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ARONSZAJN ORDERINGS

Dedicated in memory of Professor Đuro R. Kurepa

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The purpose of this note is to expose a new class of spaces which tests many of the classical covering properties where previously various singular sets of real numbers have been used. Since the existence of many of the singular sets of reals cannot be proved without additional axioms, one of the advantages of our approach is that it needs no such axioms. The classical covering properties that we have in mind have all their roots in the notion of *strong measure zero* sets of reals by E. Borel [**8**]. More generally, a metric space X is said to be of strong measure zero if for every sequence $\{\varepsilon_n\}$ of positive real numbers, X can be covered by a sequence $\{B_n\}$ of open balls such that each B_n has diameter $\leq \varepsilon_n$. This property in the literature is also known under the name of *property* C. Since this is clearly a strictly metric notion which is not preserved by taking continuous images it is natural to seek a purely topological notion which when restricted to metric spaces would reduce to the property C. One of the most prominent such attempts is the Rothberger's notion of *property* C'' (see [**29**], [**21**]) which says that for every sequence $\{\mathcal{U}_n\}$ of open covers of X one can choose $U_n \in \mathcal{U}_n$ for each n such that $X = \bigcup_{n=1}^{\infty} U_n$. This can also be expressed in game-theoretic terms using the following game $G^*(X)$ between two players I and II: Player I chooses an open cover $\mathcal{U}_n$ of X and II responds by choosing $U_n \in \mathcal{U}_n$. Player II wins the play iff $\bigcup_{n=1}^{\infty} U_n = X$; otherwise I wins. This game was considered by F. Galvin [**9**] who showed it to be equivalent to a better known dual game called the *point-open game* in which II chooses a point x_n of X and I responds by an open neighborhood U_n of x_n and in which II wins the play iff $\bigcup_{n=1}^{\infty} U_n = X$; otherwise I wins. Note that if the crucial question about the property C is whether it is as restrictive as the countability requirement (i.e., Borel's Conjecture [**8**]), the crucial question about the point open game is about

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the existence of winning strategies for one of the players i.e., whether the game is determined in the class of metric spaces or in some larger class such as, say, the class of first countable spaces (see [**9**; p. 448]). The class of first countable spaces seems more natural in this context than any other because of a theorem of Telgarsky [**32**] which says that II wins $G(X)$ iff II wins $G(X_\delta)$, where X_δ is the space on X with the topology generated by all G_δ subsets of X. So, in particular, II does not have a winning strategy in $G(X)$ for every uncountable first-countable space X. To state the most restrictive covering property of our interest here, for a family $\mathcal{U}$ and a positive integer n, let $\mathcal{U}^{\cdot n}$ denote the family of cubes U^n of members U of $\mathcal{U}$. We shall say that $\mathcal{U}$ is an *n-cover of X* if $\mathcal{U}^{\cdot n}$ covers X^n. If $\mathcal{U}$ is an n-cover of X for every positive integer n then we shall say that $\mathcal{U}$ is an *ω-cover* of X. A space X is said to have *property* γ if for every open ω-cover $\mathcal{U}$ of X there is a sequence $\{U_n\}$ of elements of $\mathcal{U}$ such that $X = \bigcup_m \bigcap_{n>m} U_n$, i.e. every element of X is in all but finitely many U_n's. Let $\underline{\lim}\, U_n$ be the short notation for $\bigcup_m \bigcap_{n>m} U_n$. This property was introduced by E. Pytkeev [**26**] and J. Gerlitz [**11**] during the course of proving that several convergence-type properties of the function space $\mathcal{C}_p(X)$ are equivalent. In particular, it has been shown that property γ of a completely regular space X is equivalent to the following convergence property: if a family F of continuous real functions on X accumulates to 0 in the topology of pointwise convergence then it contains a sequence $\{f_n\}$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all x in X. Note that all these properties strengthen the more familiar topological property of Lindelöf stating that every open cover of X contains a countable subcover. For example, it should be clear that property γ implies the Lindelöf property of every finite power of X since this is readily seen to be equivalent to the statement that every open ω-cover of X contains a countable ω-subcover. This leads us to the rich subject on preservation of paracompactness and the Lindelöf property in finite and countable products ([**19**], [**23**], [**24**]). The close relationship between these two subjects becomes more apparent if one realizes that most of the examples produced in these two areas are based on an old idea of A. Besicovitch [**6**] of a *concentrated set* of reals: a set A is concentrated around a set B if every open set which contains B contains also all but countably many elements of A. We refer the reader to [**36**; §3] where it is shown that if there is an uncountable set of reals concentrated around the rationals then there exist Michael and Sorgenfrey "lines" witnessing many of the Examples of [**19**]. Needless to say that Besicovitch introduced his notion of concentrated sets in order to study strong measure zero sets (see [**7**]), and that concentrated sets are used more recently in connection with the property γ discussed above (see [**10**]).

Our basic idea is to test these covering properties using a class of so-called *Aronszajn orderings* which have never been previously fully used as topological objects because of the general preoccupation with the extremely difficult task of classifying them as ordered structures rather than as topological spaces. Indeed, the problem of whether there can be a finite list of Aronszajn orderings such that any other such ordering contains one from the list is one of the major problems in this part of mathematics. A pleasing feature of the topological analysis is that it essentially uses all of the knowledge about Aronszajn orderings accumulated so far. For example, in §1 we show that the general Aronszajn orderings have only some of the covering properties discussed above. To get them all one needs to go to a deeper class of such orderings, the class of linear orderings whose cartesian squares are the unions of countably many chains (see §2). Even this subclass is too general, and in §§3 and 4 we need to go to the class of orderings discovered in [**35**] which are generated by coherent sequences $\{e_\alpha\}$ of finite-to-one enumerations of countable ordinals. The coherent sequences $\{e_\alpha\}$ are quite versatile mathematical objects indeed. We have shown this on several previous occasions (see for example, [**34**], [**35**], [**36**] and [**37**]). Besides the applications of §§3 and 4 we shall use them here in §5 for building examples of homogeneous nonreversible continua, or homogeneous uniform Eberlein compacta i.e., homogeneous weakly compact subspaces of Hilbert spaces.

