2}$$

where as usual $\gamma = 0.577215\ldots$ is Euler's constant. The constants a_2, a_1 and a_0 are more complicated, and were not stated explicitly in [**3**]. Heath-Brown's bound for $E_2(T)$ was improved to

$$E_2(T) = O(T^{2/3}\log^C T) \qquad (C > 0) \tag{1.3}$$

AMS Subject Classification (1991): Primary 11 M 06 in [**9**] by Motohashi and the author (see also [**7**]), where it was also proved that

$$E_2(T) = \Omega(T^{\frac{1}{2}}), \tag{1.4}$$

and as usual $f = \Omega(g)$, $g > 0$ means that $\lim_{x\to\infty} f(x)/g(x) \neq 0$. Therein it was also shown that

$$\int_0^T E_2(t)dt = O(T^{3/2}). \tag{1.5}$$

Recently Motohashi [**15**] improved (1.4) to $E_2(T) = \Omega_\pm(T^{\frac{1}{2}})$, and in [**10**] Motohashi and the author established that, with some $C > 0$,

$$\int_0^T E_2^2(t)dt = O(T^2 \log^C T). \tag{1.6}$$

In general, one can define the error term function $E_k(T)$ by the relation

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = TP_{k^2}(\log T) + E_k(T), \tag{1.7}$$

where $k \geq 1$ is a fixed integer, and $P_{k^2}(y)$ is a suitable polynomial in y of degree k^2. Apart from the classical case $k = 1$ (see [**6**] and [**7**] for an extensive discussion) and $k = 2$, our knowledge about the general $E_k(T)$ (see Ch. 4 of [**7**]) is very modest. At present it is not known $E_k(T) = o(T)$ as $T \to \infty$ for any $k \geq 3$, and in fact it is not clear how to define properly the coefficients of $P_{k^2}(y)$ for $k \geq 3$.

Instead at (1.7) one may look at the related formula of the Laplace transform (see Ch. 7 of Titchmarsh [**16**])

$$\int_0^\infty e^{-\delta t}\zeta\left|\left(\frac{1}{2}+it\right)\right|^{2k} dt \qquad (\delta \to 0+,\ \ k \geq 1 \ \text{ an integer}), \tag{1.8}$$

since Laplace transforms of many functions are easier to handle than the original functions. Kober [**11**] proved that

$$\int_0^\infty e^{-2\delta t}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 dt = \frac{\gamma - \log(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^N c_n\delta^n + O(\delta^{N+1}) \tag{1.9}$$

for $\delta \to 0+$, any fixed integer $N \geq 1$ and suitable constants c_n. This is much sharper than the corresponding asymptotic formula (1.7) when $k = 1$. Atkinson [**1**] obtained

$$\int_0^\infty e^{-\delta t}\left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \tag{1.10}$$

$$= \frac{1}{\delta}(A\log^4\delta^{-1} + B\log^3\delta^{-1} + C\log^2\delta^{-1} + D\log\delta^{-1} + E) + L_2(\delta^{-1})$$

as $\delta \to 0+$ with

$$A = 1/(2\pi^2), \quad B = \pi^{-2}(2\log(2\pi) - 6\gamma + 24\zeta'(2)\pi^{-2}) \tag{1.11}$$

and $L_2(1/\delta) \ll (1/\delta)^{13/14+\varepsilon}$ for any given $\varepsilon > 0$. Atkinson's proof used bounds for the function $E(x,r)$ in (4.6). In his work it was indicated on p. 185 how a better bound for $E(x,r)$, which depends on bounds for Kloosterman sums, will lead to the better bound

$$L_2\left(\frac{1}{\delta}\right) \ll \left(\frac{1}{\delta}\right)^{8/9+\varepsilon} \qquad (\delta \to 0+). \tag{1.12}$$

In analogy with (1.7) we define (writing in (1.8) $T = 1/\delta$ with $T \to \infty$) the function $L_k(T)$ by the relation

$$\int_0^\infty e^{-t/T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt = TQ_{k^2}(\log T) + L_k(T), \tag{1.13}$$

where $k \geq 1$ is a fixed integer, $Q_{k^2}(y)$ is a suitable polynomial in y of degree k^2, and one should have $L_k(T) = o(T)$ as $T \to \infty$. At present (analogously to $E_k(T) = o(T)$) the relation $L_k(T) = o(T)$ is not known to hold for $k \geq 3$.

