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ON THE CONVERGENCE OF A MULTICOMPONENT ALTERNATING DIRECTION DIFFERENCE SCHEME

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Abstract. We consider a multicomponent finite–difference scheme (FDS) for solving the heat conduction equation with several variables. Some convergence rate estimates consistent with the smoothness of data are obtained.

We consider the first initial–boundary value problem (IBVP) for the heat conduction equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = \Delta u + f\,, \qquad (x,\,t) \in Q = \Omega \times (0,\,T) = (0,\,1)^n \times (0,\,T)\,, \\ &u(x,\,0) = u_0(x)\,, \qquad x \in \Omega\,, \\ &u(x,\,t) = 0, \qquad x \in \Gamma = \partial\Omega\,, \quad t \in (0,\,T)\,. \end{aligned} \tag{1}$$

We assume that the generalized solution of IBVP (1) belongs to the anisotropic Sobolev space $W_2^{s,\,s/2}(Q)$, $s \geq 1$ [1]. In this case there exist a trace $u|_{t=t'} \in W_2^{s-1}(\Omega) \subset L_2(\Omega)$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω, preserving the Sobolev class.

Let $\overline{\omega}$ be a uniform mesh in $\overline{\Omega}$ with the step size h. Let us set $\omega = \overline{\omega} \cap \Omega$, $\gamma = \overline{\omega} \setminus \omega$ and $\omega_i = \omega \cup \{x = (x_1,\,\ldots\,,\,x_n) \in \gamma \,|\, x_i = 0\}$. Let $\overline{\theta}$ be a uniform mesh on $[0,\,T]$ with the step size τ, $\theta = \overline{\theta} \cap (0,\,T)$, $\theta^- = \theta \cup \{0\}$ and $\theta^+ = \theta \cup \{T\}$. Finally, let $\overline{Q}_{h\tau} = \overline{\omega} \times \overline{\theta}$. For a function v defined on the mesh $\overline{Q}_{h\tau}$ we introduce the finite–difference operators v_{x_i}, $v_{\bar{x}_i}$, v_t and $v_{\bar{t}}$ in a usual manner [2]. Let us denote $v = v(x,\,t)$ and $\hat{v} = v(x,\,t+\tau)$.

Let H_h be the set of discrete functions defined on the mesh $\overline{\omega}$, which vanish on γ. Let us denote

$$\Lambda_i v = \begin{cases} -v_{x_i \bar{x}_i}\,, & x \in \omega \\ 0\,, & x \in \gamma \end{cases} \qquad \text{and} \qquad \Lambda v = \sum_{i=1}^{n} \Lambda_i v\ .$$

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We introduce the following discrete inner product

$$(v,\, w)_\omega = h^n \sum_{x \in \omega} v(x)\, w(x)$$

and norms

$$\|v\|_\omega = (v,\, v)_\omega^{1/2} = \Big(h^n \sum_{x\in\omega} v^2(x)\Big)^{1/2} \qquad \text{and} \qquad \|v\|_{\omega_i} = \Big(h^n \sum_{x\in\omega_i} v^2(x)\Big)^{1/2} .$$

Λ_i and Λ are linear, selfadjoint, commutative and positive operators on H_h. Therefore, the "energy" norms

$$\|v\|_{\Lambda_i} = (\Lambda_i v,\, v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i} \qquad \text{and} \qquad \|v\|_{\Lambda_i^{-1}} = (\Lambda_i^{-1} v,\, v)_\omega^{1/2}$$

can be defined.

With T_i and T_t^+ we denote the Steklov averaging operators in space variables x_i and time variable t (see [3])

$$T_i f(x,\, t) = \frac{1}{h} \int_{x_i - h/2}^{x_i + h/2} f(x_1,\, \dots,\, x_i',\, \dots,\, x_n,\, t)\, dx_i' \ ,$$
$$T_t^+ f(x,\, t) = \frac{1}{\tau} \int_t^{t+\tau} f(x_1,\, \dots,\, x_n,\, t')\, dt' \ .$$

Finally, C will stand for a generic positive constant, independent of h and τ.