1. A-lines

Since our purpose here is to test the above properties with a class of spaces which is relatively close to the real line such as, for example, the class of first countable linear orderings and since we would like to do this without any use of special axioms, we find ourselves with little room to maneuver. The Lindelöf property restricts all well-ordered or conversely well-ordered subsets to be countable. The preservation of the covering properties under continuous maps restricts all subsets which can be order-isomorphically embedded into the reals to be also countable. Thus we must work with the class of *Aronszajn orderings* or *A-lines* as we shall call them henceforth. This class was introduced and proved to be nonempty by N. Aronszajn and Dj. Kurepa in [**15**] and [**16**]. So, let A be a given A-line and let D be a countable subset of A. Then the complement of D in A is split into a family T_D of convex sets, the equivalence classes of the relation in which x and y are equivalent if no member of D separates them. The members of T_D will also be called the *complementary intervals of* D. Note that T_D must be countable or else A would contain a subset order-isomorphic to a set of reals. For two countable subsets A_0 and A_1 we say that A_1 properly extends A_0 if $A_0 \subseteq A_1$ and A_1 has points in every of the complementary intervals of A_0. A *proper decomposition* of A is a sequence A_α $(\alpha < \omega_1)$ of countable subsets of A such that A_β properly extends A_α if $\alpha < \beta$, and such that $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ if δ is a limit ordinal. Note that *every* ω_1-sequence of properly extending countable subsets of A must have union equal to A. On the other hand, trivially no properly extending sequence can have length $> \omega_1$, so there is only one possibility for the length of any proper decomposition of A. Given such decomposition $\{A_\alpha\}$ let T be the union of T_{A_α}'s. This is the *partition tree* of A associated with $\{A_\alpha\}$. Needless to say this is an *Aronszajn tree* i.e., a tree with countable levels and no uncountable chains. Note also that T uniquely determines the proper decomposition $\{A_\alpha\}$ of A, so their roles will be frequently interchanged below. We shall also simplify the notation and write T_α for T_{A_α}. If δ is a countable limit ordinal, then we say that the decomposition $\{A_\alpha\}$ is *continuous at* δ if every complementary interval of A_δ has a minimal element. This notion should be more accurately called δ-continuous from the *left* since one may also define its right analogue by requiring that every convex set from T_{A_δ} has a maximal element. We shall say that A is *stationary* (or more precisely *left-stationary*) if and only if every proper decomposition $\{A_\alpha\}$ of A if (left) continuous at some countable limit ordinal. Note that if A is a stationary A-line then the set of continuities of every proper decomposition of A is a stationary subset of ω_1 explaining thus our choice of the name for this concept. It turns out that this notion is the key to many of the covering properties of our interest here as we shall now see.

THEOREM 1. *The point-open game played on a stationary A-line is not determined.*

Proof. We have already mentioned the fact that II cannot have a winning strategy on any uncountable first countable space (see [**9**], [**32**]), so we shall concentrate on showing that if A is a stationary A-line then I doesn't have a winning strategy in $G^*(A)$. So let σ be a given strategy for I in $G^*(A)$ and let $\{A_\alpha\}$ be a fixed proper decomposition of A. Choose a countable elementary submodel M of some large enough structure containing all these objects such that if δ is its intersection with ω_1 then $\{A_\alpha\}$ is continuous at δ. Let $\{a_n\}_{n=1}^\infty$ be an enumeration of all elements of A_δ as well as all the minimums of complementary intervals of A_δ. Let $\mathcal{U}_1 = \sigma(\emptyset)$. Then $\mathcal{U}_1 \in M$ and $\mathcal{U}_1$ is an open cover of A. Having chosen $U_i \in \mathcal{U}_i \cap M$ for $i < n$, let $\mathcal{U}_n = \sigma\langle U_1, \ldots, U_{n-1}\rangle$ and proceed to choosing $U_n \in \mathcal{U}_n \cap M$ as follows. If a_n is an element of $A_\delta (= A \cap M)$, choose an arbitrary $U_n \in \mathcal{U}_n \cap M$ containing a_n. So assume a_n is the minimal point of some complementary interval t of A_δ. Since a_n is a left-limit point of A_δ there is x in A_δ such that the interval $(x, a_n]$ is included in some element of $\mathcal{U}_n$. Let

$$B = \{b \in A : (x, b] \text{ is included in some member of } \mathcal{U}_n\} .$$

Then $B \in M$ and B is a convex set in A which contains a_n.

CLAIM. $\sup t < \sup B$

Proof. Otherwise, let P be the set of all complementary intervals v of T such that

$$\inf v \leq \sup B \leq \sup v$$

. Then $P \in M$ and $t \in P$. Clearly every two members of P must intersect, so P is a chain of T. It should also be clear that P must be uncountable giving us the desired contradiction.

By the Claim, there exists b in $B \cap M$ such that $b > \sup t$. Then there is U_n in $\mathcal{U}_n \cap M$ which includes $(x, b]$. It follows that U_n not only contains a_n but it also covers the whole convex set t. This completes the inductive step of the process which clearly produces a play $\langle \mathcal{U}_1, U_1, \mathcal{U}_2, U_2, \ldots\rangle$ of $G^*(A)$ in which I uses the strategy σ. But the sequence of U_n's is chosen in such a way that it covers A, so the player II wins this particular play showing that σ cannot be a winning strategy of I. This finishes the proof.

Remark. The problem of indeterminacy of the point-open game was first considered by F. Galvin [**9**] who showed, assuming the Continuum Hypothesis, that there is an undetermined set of reals concentrated around the rationals. Latter I. Reclaw [**27**] showed that the game is undetermined on any uncountable set of reals concentrated around any dense set of reals. By joint efforts E. van Douwen and R. Telgarsky (see [**32**] or [**33**; §4]) showed that a space of Dj. Kurepa [**17**] (as rediscovered by R. Pol [**22**]) can serve as an example of P-space (a space in which G_δ sets are open) for which the point-open game is not determined.

COROLLARY 2. *Every stationary A-line has the property C''.*

Remark. Since the property C'' is preserved under continuous images it is worth pointing out the stronger version of Theorem 1 which is given by the above proof. Let $(A, \leftarrow)$ be the set A with the topology generated by the intervals of the form $(x, y]$ where $x < y$ in A. Clearly $\leftarrow$ is a refinement of the order topology of A and the proof of Theorem 1 shows that even the space $(A, \leftarrow)$ has the property C''. This kind of generality in our proofs will be quite frequent henceforth.

THEOREM 3. *Every continuous function $f : A \to X$ from a stationary A-line into a separable metric space has a countable range.*

Proof. Let $\{A_\alpha\}$ be a proper decomposition of A, let T be the corresponding partition tree, and let S be the stationary set of continuities of $\{A_\alpha\}$. Let B be the set of all a in A for which there is a t_a in T such that $\min t_a = a$. (Note that there is a substantial quantity of such elements of A e.g., the minimums of the intervals from T_δ for δ in S are all members of B.) For an integer n, let

$$B_n = \left\{ a \in B : \text{diam}\,(f''t_a) \geq 1/n \right\} .$$

It suffices to show that each B_n is countable. This follows immediately from the remark above since the property C'' of $(A, \leftarrow)$, in particular, implies that every uncountable subset of A has a complete left-accumulation point in A. This completes the proof.