2. Statement of results

A comparison of the asymptotic formulas (1.1) and (1.10) shows that $a_4 = A = 1/(2\pi^2)$, but that $a_3 \neq B$. One actually has

$$B = a_3 + 2\pi^{-2}(1 - \gamma), \tag{2.1}$$

the reason for which will become clear later. Recent advances concerning $E_2(T)$ make it possible to improve (1.12), and we have

THEOREM 1. *If L_2 is defined by (1.10), then as $T \to \infty$ one has*

$$L_2(T) = O(T^{1/2}). \tag{2.2}$$

This result gives a substantial improvement over Atkinson's exponent $8/9 + \varepsilon$ in (1.12). The exponent $1/2$ in (2.2) is the limit of the method of proof, which is based on the use of (1.5). One can prove, more generally, the following

THEOREM 2. *Suppose $k \geq 2$ is a fixed integer and*

$$\int_0^T E_k(t)dt = O(T^{c_k}) \tag{2.3}$$

holds for some $c_k > 0$. If L_k is defined by (1.13), then as $T \to \infty$

$$L_k(T) = O(T^{c_k - 1}). \tag{2.4}$$

Moreover the coefficients of $Q_{k^2}(y)$ can be expressed as linear combinations of the coefficients of $P_{k^2}(y)$, defined by (1.7).

The remaining aim of this paper is to provide explicit expressions for the coefficients a_2, a_1 and a_0 in (1.1) that were not stated explicitly by Heath-Brown [**3**]. From the nature of the problem it is clear that the expressions for these coefficients will be more complicated than the expression (1.2) for a_3. For this reason they will not be stated here as a theorem, but will be dealt with in section 4, where all the appropriate notation will be introduced. From the proof of Theorem 2 in section 3 it will be clear that we can evaluate explicitly C, D and E in (1.10) as linear combinations of the $a_j's$. Conversely, if we know explicitly the coefficients of $Q_{k^2}(y)$, then it is not difficult to see that the coefficients of $P_{k^2}(y)$ can be written as linear combinations of the coefficients of $Q_{k^2}(y)$.

3. The Laplace transform of the $2k$-th moment

It is enough to prove Theorem 2, since Theorem 1 is its consequence because, by (1.5), we have $c_2 = 3/2$ in (2.3) for $k = 2$. Let

$$I_k(T) = \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt = TP_{k^2}(\log T) + E_k(T)$$

with

$$P_{k^2}(y) = \sum_{j=0}^{k^2} a_j y^j, \qquad a_j = a_j(k). \tag{3.1}$$

Then integration by parts gives

$$\begin{aligned}\int_0^\infty e^{-t/T}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} dt &= T^{-1}\int_0^\infty e^{-t/T} I_k(t)dt \\ &= T^{-1}\int_0^\infty e^{-t/T} tP_{k^2}(\log t)dt + T^{-1}\int_0^\infty e^{-t/T}E_k(t)dt \\ &= T\int_0^\infty e^{-x}xP_{k^2}(\log x + \log T)dx + T^{-2}\int_0^\infty e^{-t/T}\left(\int_0^t E_k(y)dy\right)dt = I' + I'',\end{aligned} \tag{3.2}$$

say. Inserting (3.1) in the expression for I' we obtain

$$\begin{aligned}I' &= T\int_0^\infty e^{-x}x\sum_{j=0}^{k^2} a_j(\log x + \log T)^j dx \\ &= T\int_0^\infty e^{-x}x\sum_{j=0}^{k^2} a_j \sum_{i=0}^{j}\binom{j}{i}\log^i T\cdot\log^{j-i}x\cdot dx\end{aligned}$$

$$= T\sum_{j=0}^{k^2} a_j \sum_{i=0}^{j}\binom{j}{i}\log^i T\left(\int_0^\infty e^{-x}x\cdot\log^{j-i}x\cdot dx\right).$$