We approximate IBVP (1) with the following alternating direction finite–difference scheme (see [4], [5])

$$\begin{gathered} (I + \sigma\tau\, \Lambda_i)\, v_t^i + \sum_{j=1}^n \Lambda_j v^j = \widetilde{f} \equiv T_1^2 \cdots T_n^2 T_t^+ f\, , \qquad t \in \theta^- \, , \\ v^i\big|_{t=0} = T_1^2 \cdots T_n^2 u_0\, , \qquad i = 1,\, 2,\, \dots,\, n \end{gathered} \tag{2}$$

where σ is a free weight parameter and $Iv \equiv v$. FDS (2) represents a system of n unknown mesh functions v^i. They can be determined paralelly, contrary to the other variants of the alternating direction method, such as the factorized scheme

$$(I + \sigma\tau\, \Lambda_1) \cdots (I + \sigma\tau\, \Lambda_n)\, v_t + \Lambda v = f\, .$$

The errors defined as $z^i = T_1^2 \cdots T_n^2\, u - v^i$ satisfy the FDS

$$\begin{gathered} (I + \sigma\tau\, \Lambda_i)\, z_t^i + \sum_{j=1}^n \Lambda_j z^j = \varphi^i\, , \qquad t \in \theta^- \, , \\ z^i\, \big|_{t=0} = 0\, , \qquad i = 1,\, 2,\, \dots,\, n \end{gathered} \tag{3}$$

where

$$\varphi^i = \varphi + \Lambda\chi, \qquad \chi = \sigma\tau T_1^2 \cdots T_n^2 u_t,$$
$$\varphi = \sum_{j=1}^{n} \Lambda_j \eta^j, \qquad \eta^j = \left(\prod_{l\neq j} T_l^2\right)\left(T_j^2 u - T_t^+ u\right).$$

To prove the stability and the convergence of the FDS (2) we represent the equation (3) in the matrix form (see also [6])

$$(\mathbf{I} + \sigma\,\tau\,\Lambda)\,\mathbf{z}_t + \mathbf{E}\,\Lambda\mathbf{z} = \Phi\,, \qquad t \in \theta^-\,; \qquad \mathbf{z}|_{t=0} = \mathbf{0}\,, \tag{4}$$

where $\mathbf{z} = (z^1, \ldots, z^n)^T$, $\Phi = (\varphi^1, \ldots, \varphi^n)^T$, $\mathbf{I} = \operatorname{diag}(I, \ldots, I)$, $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_n)$ and

$$\mathbf{E} = \begin{pmatrix} I & I & \ldots & I \\ I & I & \ldots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \ldots & I \end{pmatrix}.$$

Let us also define the inner product and norms of vector–functions

$$(\mathbf{z},\, \mathbf{w}) = \sum_{i=1}^{n} (z^i,\, w^i)_\omega\,, \qquad \|\mathbf{z}\| = (\mathbf{z},\, \mathbf{z})^{1/2}\,, \qquad \|\mathbf{z}\|_\Lambda = (\Lambda\mathbf{z},\, \mathbf{z})^{1/2}\,.$$

Applying operator Λ to (4) we obtain a FDS in canonical form (see [2])

$$\mathbf{B}\,\mathbf{z}_t + \mathbf{A}\,\mathbf{z} = \Psi\,, \tag{5}$$

where $\mathbf{A} = \Lambda\,\mathbf{E}\,\Lambda = \mathbf{A}^* \geq \mathbf{0}$, $\mathbf{B} = \Lambda + \sigma\,\tau\,\Lambda^2 = \mathbf{B}^* > \mathbf{0}$ i $\Psi = \Lambda\Phi$. The FDS (5) is stable for

$$\mathbf{B} - 0.5\,\tau\,\mathbf{A} > \mathbf{0}\,.$$

For $\sigma \geq n/2$ we have

$$\begin{aligned}
\big((\mathbf{B} - 0.5\,\tau\,\mathbf{A})\,\mathbf{z},\, \mathbf{z}\big) &= (\Lambda\,\mathbf{z},\, \mathbf{z}) + \sigma\,\tau\,(\Lambda\,\mathbf{z},\, \Lambda\,\mathbf{z}) - 0.5\,\tau\,(\mathbf{E}\,\Lambda\,\mathbf{z},\, \Lambda\,\mathbf{z}) \\
&= \sum_{i=1}^{n} (\Lambda_i\, z^i,\, z^i)_\omega + \sigma\,\tau \sum_{i=1}^{n} (\Lambda_i\, z^i,\, \Lambda_i\, z^i)_\omega - 0.5\,\tau \left\| \sum_{i=1}^{n} \Lambda_i\, z^i \right\|_\omega^2 \\
&= \sum_{i=1}^{n} \|z^i\|_{\Lambda_i}^2 + \tau\,(\sigma - n/2) \sum_{i=1}^{n} \|\Lambda_i z^i\|_\omega^2 + 0.5\,\tau \sum_{i=2}^{n} \sum_{j=1}^{i-1} \|\Lambda_i\, z^i - \Lambda_j\, z^j\|_\omega^2 \\
&\geq \sum_{i=1}^{n} \|z^i\|_{\Lambda_i}^2 = \|\mathbf{z}\|_\Lambda^2\,,
\end{aligned}$$

which means that

$$\mathbf{B} - 0.5\,\tau\,\mathbf{A} \geq \Lambda > \mathbf{0}\,,$$

and, consequently, FDS (5) is stable.