To find an interpretation of this result in terms of the function space $\mathcal{C}_p(A)$, choose a sufficiently closed ordinal δ in S e.g., an accumulation point of S. Then we can find a continuous retraction

$$r_\delta : A \to A_\delta^+$$

where

$$A_\delta^+ = A_\delta \cup \{\min t : t \in T_\delta\} .$$

To see that such a retraction r_δ exists, map every t of T_δ into its minimum if t has no maximums; otherwise find a cut c of t not realizable in A such that $\min(t) < c < \max(t)$ and map everything in t below c into $\min(t)$ and everything in t above c into $\max(t)$. It should be clear that so defined map is continuous. What we have shown in the proof of Theorem 3 is that for every f in $\mathcal{C}_p(A)$ there exist δ in S such that

$$f = (f \restriction A_\delta^+) \circ r_\delta .$$

Moreover, we may assume that δ is chosen so that f is constant on any of the complementary intervals of A_δ. The set $\mathcal{F}_\delta$ of all f in $\mathcal{C}_p(A)$ for which a given

$\delta \in S$ works is clearly second countable being isomorphic to a G_δ-subset of $\mathcal{C}_p(A_\delta^+)$. Note also that each $\mathcal{F}_\delta$ is a closed topological vector subspace of $\mathcal{C}_p(A)$. All this leads to the following interesting decomposition property of $\mathcal{C}_p(A)$.

THEOREM 4. *If A is a stationary A-line then its function space $\mathcal{C}_p(A)$ is the union of an increasing ω_1-sequence of closed second countable vector subspaces.*

COROLLARY 5. If A is a stationary A-line, then the closure of every countable subset of $\mathcal{C}_p(A)$ is second countable.

2. C-Lines

A *Countryman line* (*C-line* in short) is an uncountable linear ordering whose cartesian square is the union of countably many chains. It is easily seen that every C-line is also an A-line and that every finite power of a C-line is also the union of countably many chains. The following result shows that they have stronger covering properties then the wider class of A-lines.

THEOREM 6. *Every stationary C-line C has the property γ, i.e., every open ω-cover $\mathcal{U}$ of C contains a sequence $\{U_n\}$ such that $C = \underline{\lim}\, U_n$.*

Proof. Let $\{C_\alpha\}$ be a fixed proper decomposition of C and let T be the corresponding partition tree of complementary intervals to the C_α's. Choose a countable elementary submodel M of some large enough structure containing all the relevant objects such that if δ is its intersection with ω_1 then $\{C_\alpha\}$ is continuous at δ. Let $\{a_{2n}\}_{n=1}^\infty$ be an enumeration of all elements of C_δ and let $\{a_{2n-1}\}_{n=1}^\infty$ be an enumeration of all minimums of the complementary intervals of C_δ. The theorem will be proved once we show that for every n there is an open set U_n in $\mathcal{U} \cap M$ which includes

$$\{a_2, a_4, \ldots, a_{2n}\} \cup \Big(\bigcup_{i=1}^n t_{2i-1}\Big)$$

where t_{2i-1} is the complementary interval of C_δ whose minimum is equal to a_{2i-1}. Applying the fact that $\mathcal{U}$ is a $2n$-cover of C, choose U in $\mathcal{U}$ containing $\{a_1, a_2, \ldots, a_{2n}\}$ and for each $1 \le i \le n$ choose $x_i < a_{2i-1}$ in C_δ $(= C \cap M)$ such that $(x_i, a_{2i-1}] \subseteq U$. Let Y be the set of all $\langle y_1, \ldots, y_n\rangle$ in C^n such that $x_i < y_i$ $(1 \le i \le n)$ and such that

$$\{a_2, a_4, \ldots, a_{2n}\} \cup \Big(\bigcup_{i=1}^n (x_i, y_i]\Big)$$

is covered by a single element of $\mathcal{U}$. Then $Y \in M$ and $\langle a_1, a_3, \ldots, a_{2n-1}\rangle \in Y$. Since C^n is the union of countably many chains there is a chain $Y_0 \subseteq Y$ such that $Y_0 \in M$ and $\langle a_1, a_3, \ldots, a_{2n-1}\rangle \in Y_0$.

CLAIM. *There is $\langle y_1, \ldots, y_n\rangle$ in $Y_0 \cap M$ such that $a_{2i-1} < y_i$ for all $1 \le i \le n$.*

Proof. Note that it suffices to find such $\langle y_1, \ldots, y_n\rangle$ with the property that $a_1 < y_1$ because the other inequalities will follow from this one and the fact that Y_0

is a chain of C^n. So, let c be the supremum (taken in the Dedekind completion of C) of the set of all first coordinates of elements of Y_0. Note that since C contains no uncountable well-ordered subsets this supremum is equal to the supremum of the projection of $Y_0 \cap M$ on the first coordinate. Note also that $c \in M$ and that $a_1 \leq c$. Since a_1 is not a member of M the equality is impossible, so we have $a_1 < c$. Since $c = \sup \pi_1''(Y_0 \cap M)$, there is $\langle y_1, \ldots, y_n \rangle$ in $Y_0 \cap M$ such that $a_1 \leq y_1$ and this proves the Claim.

Choose $\langle y_1, \ldots, y_n \rangle$ as in the Claim. The proof of Theorem 6 is finished if we show that t_{2i-1} is included in $(x_i, y_i]$ for all $1 \leq i \leq n$ or equivalently that $\sup t_{2i-1} \leq y_i$ for all $1 \leq i \leq n$. But this is clearly so since $a_i = \min t_{2i-1} < y_i$ and since the point y_i, being a member of C_δ, cannot split the complementary interval t_{2i-1}. The proof of Theorem 6 is thus completed.