But $\Gamma^{(k)}(z) = \int\limits_0^\infty e^{-t}t^{z-1}(\log t)^k dt$, for Re$z > 0$ and $k \ge 0$ which gives

$$I' = T\sum_{j=0}^{k^2} a_j\sum_{i=0}^{j}\binom{j}{i}\Gamma^{(j-i)}(2)\cdot\log^i T = T\sum_{i=0}^{k^2} b_i\log^i T \tag{3.3}$$

with

$$b_i = b_i(k) = \sum_{j=i}^{k^2}\binom{j}{i}a_j\Gamma^{(j-i)}(2) \qquad (i = 0, 1, \dots, k^2), \tag{3.4}$$

so that the coefficients of Q_{k^2} are linear combinations of the coefficients of P_{k^2}. By using (2.3) we obtain

$$\begin{aligned} I'' &= T^{-2}\int_0^\infty e^{-t/T}\left(\int_0^t E_k(y)dy\right)dt \ll T^{-2}\int_0^\infty e^{-t/T}t^{c_k}dt \\ &= T^{c_k-1}\Gamma(c_k+1) \ll T^{c_k-1}, \end{aligned} \tag{3.5}$$

so that Theorem 2 follows from (1.13) and (3.2)–(3.5). Note that in the particular case $k = 2$ (3.4) yields

$$\begin{aligned} &A = b_4 = a_4 = 1/(2\pi^2), \;\; B = b_3 = 4a_4\Gamma'(2) + a_3\Gamma(2), \\ &C = b_2 = 6a_4\Gamma''(2) + 3a_3\Gamma'(2) + a_2\Gamma(2), \\ &D = b_1 = 4a_4\Gamma^{(3)}(2) + 3a_3\Gamma''(2) + 2a_2\Gamma'(2) + a_1\Gamma(2), \\ &E = b_0 = a_4\Gamma^{(4)}(2) + a_3\Gamma^{(3)}(2) + a_2\Gamma''(2) + a_1\Gamma'(2) + a_0\Gamma(2). \end{aligned} \tag{3.6}$$

Since $\gamma = -\int_0^\infty e^{-x}\log x dx = -\Gamma'(1)$, it follows by an integration by parts that $-\gamma = -\int_0^\infty x(e^{-x}\log x)'dx = \Gamma'(2) - 1$. Thus $\Gamma'(2) = 1 - \gamma$, and (3.6) yields $B = b_3 = 4\cdot\frac{1}{2\pi^2}(1-\gamma) + a_3$, which is (2.1). Conversely, if the $b_j's$ are known, then (3.4) is a system of k^2+1 linear equations in k^2+1 unknowns $a_0, a_1, \dots, a_{k^2}$ with a triangular determinant whose value is $(\Gamma(2))^{k^2+1} = 1$, so that the $a_j's$ can be uniquely expressed as linear combinations of the $b_j's$.

4. The coefficients of the main term in the fourth moment formula

There are several ways to obtain explicitly the coefficients a_j in (1.1). This can be achieved by following the proofs of the fourth moment formula (see Ch. 4 of [**7**] or [**13**]). Here we shall follow the method of Heath-Brown [**3**], who showed that the main term $TP_4(\log T)$ in (1.1) consists of two parts: the part coming from the "diagonal" terms, and the part coming from the "non-diagonal" terms of a sum involving the number of divisors function $d(n)$.

The diagonal terms furnish an expression of the form $TR_4(\log T)$, and the non-diagonal terms the expression $TQ_2(\log T)$. Here $R_4(x)$ and $Q_2(x)$ denote suitable polynomials of degree four and two, respectively. It will turn out that the coefficients of $Q_2(x)$ have a more complex form than those of $R_4(x)$. Thus we shall have

$$P_4(x) = R_4(x) + Q_2(x). \tag{4.1}$$

As on p. 403 of [**3**] the diagonal terms make a contribution which is equal to

$$\begin{aligned}2 \sum_{m \le T/(2\pi)} & d^2(m)m^{-1}(T - 2\pi m) \\ &= \frac{1}{2\pi i}\left\{\int\limits_{1-i\infty}^{1+i\infty} 4\pi\zeta^4(s+1)\zeta^{-1}(2s+2)\left(\frac{T}{2\pi}\right)^{s+1}\frac{ds}{s(s+1)}\right\}.\end{aligned}$$