Using energy method, multiplying (5) by $2\,\tau\,\mathbf{z}_t$, we obtain the a priory estimate

$$\max_{t\in\theta^+}\|\mathbf{z}\|^2_{\mathbf{A}}+\tau\sum_{t\in\theta^-}\|\mathbf{z}_t\|^2_\Lambda\le C\,\tau\sum_{t\in\theta^-}\|\Phi\|^2_\Lambda\,,$$

where $\|\mathbf{z}\|^2_{\mathbf{A}}=(\mathbf{A}\,\mathbf{z},\,\mathbf{z})$, or, in expanded form

$$\|\mathbf{z}\|^2_3\equiv\max_{t\in\theta^+}\|\sum_{i=1}^n\Lambda_i z^i\|^2_\omega+\sum_{i=1}^n\tau\sum_{t\in\theta^-}\|z^i_t\|^2_{\Lambda_i}\le C\sum_{i=1}^n\tau\sum_{t\in\theta^-}\|\varphi^i\|^2_{\Lambda_i}\,. \tag{6}$$

Others standard a priori estimates (see [2]) do not hold because the operators $\mathbf{A}$ and $\mathbf{B}$ do not commute.

Further

$$\|\varphi^i\|_{\Lambda_i}\le\sum_{j=1}^n\|\eta^j_{x_j\bar{x}_jx_i}\|_{\omega_i}+\|\chi_{x_i\bar{x}_ix_i}\|_{\omega_i}\,.$$

The value of $\eta^j_{x_j\bar{x}_jx_i}$ in the node $(x,\,t)\in\omega_i\times\theta^-$ is a bounded linear functional of $u\in W_2^{s,\,s/2}(e)$, where $e=\prod_{l=1}^n(x_l-2h,\,x_l+2h)\times(t,\,t+\tau)$ and $s\ge 1$. Moreover, $\eta^j_{x_j\bar{x}_jx_i}$ vanishes on the functions of the form $u=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\,t^\beta$, $\alpha_1+\ldots+\alpha_n+2\,\beta\le 4$. Using the Bramble–Hilbert lemma, in the same manner as in [3], for $\tau\sim h^2$, we obtain

$$|\eta^j_{x_j\bar{x}_jx_i}|\le C\,h^{s-4-n/2}\,|u|_{W_2^{s,\,s/2}(e)}\,,\qquad 3\le s\le 5\,.$$

From here, by summation over the mesh, follows

$$\Big\{\tau\sum_{t\in\theta^-}\|\eta^j_{x_j\bar{x}_jx_i}\|_{\omega_i}\Big\}^{1/2}\le C\,h^{s-3}\,\|u\|_{W_2^{s,\,s/2}(Q)}\,,\qquad 3\le s\le 5\,.$$

In the same manner we can estimate $\chi_{x_i\bar{x}_ix_i}$. From these estimates and the inequality (6) follows

$$\|\mathbf{z}\|_3\le C\,h^{s-3}\,\|u\|_{W_2^{s,\,s/2}(Q)}\,,\qquad 3\le s\le 5\,. \tag{7}$$

Remark. In same cases the assumption $\tau\sim h^2$ can be omited. For example, using representations

$$\eta^j=\Big(\prod_{l=1}^n T_l^2\Big)\frac{1}{\tau}\int_t^{t+\tau}\int_{t'}^t\frac{\partial u(x,\,t'')}{\partial t}\,dt''\,dt'$$
$$+\Big(\prod_{l\ne j}T_l^2\Big)\,T_t^+\,\frac{1}{h}\int_{x_j-h}^{x_j+h}\int_{x_j'}^{x_j}(x_j''-x_j')\Big(1-\frac{|x_j'-x_j|}{h}\Big)\times$$
$$\times\,\frac{\partial^2u(x_1,\,\ldots\,,x_j'',\,\ldots\,,x_n,\,t)}{\partial x_j^2}\,dx_j''\,dx_j'\,,$$

$$\chi=\sigma\,\tau\,\Big(\prod_{l=1}^n T_l^2\Big)\,T_t^+\,\frac{\partial u}{\partial t}$$

we directly obtain

$$|\eta^j_{x_j\bar{x}_j x_i}|\,, \quad |\chi_{x_i\bar{x}_i x_i}| \le C\,\frac{h^2+\tau}{\sqrt{h^n\,\tau}}\,\|u\|_{W_2^{5,\,5/2}(e)}\,.$$