Corollary 7. *Let X be a stationary C-line. Then the function space $\mathcal{C}_p(X)$ is the union of an increasing ω_1-sequence of closed second countable vector subspaces of $\mathcal{C}_p(X)$. It follows that the closure of every countable subset of $\mathcal{C}_p(X)$ is second countable. Moreover, if a family $\mathcal{F}$ of functions from $\mathcal{C}_p(X)$ has the constantly 0 function in its closure then it contains a sequence $\{f_n\}$ which pointwise converges to 0.*

Proof. The decomposition of $\mathcal{C}_p(X)$ has already been discussed in the previous section. The deep work of Pytkeev [**26**] and Gerlitz-Nagy [**11**] shows that the convergence property and some of its weaker forms (sequentiallity and being a k-space) are in fact equivalent to the property γ of X. Since we know that the closure of every countable subset of $\mathcal{C}_p(X)$ is second countable it suffices to find a countable $\mathcal{F}_0 \subseteq \mathcal{F}$ having $\bar{0}$ in its closure, so we could use an even older work from that subject (see [**25**]). But for the convenience of the reader, we present a direct deduction. For this purpose we choose a sequence $\{x_n\}$ of distinct elements of X. Let $\mathcal{U}$ be the set of all open subsets of X of the form

$$U(f,n) = \{x \in X : x \neq x_n \text{ and } |f(x)| < 2^{-n}\}$$

where $f \in \mathcal{F}$ and $n \in \mathbb{N}$. By our assumption about $\mathcal{F}$, $\mathcal{U}$ is an ω-cover of X, so there exist $\{U(f_i, n_i)\} \subseteq \mathcal{U}$ such that $\underline{\lim}\, U(f_i, n_i) = X$. It is easily checked that $n_i \to \infty$ which in turn gives $f_n \to 0$. This completes the proof.

Remark. We are again in the situation to note that the proof of Theorem 6 works equally well for the arrow space $(C, \leftarrow)$ in place of the line C, so we have here a possibly stronger result that even the space $(C, \leftarrow)$ has the property γ as well as the other properties stated in Corollary 7.

3. Sums and Products

Many of the classical covering properties are countably additive but we shall now see that the property γ is not, even in the class of Aronszajn orderings. All our examples will be suborderings of a single C-line obtained by lexicographically ordering an Aronszajn tree of the following kind (see [**35**]): Let $\mathbb{Z}^-$ be the set of negative integers and choose a sequence

$$e_\alpha : \alpha \to \mathbb{Z}^- \quad (\alpha < \omega_1)$$

of finite-to-one maps such that for all $\alpha < \beta$, $e_\alpha =^* e_\beta \restriction \alpha$ i.e., the functions e_α and $e_\beta \restriction \alpha$ differ only on a finite number of places. For $\alpha < \omega_1$, let

$$T_\alpha(e) = \{t : \alpha \to \mathbb{Z}^- : \ t =^* e_\alpha\}$$

and let $T(e)$ be the union of $T_\alpha(e)$'s. We shall use the notation $A(e)$ for the C-line $T(e)$, $<_{lex}$. Note that $A(e)$ has a natural decomposition $\{A_\alpha(e)\}$, where $A_\alpha(e)$ is the union of $T_\xi(e)$ for $\xi < \alpha$. The tree $T(e)$ can naturally be identified with the corresponding partition tree of $A(e)$ via the correspondence which sends an element t of $T(e)$ into the convex set

$$\{s \in T(e) : \ t \subseteq s\}$$

of $A(e)$. Note that t is the minimal element of the set so the decomposition $\{A_\alpha(e)\}$ is δ-continuous for every countable ordinal δ.

THEOREM 8. *There exist two C-lines B_0 and B_1 with property γ whose sum $B = B_0 + B_1$ does not have this property. In fact, the square of B has an uncountable closed discrete subspace so it is neither Lindelöf nor normal.*

Proof. Choose two disjoint stationary sets S_0 and S_1 of countable limit ordinals. Let B_0 (resp. B_1) be the set of all t in $T(e)$ such that, $\ell(t)$, the length of t, is either an element of S_0 (resp. S_1) or it is an even (resp. odd) successor ordinal. Note that the decomposition $\{A_\alpha(e)\}$ when restricted to B_i is continuous at every δ in S_i, so B_i is a stationary C-line for every $i < 2$. By Theorem 6, B_0 and B_1 have property γ, but we shall prove that their product $B_0 \times B_1$ fails to have the Lindelöf property or to be a normal space.

CLAIM 1. *$B_0 \times B_1$ contains an uncountable closed discrete subspace.*

Proof. Consider the following subset of the product $B_0 \times B_1$:

$$D = \left\{ \left\langle e_\delta, e_\delta^\frown\langle -1\rangle \right\rangle : \ \delta \in S_0 \right\} .$$

Let $\langle a, b\rangle$ be an arbitrary element of the product and suppose there is a sequence $\{\delta_i\}$ of distinct elements of S_0 such that $a = \lim e_{\delta_i}$ and $b = \lim e_{\delta_i}^\frown\langle -1\rangle$. Moreover, we may assume the convergence is monotonic. If the convergence to a is from the left then by the very nature of the lexicographical ordering it must be also that

$$a = \lim_{i\to\infty} e_{\delta_i}^\frown\langle -1\rangle$$

i.e., that $a = b$ contradicting the fact that B_0 and B_1 are disjoint. It follows that we may assume that $a \subset e_{\delta_i}$ for all i. But in order that such sequences converge to a it must be that

$$\lim_{i\to\infty} e_{\delta_i}(\alpha) = -\infty$$

where α is the length of a. Since $e_{\delta_i}^\frown\langle -1\rangle$'s extend e_{δ_i}'s it follows that they also converge to a, a contradiction. This shows that D is a closed discrete subset of $B_0 \times B_1$.

Claim 2. *$B_0 \times B_1$ is not a normal space.*

Proof. By renaming the elements of the coherent sequence $\{e_\alpha\}$ assume that no e_α takes the values -2 and -1. For δ in S_0, let

$$I_\delta = \left\{ t \in B_1 : \ t \supset e_\delta^\frown\langle -2\rangle \text{ or } t \supseteq e_\delta^\frown\langle -1\rangle \right\} .$$

Then I_δ is a convex open subset of B_1 which contains $e_\delta^\frown\langle -1\rangle$. By our assumption about e_α's the sets I_δ are disjoint. So, we have a disjoint family

$$B_0 \times I_\delta \quad (\delta \in S_0)$$

which separates the closed discrete set D and the Claim will be proved if we show that D cannot be discretely separated. To see this, let $J_\delta \times K_\delta$ $(\delta \in S_0)$ be a given sequence of products of intervals which separates D. We may assume that x_δ, the left end-point of J_δ, has length $< \delta$ (since $\{A_\alpha(e) \cap B_0\}$ is continuous at δ). By the PDL there is stationary $S \subseteq S_0$ and x in B_0 such that $x_\delta = x$ for all $\delta \in S$. Since B_1 has, in particular, the Lindelöf property we can choose a complete accumulation point y of $e_\delta^\frown\langle -1\rangle$ $(\delta \in S)$ in B_1. It follows that every neighborhood of $\langle x, y\rangle$ in $B_0 \times B_1$ intersects uncountably many rectangles from the given family $J_\delta \times K_\delta$ $(\delta \in S_0)$ showing its non-discreteness. This completes the proof of Theorem 8.