The term $TR_4(\log T)$ will be the residue of the simple pole (of the integrand in curly brackets) at $s = 0$. Near $s = 0$ we have the expansions

$$\zeta^{-1}(2s+2) = \zeta^{-1}(2) - \frac{2s\zeta'(2)}{\zeta^2(2)} + c_2 s^2 + c_3 s^3 + \ldots$$

with

$$\begin{aligned} c_k &= \frac{2^k}{k!}\sum_{n=1}^{\infty}\mu(n)(-\log n)^{\kappa} n^{-2} = \frac{1}{k!}\left\{\frac{d^k}{ds^k}(\zeta^{-1}(2s+2))\right\}\Bigg|_{s=0}, \\ \frac{1}{s+1} &= 1 - s + s^2 - s^3 + \ldots, \qquad (4.2) \\ \left(\frac{T}{2\pi}\right)^{s+1} &= \frac{T}{2\pi}\left\{1 + s\log\left(\frac{T}{2\pi}\right) + \frac{s^2}{2!}\log^2\left(\frac{T}{2\pi}\right) + \frac{s^3}{3!}\log^3\left(\frac{T}{2\pi}\right) + \ldots\right\}, \\ \zeta^4(s+1) &= \frac{1}{s^4} + \frac{4\gamma}{s^3} + \frac{b_{-2}}{s^2} + \frac{b_{-1}}{s} + b_0 + b_1 s + b_2 s^2 + \ldots. \end{aligned}$$

The coefficients $b_k (k \ge -2)$ may be found from the relation

$$\zeta^4(s+1) = \left(\frac{1}{s} + \gamma_0 + \gamma_1 s + \gamma_2 s^2 + \gamma_3 s^3 + \ldots\right)^4, \tag{4.3}$$

where one has (see Theorem 1.3 of [**6**])

$$\gamma_0 = \gamma, \gamma_k = \frac{(-1)^k}{k!}\lim_{N\to\infty}\left(\sum_{m \le N}\frac{\log^k m}{m} - \frac{\log^{k+1} N}{k+1}\right).$$

Israilov [**5**] calculated

$$\gamma_1 = 0.072815846\ldots, \quad \gamma_2 = -0.004845182\ldots, \quad \gamma_3 = -0.000342305\ldots,$$

and Euler's constant $\gamma = \gamma_0$ is of course known with much greater accuracy,

$$\gamma = 0.5772\ 15664\ 90153\ 28606\ 06512\ldots.$$

From (4.2) and (4.3) we obtain by comparing the coefficients

$$b_{-2} = 4\gamma_1 + 6\gamma^2, \;\; b_{-1} = 4\gamma_2 + 12\gamma\gamma_1 + 4\gamma^3,$$
$$b_0 = 4\gamma_3 + 12\gamma\gamma_2 + 6\gamma_1^2 + 12\gamma^2\gamma_1 + \gamma^4.$$

Since the residue is the coefficient of s^{-1}, we obtain

$$\begin{aligned} &TR_4(\log T) \\ &= \frac{2T}{\zeta(2)}\left\{\frac{1}{24}\log^4\left(\frac{T}{2\pi}\right) + \frac{a}{6}\log^3\left(\frac{T}{2\pi}\right) + \frac{b}{2}\log^2\left(\frac{T}{2\pi}\right) + c\log\left(\frac{T}{2\pi}\right) + d\right\} \end{aligned} \tag{4.4}$$

with

$$\begin{aligned} a &= 4\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}, \;\; b = 1 + \frac{2\zeta'(2)}{\zeta(2)} + c_2\zeta(2) + 4\gamma\left(-1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + b_{-2}, \\ c &= -1 - \frac{2\zeta'(2)}{\zeta(2)} + (c_3 - c_2)\zeta(2) + 4\gamma\left(1 + c_2\zeta(2) + \frac{2\zeta'(2)}{\zeta(2)}\right) \\ &\quad + b_{-2}\left(-1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + b_{-1}, \\ d &= 1 + \frac{2\zeta'(2)}{\zeta(2)} + (c_2 - c_3 + c_4)\zeta(2) + 4\gamma\left(-1 - \frac{2\zeta'(2)}{\zeta(2)} + (c_3 - c_2)\zeta(2)\right) \\ &\quad + b_{-2}\left(1 + \frac{2\zeta'(2)}{\zeta(2)} + c_2\zeta(2)\right) + b_{-1}\left(-1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + b_0. \end{aligned} \tag{4.5}$$