From here, similarly as in a previous case, follows

$$\|\mathbf{z}\|_3 \le C\,(h^2+\tau)\,\|u\|_{W_2^{5,\,5/2}(Q)}\,.$$

Another group of convergence rate estimates can be obtained in the following way. Applying $\Lambda_i\,(I+\sigma\tau\Lambda_i)^{-1}$ to (3), after summation on i we obtain

$$z_t + Az = \psi\,, \qquad t\in\theta^-\,; \qquad z|_{t=0}=0\,, \tag{8}$$

where

$$z=\Lambda^{-1}\sum_{i=1}^{n}\Lambda_i z^i\,, \qquad A=\sum_{i=1}^{n}A_i=\sum_{i=1}^{n}\Lambda_i(I+\sigma\tau\,\Lambda_i)^{-1}\,, \qquad \psi=\Lambda^{-1}\sum_{i=1}^{n}A_i\varphi^i\,.$$

For $\sigma\ge n/[2\,(1-\alpha)]\,,\ 0<\alpha<1\,,$ we have $0<\alpha\,I\le I-0.5\,\tau\,A\le I\,,$ so the FDS (8) is absolutelly stable.

The operators A and Λ satisfy the relations $A\le\Lambda$ and $\Lambda^{-1}\le A^{-1}$. In the case when $\tau\sim h^2$ we also have $\beta\,\Lambda_i\le A_i\,,\ \beta\,\Lambda\le A\,,\ 0<\beta<1\,.$ Using these relations, the energy method [2] and the Fourier expansion in t (see [7]) we obtain the a priory estimates

$$\|z\|_0^2 \equiv \tau\sum_{t\in\theta^-}\left\|\frac{\hat{z}+z}{2}\right\|_\omega^2 \le C\,\tau\sum_{t\in\theta^-}\|\Lambda^{-1}\psi\|_\omega^2\,, \tag{9}$$

$$\begin{aligned}\|z\|_1^2 \equiv \max_{t\in\theta^+}\|z\|_\omega^2+\tau\sum_{t\in\theta^-}\left\|\frac{\hat{z}+z}{2}\right\|_\Lambda^2+\tau^2\sum_{t,t'\in\overline{\theta},\ t\ne t'}\left\|\frac{z(\,\cdot\,,t)-z(\,\cdot\,,t')}{t-t'}\right\|_\omega^2 \\ \le C\,\tau\sum_{t\in\theta^-}\|\psi\|_{\Lambda^{-1}}^2\,,\end{aligned} \tag{10}$$

$$\|z\|_2^2 \equiv \max_{t\in\theta^+}\|z\|_\Lambda^2+\tau\sum_{t\in\theta^-}\left\|\Lambda\frac{\hat{z}+z}{2}\right\|_\omega^2+\tau\sum_{t\in\theta^-}\|z_t\|_\omega^2 \le C\,\tau\sum_{t\in\theta^-}\|\psi\|_\omega^2\,. \tag{11}$$

Further

$$\|\Lambda^{-1}\psi\|_\omega \le \sum_{j=1}^{n}\|\eta^j\|_\omega + n\,\|\chi\|_\omega\,, \tag{12}$$

$$\|\psi\|_{\Lambda^{-1}} \le \sum_{j=1}^{n}\Big(\|\eta^j_{x_j}\|_{\omega_j}+\|\chi_{x_j}\|_{\omega_j}\Big)\,, \tag{13}$$

$$\|\psi\|_\omega \le \sum_{j=1}^{n}\Big(\|\eta^j_{x_j\bar{x}_j}\|_\omega+\|\chi_{x_j\bar{x}_j}\|_\omega\Big)\,. \tag{14}$$

In such a way, the problem of deriving the convergence rate estimate for FDS (8), or (2), is now reduced to estimation of η^j, χ, $\eta^j_{x_j}$, χ_{x_j}, $\eta^j_{x_j\bar{x}_j}$ and $\chi_{x_j\bar{x}_j}$. Using the Bramble–Hilbert lemma, in the same manner as in the previous case, from (9–14) we obtain

$$\|z\|_0 \le C\, h^s\, \|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 1 \le s \le 2\,, \tag{15}$$

$$\|z\|_1 \le C\, h^{s-1}\, \|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 1 \le s \le 3\,, \tag{16}$$

$$\|z\|_2 \le C\, h^{s-2}\, \|u\|_{W_2^{s,\,s/2}(Q)}\,, \qquad 2 \le s \le 4\,. \tag{17}$$

The convergence rate estimates (7), (15–17) are consistent with the smoothnes of data. In such a way, results for standard FDSs for parabolic problems with solutions in the Sobolev classes $W_2^{s,\,s/2}$ (see [3], [7]) are extended to the new class of multicomponent alternating direction difference schemes. In [4], [5] the convergence of these schemes is proved for the problems with smooth solutions ($u \in C^{2k,\,k}$).

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