It should be clear that we can make also the following variation of Theorem 8.

Theorem 9. *For every positive integer n there is a family $B_0, \dots, B_n$ of C-lines such that the sum of any subfamily of size at most n has the property γ but the sum $B = B_0 + \cdots + B_n$ does not have this property. In fact, the product $B_0 \times \cdots \times B_n$ contains an uncountable closed discrete subset so it is neither Lindelöf nor normal.*

To mention yet another striking property of this class of C-lines, set

$$T^0(e) = T(e) \cup \left\{ t^\frown\langle 0\rangle : \ t \in T(e) \right\} .$$

Let $A^0(e)$ be the C-line obtained by lexicographically ordering the tree $T^0(e)$. What we get now is that for every limit node t of $T^0(e)$ (which actually must be an element of $T(e)$), the convex set

$$I_t = \left\{ s \in T^0(e) : \ t \subseteq s \right\}$$

has both a minimal and a maximal element. In fact, $t = \min I_t$ and $t^\frown\langle 0\rangle = \max I_t$. It follows that the naturally defined decomposition $\{A^0_\alpha(e)\}$ is both left and right continuous at every limit level δ. The results (and proofs) of the previous section show that both of the arrow topologies $\leftarrow$ and $\rightarrow$ on $A^0(e)$ have property γ, so we have the following

Theorem 10. *There is a C-line A such that both of its arrow topologies have property γ.*

Remarks. The examples of this section can serve as linearly ordered examples distinguishing the property γ from the other weaker covering properties which are usually summable. At the same time these lines show nonproductiveness of all these properties. For example, consider the weakest such property considered in this paper, the Lindelöf property. Then Theorem 8 says that there exist two C-lines B_0 and B_1 which have the Lindelöf property in all finite powers but $B_0 \times B_1$ fails to be even a normal space. Note also that Theorem 9 says that for every positive integer n there is a C-line B such that B^n is Lindelöf but B^{n+1} is not normal. Also, Theorem 10 says that there is a C-line A such that $(A, \leftarrow)$ and $(A, \rightarrow)$ are Lindelöf in all finite powers while the diagonal of A^2 is an uncountable closed discrete subset of the product $(A, \leftarrow) \times (A, \rightarrow)$. These are the first *linearly orderable* examples of this sort in the rather rich and old subject of the preservation of paracompactness and the Lindelöf property in products. One of the first classical examples in this subject is the arrow space of the real line. Its square contains the second diagonal as a closed discrete set and it is not a normal space (see [**31**]). To show that higher powers can also behave unpredictably E. Michael [**19**] has constructed using the Continuum Hypothesis some very special sets of reals whose arrow spaces do show the complex behavior in all finite powers. It is now known that such sets of reals cannot be constructed without additional axioms (see [**36**; §8]), so the C-line of our Theorem 10 is a reasonable substitute for these objects. Michael [**19**] also considers certain subspaces of the Alexandroff duplicate of $\mathbb{R}$ (see [**1**]). These are the well known Michael's lines (not linearly orderable!). It is interesting that the key idea behind Michael's lines is again the classical idea of concentrated sets of reals of A. Besicovitch [**6**] originated in the course of study of strong measure zero sets.

4. A-trees

An *Aronszajn tree* (A-tree in short) is a tree of height ω_1 which has countable levels but no uncountable chains i.e., the tree witnessing the failure of "König uncountability lemma". They are the partition trees of A-lines and the only way one gets these lines. In [**34**] and [**37**] we have shown that with the right topology they can serve as a source of very interesting examples. To define the topology let us say that a node t of a tree T is *isolated* if either t has an immediate predecessor (i.e, it is of a successor height) or it is a limit node and there is $s \neq t$ in T having the same set of strict predecessors. Let T^0 denote the set of isolated nodes of T. The topology of T that we consider is the topology generated by the family

$$V_t = \{s \in T : t \leq s\} \quad (t \in T^0)$$

as clopen subbasis. Thus a basic open neighborhood of a node t looks like this

$$B_s^F(t) = \{u \in T : u \geq s \text{ and } u \not\geq v \text{ for all } v \in F\},$$

for some $s \leq t$ in T^0 and a finite set F of immediate successors of t in T. We shall say that T is *continuous* at a limit level δ if different nodes of the δth level of T

have different sets of predecessors. A tree T is *stationary* if the sets of levels on which T is continuous is stationary. The following summarizes some of the facts proved in [**34**] and [**37**] about this topology.

THEOREM 11.

(a) *Every stationary A-tree is a first countable space with property γ, i.e., every open ω-cover of T contains a sequence $\{U_n\}$ such that $T = \underline{\lim}\, U_n$.*
(b) *The point-open game played on T is not determined.*
(c) *Every continuous function from T into a separable metric space has countable range.*
(d) *If $\mathcal{F} \subseteq \mathcal{C}_p(T)$ accumulates to the constantly 0 map then it contains a sequence $\{f_n\}$ which pointwise converges to 0.*
(e) *The function space $\mathcal{C}_p(T)$ is the increasing union of a family of closed separable metric vector subspaces.*

Hence, we have all the analogues of the results proved above for the class of A-lines. The topology on the A-tree, however, seems to be more natural and flexible as some of the proofs indicate. For example, note that the decomposition of (e) is generated by a sequence of retractions r_δ for δ a limit node of continuity of T. The r_δ is simply the projection $f_\delta(t) = t \restriction \delta$ from T onto the set $T_{\leq\delta}$ of nodes of T of height $\leq \delta$. The decomposition works since the proof of (c) shows that for every such continuous map f there is a level δ of continuity of T such that f is constant on any V_t for t from that level. In [**34**; p. 585] we claimed that the tree $T(e)$ (of the previous section) admits a weaker separable metric topology. This of course contradicts Theorem 11 (c) which first appeared as a Claim on p. 149 of our second paper [**37**]. It is interesting that we have overlooked the fact that these two Claims contradict each other and we thank W. Fleissner for an inquiry which made us realize this. The Aronszajn tree space of [**34**] has been recently used by K. Alster and R. Pol ([**2**], [**3**]) to answer an old question of E. Michael about the existence of a nonproductively Lindelöf space whose product with every hereditarily Lindelöf space is Lindelöf. In our terminology, K. Alster and R. Pol proved that the product of every stationary $\mathbb{R}$-embeddable (see [**33**]) A-tree with every hereditarily Lindelöf space is Lindelöf. We offer now a definite result in this direction.