Expanding $\log^j(T/2\pi) = (\log T - \log 2\pi)^j$ by the binomial theorem we obtain ($TR_4(\log T) = g_1(T)$ in Heath-Brown's notation)

$$\begin{aligned} TR_4(\log T) = \frac{T}{\pi^2}\Big\{&\frac{1}{2}\log^4 T + (2a - 2\log(2\pi))\log^3 T + (3\log^2(2\pi) - 6a\log(2\pi) \\ &+ 6b)\log^2 T + (-2\log^3(2\pi) + 6a\log^2(2\pi) - 12b\log(2\pi) + 12c)\log T \\ &+ \left(\frac{1}{2}\log^4(2\pi) - 2a\log^3(2\pi) + 6b\log^2(2\pi) - 12c\log(2\pi) + 12d\right)\Big\}. \end{aligned}$$

The coefficients a_4 and a_3 in (1.1) are the coefficients of $\log^4 T$ and $\log^3 T$, respectively, and so by the first formula in (4.5) they are

$$a_4 = \frac{1}{2\pi^2}, \;\; a_3 = \frac{2a - 2\log(2\pi)}{\pi^2} = \frac{8\gamma - 2 - 24\zeta'(2)\pi^{-2} - 2\log(2\pi)}{\pi^2}.$$

These are the same values that were obtained by Ingham and Heath-Brown.

For the remaining part $TQ_2(\log T)$ in the main term one has (see p. 404 of [3]) that it is the main term in the asymptotic formula for

$$f_0(T) \sim 2\mathrm{Re}\left\{\sum_{r=1}^{R}(ir)^{-1}S_r\right\} \qquad (R \to \infty),$$

where $S_r \sim \int_0^{T/(2\pi)} m'(x,r)e^{iTr/x}dx$. Here $m(x,r)$ stands for the main term in the asymptotic formula for the so-called binary additive divisor problem (see Motohashi [**14**] for an extensive discussion), namely

$$\sum_{n\le x} d(n)d(n+r) = m(x,r) + E(x,r), m(x,r) = x\sum_{j=0}^{2} c_{3-j}(r)\log^j x. \tag{4.6}$$

We have $m'(x,r) = d_0(r)\log^2 x + d_1(r)\log x + d_2(r)$ with

$$d_0(r) = c_1(r), d_1(r) = c_2(r) + 2c_1(r), d_2(r) = c_2(r) + c_3(r). \tag{4.7}$$

In Theorem 2 of [**3**] Heath-Brown evaluated the constants $c_i(r)$ in (4.6), but his expressions are cumbersome. We find it more expedient to use the expressions given by Balakrishnan and Sengupta [2], and from these expressions and (4.7) we obtain, with the notation

$$\sigma_z(n) = \sum_{d|n} d^z, \quad \sigma'_z(n) = \frac{d}{dz}(\sigma_z(n)) = \sum_{d|n} d^z\log d, \quad \sigma''_z(n) = \sum_{d|n} d^z\log^2 d,$$

the following:

$$\begin{aligned} d_0(r) &= \frac{\sigma_{-1}(r)}{\zeta(2)}, d_1(r) = d_0(r)\left\{4\gamma - 4\frac{\zeta'}{\zeta}(2) - 4\frac{\sigma'_{-1}}{\sigma_{-1}}(r)\right\},\\ d_2(r) &= d_0(r)\Bigg\{4\gamma - 1 - 4\frac{\zeta'}{\zeta}(2) - 4\frac{\sigma'_{-1}}{\sigma_{-1}}(r) - 4\frac{\zeta''}{\zeta}(2) - 4\left(\frac{\zeta'}{\zeta}(2)\right)^2\\ &\quad + 4\frac{\sigma''_{-1}}{\sigma_{-1}}(r) - 4\left(\frac{\sigma'_{-1}}{\sigma_{-1}}(r)\right)^2 + \left(2\gamma - 1 - 2\frac{\zeta'}{\zeta}(2) - 2\frac{\sigma'_{-1}}{\sigma_{-1}}(r)\right)^2\Bigg\}. \end{aligned} \tag{4.8}$$

Further, after a change of variable, we have

$$\begin{aligned} S_r &\sim \int_0^{T/(2\pi)} \{d_0(r)\log^2 x + d_1(r)\log x + d_2(r)\}e^{iTr/x}dx\\ &= Tr\int_{2\pi r}^{\infty} \{d_0(r)\log^2\left(\frac{Tr}{y}\right) + d_1(r)\log\left(\frac{Tr}{y}\right) + d_2(r)\}e^{iy}\frac{dy}{y^2}. \end{aligned}$$