THEOREM 12. *Let T be a stationary A-tree. Then the product of T with every hereditarily Lindelöf space is Lindelöf iff T contains no Souslin subtree.*

Proof. Suppose first that T has a Souslin subtree S which we may assume to be downward closed and have the property that each of its nodes has uncountably many successors in S. Choose a closed and unbounded set C of countable limit ordinals such that if $S(C)$ is the set of elements of S with heights in C, then every node of the subtree $S(C)$ has infinitely many immediate successors in $S(C)$. For every s in $S(C)$, we fix an immediate successor s^+ in the tree S (rather than $S(C)$). For s in $S(C)$, let

$$B_s = \{t \in S(C) : \ t = s \text{ or } t \geq s^+\}\ .$$

We consider $S(C)$ with topology generated by B_s $(s \in S(C))$ as clopen subbasis. We leave the reader to check that with this topology $S(C)$ is a hereditarily Lindelöf space whose product with T is not Lindelöf since it contains the diagonal of $S(C)$ as an uncountable closed discrete subset.

Suppose now that T contains no Souslin subtrees and that X is a given hereditarily Lindelöf space. Let $\mathcal{W}$ be a given open cover of $T \times X$. We need to find a countable subcover of $\mathcal{W}$. It will be easier to find a *ccc* partial ordering $\mathcal{P}^*$ which forces $\mathcal{W}$ to have a countable subcover $\dot{\mathcal{W}}_0$. Note that this will be sufficient since the set of all $W \in \mathcal{W}$ which are forced by some condition from $\mathcal{P}^*$ to be in $\dot{\mathcal{W}}_0$ will be desired countable subcover of $\mathcal{W}$.

Let $\mathcal{P}$ be the poset of all finite antichains of T and let $\mathcal{P}^*$ be the set of all sequences $\{p_i\}$ of elements of $\mathcal{P}$ such that $p_i = \emptyset$ for all but finitely many i's. Then our assumption that T contains no Souslin subtree translates into the fact that $\mathcal{P}$ and therefore $\mathcal{P}^*$ is a poset with property K (see [**33**; §9]). It is easily seen that this strong chain condition preserves the hereditary Lindelöfness of X. Note that in the forcing extension of $\mathcal{P}^*$ we still have the hypothesis about X and T satisfied and, moreover, we may take advantage of the fact that tree T is now special i.e., that there is an $f : T \to \omega$ such that $f^{-1}(i)$ is an antichain of T for all i. So, working in this extension, choose a countable elementary submodel M of some large enough structure containing all these objects such that if δ is its intersection with ω_1 then T is continuous at level δ.

Claim. *$\mathcal{W} \cap M$ covers $T \times X$.*

Proof. Let $\langle u, x\rangle$ be a given element of the product. We need to find a member of $\mathcal{W} \cap M$ which contains this point. Clearly, we may assume that u has height $\geq \delta$ since otherwise we are easily done using the Lindelöf property of X. Let t' be the projection of u on the δth level of T and let $n = f(t')$. Then we can find basic open neighborhoods $B_s^{F'}(t')$ and U_x of t' and x, respectively, such that

$$B_s^{F'}(t') \times U_x$$

is included in a member of the cover $\mathcal{W}$. Let $\mathcal{U}$ be the set of all open subsets U of X for which there exist $t \in T$ with $f(t) = n$ and a finite set F of immediate successors of t such that $B_s^F(t) \times U$ is included in a member of $\mathcal{W}$. Being definable from a sequence of elements of M, $\mathcal{U}$ is clearly an element of M. Note that U_x is a member of $\mathcal{U}$ as witnessed by t' and F'. Since X is hereditarily Lindelöf, $\mathcal{U}$ and $\mathcal{U} \cap M$ have the same unions, so we can fix a $U \in \mathcal{U} \cap M$ containing x. Then for some t in $T \cap M$ and some finite set F of immediate successors of t, the product $B_s^F(t) \times U$ is included in a member W of $\mathcal{W} \cap M$. So, the proof of the Claim is finished once we show that $B_s^F(t) \times U$ contains $\langle u, x\rangle$ which by the choice of U reduces to showing that u is a member of the basic open set $B_s^F(t)$. So, we have to check the following:

(a) $u \geq s$, and
(b) $u \not\geq v$ for all $v \in F$.

The fact (a) follows from $u \geq t' > s$. If (b) fails for some v, then we would have that $u \geq v > t$. It follows that t and t' are predecessors of u, so they must be comparable. But this contradicts the fact that $f(t) = f(t') = n$ and finishes the proof.

While Theorem 12 seems to be satisfactory result, we would still like to know more about the class of spaces X for which the product $T \times X$ is Lindelöf, independently on whether T contains a Souslin subtree or not. One of the striking properties of the space $S(C)$ from the above proof is that it is *separated* i.e., that the neighborhood assignment B_s $(s \in S(C))$ has the property that $t \notin B_s$ whenever $s \not\leq t$ so in particular $S(C)$ is not a separable space. Hence, a possible candidate would be the class $\mathcal{H}$ of all hereditarily Lindelöf and hereditarily separable spaces.

THEOREM 13. *Let T be a stationary A-tree. Then for every X in $\mathcal{H}$, the product $T \times X$ is Lindelöf.*

Proof. As in the proof of Theorem 12 it suffices to show that the poset $\mathcal{P}^*$ preserves the hereditary Lindelöfness of every space $X \in \mathcal{H}$. Otherwise, by the standard counting arguments we would be able to find two ω_1 sequences $\{F_\alpha\}$ and $\{(x_\alpha, V_\alpha)\}$ such that:

(a) F_α's are finite subsets of T of the same size n.
(b) if $\alpha < \beta$ then the height of every node of F_α is smaller than the height of every node from F_β,
(c) x_α's are elements of X and V_α's are their neighborhoods,
(d) if $\alpha < \beta$ and if $x_\beta \in V_\alpha$ then there must be $s \in F_\alpha$ and $t \in F_\beta$ such that $s < t$.

Our assumption about X, in particular means that we can find an uncountable set $I \subseteq \omega_1$ such that for every finite $J \subseteq I$ the set

$$P_J = \{\beta : x_\beta \in V_\alpha \text{ for all } \alpha \in J\}$$

is uncountable. Let $\mathcal{F}$ be a uniform ultrafilter on ω_1 containing the sets P_J ($J \in [I]^{<\omega}$). By (d) and by some counting there exist $1 \leq i,\, j \leq n$ and uncountable $I_0 \subseteq I$ such that for every α in I_0, there is a set $R_\alpha \in \mathcal{F}$ such that for every β in R_α, the ith element of F_α is less than the jth element of F_β. (Here we are assuming to have fixed in advance an enumeration $\{t_{\alpha\ell}\}_{\ell=1}^{n}$ of F_α for every α.) Let

$$Z = \{t_{\alpha i} :\ \alpha \in I_0\}\ .$$

Then every two elements $t_{\xi i}$ and $t_{\eta i}$ of Z are comparable for they are dominated by $t_{\beta j}$ for every $\beta \in R_\xi \cap R_\eta$. It follows that Z is an uncountable chain of T, a contradiction.

COROLLARY 14. *If T is a stationary A-tree then $T \times X$ is Lindelöf for every separable metric space X.*

We finish this section with a remark that similar preservation results can be proved for the corresponding class of stationary A-lines.

5. Homogeneity of $T(e)$

The purpose of this section is to take advantage of the fact that the tree $T(e)$ is a homogeneous object by associating it to some structure very similar to those of the previous two sections and which will inherit their homogeneities from $T(e)$.

Let $\tilde{T}(e)$ be the set of all mappings $t : \alpha \to \mathbb{Z}^-$ such that α is a countable ordinal and such that $t \restriction \xi$ is an element of $T(e)$ for all $\xi < \alpha$ or in other words, $t \restriction \xi =_* e_\xi$ for all $\xi < \alpha$. Then $\tilde{T}(e)$ ordered lexicographically is a complete linearly ordered set. It is not continuous since it contains many gaps of the form

$$\Big[t^\frown\langle m\rangle^\frown\langle -1,-1,\ldots\rangle,\ t^\frown\langle m+1\rangle\Big]$$

where $t \in T(e)$ and $m \le -2$. To get a continuum we remove the left-hand points of all the gaps of $\tilde{T}(e)$. Let $\tilde{A}(e)$ denote so obtained linearly ordered continuum. It is a first countable Aronszajn continuum (see [**33**]) which can be represented as an increasing union of an ω_1-sequence of Cantor sets

$$\tilde{A}_{\delta+1}(e) = \Big\{t \in \tilde{A}(e) :\ \ell(t) \le \delta\Big\}$$

where δ is a countable limit ordinal. To see this note that for every such δ, the level $T_\delta(e)$ of the tree $T(e)$ is order-dense in $\tilde{A}_{\delta+1}(e)$. To state our next result, recall that an ordered continuum A is called *homogeneous* if for every two pairs $x_0 < x_1$ and $y_0 < y_1$ of non-endpoints of A there exists an order-isomorphism σ such that $\sigma(x_i) = y_i$ for all $i < 2$. Also, we say that A is *reversible* if $(A, \le)$ is order-isomorphic to its reverse $(A, \ge)$. For example, the unit interval is a homogeneous reversible continuum.

THEOREM 15. *$\tilde{A}(e)$ is a homogeneous nonreversible first countable continuum of weight $\aleph_1$.*

Proof. Let us first prove that there is no order-reversing bijection $\sigma : \tilde{A}(e) \to \tilde{A}(e)$. Note that no member of the tree $T(e)$ has been removed from $\tilde{T}(e)$ in its transition to $\tilde{A}(e)$ so it makes sense looking at the images of members of $T(e)$. Since $T(e)$ ordered lexicographically is an A-line the images of its elements cannot be bounded in length. So for every limit ordinal δ we can fix t_δ in $T(e)$ of length $\ge \delta$ such that $\sigma(t_\delta)$ has length $> \delta$. Thus for every such δ there exist $f(\delta) < \delta$ which bounds all $\xi < \delta$ such that $t_\delta(\xi) \ne \sigma(t_\delta)(\xi)$. By the PDL there is uncountable set S and $\gamma < \omega_1$ such that $f(\delta) = \gamma$ for all $\delta \in S$. We may also assume that all the t_δ's for $\delta \in S$ have the same restriction on γ and similarly that all the $\sigma(t_\delta)$'s have the same restriction on γ. Moreover, we may assume that $t_\delta \restriction \delta$ $(\delta \in S)$ and $\sigma(t_\delta) \restriction \delta$ $(\delta \in S)$ are two antichains of $T(e)$. Consider two $\delta < \varepsilon$ in S and suppose that, for example, $t_\delta <_{lex} t_\varepsilon$, i.e., that $t_\delta(\xi) < t_\varepsilon(\xi)$ where ξ is the minimal ordinal where t_δ and t_ε disagree. Note that $\gamma \le \xi < \delta$. Let $\bar{\xi}$ be the minimal ordinal where $\sigma(t_\delta)$ and $\sigma(t_\varepsilon)$ disagree. Then again we must have that $\gamma \le \bar{\xi} < \delta$. By the choice of the pressing down map f, we have that

$$\sigma(t_\delta) \restriction [\gamma,\delta) = t_\delta \restriction [\gamma,\delta) \quad \text{and} \quad \sigma(t_\varepsilon) \restriction [\gamma,\varepsilon) = t_\varepsilon \restriction [\gamma,\varepsilon)\ .$$

It follows that $\bar{\xi} = \xi$ and that

$$\sigma(t_\delta)(\bar{\xi}) = t_\delta(\bar{\xi}) < t_\varepsilon(\bar{\xi}) = \sigma(t_\varepsilon)(\bar{\xi})$$

i.e., that $\sigma(t_\delta) <_{lex} \sigma(t_\varepsilon)$ contradicting the assumption that σ is order-reversing.

To show that $\tilde{A}(e)$ is homogeneous fix two pairs $x_0 < x_1$ and $y_0 < y_1$ of non-endpoints of $\tilde{A}(e)$. Choose a countable limit ordinal δ strictly above the lengths of x_i's and y_i's. Thus we have Cantor set $\tilde{A}_{\delta+1}(e)$, an order-dense subset $T_\delta(e)$ of it, and $x_0 < x_1$ and $y_0 < y_1$ in $\tilde{A}_{\delta+1}(e) \setminus T_\delta(e)$. So by Cantor's theorem there is an order-isomorphism $\sigma : \tilde{A}_{\delta+1}(e) \to \tilde{A}_{\delta+1}(e)$ such that

$$\sigma'' T_\delta(e) = T_\delta(e)$$

and such that $\sigma(x_i) = y_i$ for $i < 2$. Extend σ to the rest of $\tilde{A}(e)$ by the formula

$$\sigma(t) = \sigma(t \restriction \delta)^\frown t \restriction [\delta, \ell(t)) \ .$$

It is easily checked that this completes the proof of Theorem 15.