Therefore it follows that

$$TQ_2(\log T) = 2\mathrm{Re}\left\{\sum_{r=1}^{\infty}(ir)^{-1}S_r\right\} = T(e_0\log^2 T + e_1\log T + e_2)$$

with

$$\begin{aligned} e_0 &= 2\sum_{r=1}^{\infty} d_0(r)\int_{2\pi r}^{\infty}\frac{\sin x}{x^2}dx,\\ e_1 &= \sum_{r=1}^{\infty}\left\{d_0(r)\int_{2\pi r}^{\infty}(2\log r - 2\log x)\frac{\sin x}{x^2}dx + d_1(r)\int_{2\pi r}^{\infty}\frac{\sin x}{x^2}dx\right\}, \end{aligned} \tag{4.9}$$

$$e_2 = \sum_{r=1}^{\infty} \int_{2\pi r}^{\infty} \Big\{ d_0(r) \log^2 x - (2d_0(r)\log r + d_1(r)) \log x$$
$$+ d_0(r)\log^2 r + d_1(r)\log r + d_2(r) \Big\} \frac{\sin x}{x^2} dx.$$

If, for $\mathrm{Re}\, a < 1$ and $x > 0$ we introduce the standard notation

$$C(x,a) = \int_x^{\infty} t^{a-1} \cos t \cdot dt, \quad S(x,z) = \int_x^{\infty} t^{a-1} \sin t \cdot dt,$$

then we have

$$\frac{\partial S(x,a)}{\partial a} = \int_x^{\infty} t^{a-1} \log t \cdot \sin t \cdot dt, \quad \frac{\partial^2 S(x,a)}{\partial a^2} = \int_x^{\infty} t^{a-1} \log^2 t \cdot \sin t \cdot dt,$$

and

$$e^{-\frac{1}{2}\pi i a} \Gamma(a, ix) = C(x,a) - iS(x,a), \quad S(x,a) = -\mathrm{Im}\{e^{-\frac{1}{2}\pi i a}\Gamma(a,ix)\},$$
$$\Gamma(a,x) = \Gamma(a) - \gamma(a,x) = \int_x^{\infty} e^{-t} t^{a-1} dt,$$

where $\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt$ is the incomplete gamma-function. With this notation we have

$$\begin{aligned}
e_0 &= 2\sum_{r=1}^{\infty} d_0(r) S(2\pi r, -1),\\
e_1 &= 2\sum_{r=1}^{\infty} \Big\{ (2d_0(r)\log r + d_1(r)) S(2\pi r, -1) - 2d_0(r) \frac{\partial S}{\partial}(2\pi r, -1)\Big\},\\
e_2 &= 2\sum_{r=1}^{\infty} \Big\{ \big(d_0(r) \frac{\partial^2 S}{\partial^2}(2\pi r, -1) - (2d_0(r)\log r + d_1(r)\big) \frac{\partial S}{\partial}(2\pi r, -1)\\
&\qquad + (d_0(r)\log^2 r + d_1(r)\log r + d_2(r)) S(2\pi r, -1)\Big\}.
\end{aligned} \tag{4.10}$$

Hence finally from (4.1) and the above expressions we obtain

THEOREM 3. *For the coefficients a_j in (1.1) we have*

$$\begin{aligned}
a_4 &= \frac{1}{2\pi^2}, \quad a_3 = 2(4\gamma - 1 - \log(2\pi) - 12\zeta'(2)\pi^{-2})\pi^{-2},\\
a_2 &= (3\log^2(2\pi) - 6a\log(2\pi) + 6b)\pi^{-2} + e_0,\\
a_1 &= (-2\log^3(2\pi) + 6a\log^2(2\pi) - 12b\log(2\pi) + 12c)\pi^{-2} + e_1,\\
a_0 &= \left(\frac{1}{2}\log^4(2\pi) - 2a\log^3(2\pi) + 6b\log^2(2\pi) - 12c\log(2\pi) + 12d\right)\pi^{-2} + e_2,
\end{aligned}$$

where a, b, c, d are given by (4.5) and e_0, e_1, e_2 *by (4.10).*

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