Remark. The first homogeneous nonreversible Aronszajn (in fact, Souslin) continuum was constructed by Jensen [**13**] (see also [**33**; p. 269]) using some special axioms of set theory. The first construction of a homogeneous nonreversible Aronszajn continuum, without any use of special axiom, was sketched by Shelah in [**30**]. Shelah's sketch is based on his construction ([**30**]) of a C-line, the first such construction appearing in the literature. Some first countable homogeneous and nonreversible continua of a quite different sort have been constructed more recently by K. Hart and J. van Mill [**12**]. Their continua have weight and cellularity larger than $\aleph_1$.

To present yet another application of the tree $T(e)$, recall that a compact Hausdorff space is called an *Eberlein compactum*, if it is homeomorphic to a weakly compact subset of a Banach space (see [**4**], [**18**]). If X is an Eberlein compactum then the Banach space $\mathcal{C}(X)$ is weakly compactly generated which amounts to the fact that $\mathcal{C}_p(X)$ contains a dense σ-compact subset. It is in this context that we were able to use $T(e)$ in [**37**] to construct some compactly generated groups with striking convergence-type properties. This of course did not exhaust the rich combinatorial structure of the object $T(e)$ for we shall now use it to consider the homogeneity restriction in the class of Eberlein compacta. Note that this is indeed a severe restriction as any homogeneous Eberlein compactum must be first countable. The first example of a nonmetrizable homogeneous Eberlein compact space was given by J. van Mill [**20**]. The weight and therefore the cellularity of his example are equal to the continuum, the maximal possible value. Since the possible weights and especially the cellularities of homogeneous compacta are still of a largely unknown nature, it seems desirable to search for, say, homogeneous Eberlein compacta of weight (and therefore cellularity) different from the two known values.

The set $\tilde{T}(e)$ has another natural topology. To see this let $T^0(e)$ be the set of all nodes of $T(e)$ of successor lengths. For t in $T^0(e)$, set

$$\tilde{V}_t = \{u \in \tilde{T}(e) : \ t \subseteq u\}$$

and consider $\tilde{T}(e)$ with the topology generated by $\tilde{V}_t$ $(t \in T^0(e))$ as a clopen subbasis. This is the path-topology of $\tilde{T}(e)$ obtained from the Cantor cube

$$\{0,1\}^{T^0(e)}$$

by identifying the elements of $\tilde{T}(e)$ with the set of all downward closed chains of the tree $T^0(e)$. It follows that $\tilde{T}(e)$ is a first countable compact retractive space.

THEOREM 16. *$\tilde{T}(e)$ is a homogeneous Eberlein compactum of weight $\aleph_1$.*

Proof. For $n \in \mathbb{Z}^-$, let P_n be the set of all t in $T^0(e)$ whose last term is equal to n. Then $\tilde{V}_t$ $(t \in P_n)$ is a point-finite family of clopen subsets of $\tilde{T}(e)$, a fact which immediately follows from the assumption that the e_α's are finite-to-one functions. It follows that

$$\tilde{V}_t \qquad (t \in P_n,\ n \in \mathbb{Z}^-)$$

is a σ-point-finite separating family of clopen subsets of $\tilde{T}(e)$. By a theorem of H. Rosenthal [**28**] this means that $\tilde{T}(e)$ is an Eberlein compactum.

To show that $\tilde{T}(e)$ is homogeneous, let x and y be given two elements of $\tilde{T}(e)$. Choose a countable limit ordinal δ strictly larger than the lengths of x and y. The set $\tilde{T}_{\leq\delta}(e)$ of all nodes of $\tilde{T}(e)$ of length $\leq \delta$ is homeomorphic to the Cantor set and $T_\delta(e)$, the δth level of $T(e)$, is one of its countable dense subsets. It follows that we can find a homeomorphism h of $\tilde{T}_{\leq\delta}(e)$ such that

(a) $h(x) = y$, and
(b) $h''T_\delta = T_\delta$.

Extend h to the rest of $\tilde{T}(e)$ by the formula

$$h(t) = h(t \restriction \delta)^\frown t \restriction [\delta, \ell(t))\ .$$

It is easily checked that this is a well-defined map and that it is indeed a homeomorphism of $\tilde{T}(e)$. This completes the proof.

Remark. Note that the separating family $\tilde{V}_t$ $(t \in T^0(e))$ of $\tilde{T}(e)$ is, in fact, σ-disjoint. To see this, note that for each $n \in \mathbb{Z}^-$, the subtree P_n of $T^0(e)$ is the union of countably many antichains

$$P_{nk} = \{t \in P_n : t \text{ has exactly } k \text{ predecessors in } P_n\}$$

where $k \in \mathbb{Z} \setminus \mathbb{Z}^-$. Hence $\tilde{V}_t(t \in P_{nk}, n \in \mathbb{Z}^-, -k \in \mathbb{Z} \setminus \mathbb{Z}^-)$ is a σ-disjoint decomposition of the separating family $\tilde{V}_t(t \in T^0(e))$ of $\tilde{T}(e)$. It follows that $\tilde{T}(e)$ is an *uniform* Eberlein compactum i.e., that it is homeomorphic to a weakly compact subset of Hilbert space (see [**5**]).

For a countable limit ordinal δ let

$$r_\delta : \tilde{T}(e) \to \tilde{T}_{\leq\delta}(e)$$

be the natural retraction: $r_\delta(t) = t \restriction \delta$, where the restriction operation is taken in its wider meaning: $t \restriction \delta = t$ if $\delta \geq \ell(t)$. Then the space of continuous real function on $\tilde{T}(e)$ admits a similar decomposition theorem to these considered in previous sections in the case of A-lines and trees.

THEOREM 17. *For every continuous real function f on $\tilde{T}(e)$ there is a countable limit ordinal δ such that $f = (f \restriction \tilde{T}_{\leq\delta}(e)) \circ r_\delta$.*

Proof. Working as in the proof of Theorem 3, consider the sets

$$Z_n = \{t \in T(e) : \mathrm{diam}\,(f''V_t) \geq 1/n\}$$

for $n = 1, 2, \ldots$ and show that each of them is countable. The conclusion of Theorem 15 follows immediately from this.

It follows that $\mathcal{C}_p(\tilde{T}(e))$ is the union of an increasing ω_1-sequence of closed subspaces homeomorphic to the function space of the Cantor set. Similar fact can be proved about the weak topology of $\mathcal{C}(\tilde{T}(e))$. Note that the fact that $\tilde{T}(e)$ is the increasing union of Cantor sets $\tilde{T}_{\leq\delta}(e)$ means, in particular, that every closed metric subspace of $\tilde{T}(e)$ is G_δ. Thus we have at the same time answered yet another metrizability question about Eberlein compacta (see [**28**, p. 109]). The homogeneous Eberlein compactum of [**20**] does not have this additional property.

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