PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 56 (70), 1994, 90–94

REGULAR AND T–FREDHOLM ELEMENTS IN BANACH ALGEBRAS

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Communicated by Miroljub Jevtić

Abstract. Let $T : A \to B$ be an algebra homomorphism of a Banach algebra A to an algebra B. An element $a \in A$ is T–Fredholm [**2**] if $T(A) \in B^{-1}$ and $a \in A$ is regular [**3**] provided there is an element $a' \in A$ such that $a = aa'a$. We investigate regular and T-Fredholm elements in Banach algebras. As a corollary, we get a well known result [**5**, Theorem 3].

0. Introduction. Let A be an additive category. We say that a morphism $a \in A$ is invertible provided there is a morphism $a' \in A$ such that $aa' = 1$ and $a'a = 1$. The class of all invertible morphisms is denoted by A^{-1}. If $a \in A$, then a generalised inverse for a is a morphism $a' \in A$ such that $a = aa'a$. The regular morphisms of a category A form a class $\hat{A} = \{a \in A : a \in aAa\}$. The class of all idempotents is denoted by $\dot{A} = \{a \in A : a^2 = a\}$ [**3**, Definition 1.1]. If B is an additive category and $T : A \to B$ is an additive functor, then a morphism $a \in A$ is T–Fredholm if $T(a) \in B^{-1}$. The class of all T–Fredholm morphisms is denoted by $\Phi_T(A)$ [**3**, Definition 2.1]. We shall use $\mathbb{C}$ to denote the complex plane. For a subset S of $\mathbb{C}$, let S' denote the set of all points of accumulation of S. Let $T : A \to B$ be an algebra homomorphism of a Banach algebra A to an algebra B. For a subset M of A let $\mathrm{cl}M$ denote the closure of M. If $a \in A$, then $\sigma(a)$ denotes the spectrum of a. Recall [**1**] that an ideal I of A is inessential, if $x \in I \Leftrightarrow \sigma(x)' \subseteq \{0\}$. We allways assume that there are identities in A and B and $T(1) = 1$. Recall [**2**, 1.6] that a homomorphism T has the Riesz property if $T^{-1}(0)$ is an inessential ideal of A. Since A is a Banach algebra, then [**4**, 1.1]

$$\dot{A}A^{-1} = \hat{A} \cap \mathrm{cl}(A^{-1}). \tag{0.1}$$

In this note we investigate regular and T-Fredholm elements in Banach algebras. As a corollary, we get a well known result [**5**, Theorem 3].

AMS Subject Classification (1990): Primary 47 A 53

1. Results. Let $T : A \to B$ be an algebra homomorphism of a Banach algebra A to an algebra B.

LEMMA 1. *If T has the Riesz property and $a^2 - a \in T^{-1}(0)$, then there are $p \in \dot{A}$ and $a', a'' \in A$ such that*

$$p = a'a = aa', \qquad 1 - p = a''(1-a) = (1-a)a'', \qquad a - p \in T^{-1}(0).$$

Proof. If $a^2 - a \in T^{-1}(0)$, then $\sigma(a^2 - a)' \subseteq \{0\}$ and $\sigma(a)' \subseteq \{0, 1\}$. There are open subsets U_1 and U_2 of $\mathbb{C}$ such that $1 \in U_1, 0 \in U_2$, $\mathrm{cl}U_1 \cap \mathrm{cl}U_2 = \emptyset$ and $\sigma(a) \subseteq U_1 \cup U_2$. Now, if

$$f(z) = \begin{cases} 1, & \text{for } z \in U_1 \\ 0, & \text{for } z \in U_2 \end{cases}, \qquad h(z) = \begin{cases} 1/z, & \text{for } z \in U_1 \\ 0, & \text{for } z \in U_2 \end{cases},$$
$$f_1(z) = \begin{cases} 0, & \text{for } z \in U_1 \\ 1, & \text{for } z \in U_2 \end{cases}, \qquad h_1(z) = \begin{cases} 0, & \text{for } z \in U_1 \\ 1/(1-z), & \text{for } z \in U_2 \end{cases},$$
$$g(z) = z \qquad \text{and} \qquad g_1(z) = 1 - z \quad \text{for} \quad z \in U_1 \cup U_2,$$

then $f(a) = p$, $g(a) = a$, $h(a) = a'$, $f_1(a) = 1-p$, $g_1(a) = 1-a$, $h_1(a) = a''$, $p^2 = p$ and $p = aa' = a'a$, $1-p = a''(1-a) = (1-a)a''$. Hence $a-p = a(1-p)+(a-1)p = (a^2 - a)(a' - a'') \in T^{-1}(0)$. ■

LEMMA 2. *If T has the Riesz property, then*

$$\dot{A} + T^{-1}(0) = T^{-1}(\dot{B}). \tag{1.1}$$

Proof. Since the inclusion $\subseteq$ in (1.1) is obvious, it is enough to prove the opposite inclusion. If $T(a) \in \dot{B}$, then $T(a) = T(a^2)$ and $a^2 - a \in T^{-1}(0)$. By Lemma 1 there is a $p \in \dot{A}$ such that $a - p \in T^{-1}(0)$. Hence $a = p + (a-p) \in \dot{A} + T^{-1}(0)$. ■

THEOREM 3. *If $T(A) = B$ and T has the Riesz property, then*

$$\hat{A} + T^{-1}(0) = T^{-1}(\hat{B}). \tag{1.2}$$

Proof. The inclusion $\subseteq$ in (1.2) is obvious, so we have to prove the opposite inclusion. If $T(a) \in \hat{B}$, there is $b \in A$ such that $T(a)T(b)T(a) = T(a)$. For $c = ba$, we have

$$T(c^2 - c) = T(baba - ba) = T(b)[T(aba) - T(a)] = 0.$$

There are $p \in \dot{A}$ and $a' \in A$ such that $p = cc' = c'c$ and $p - c \in T^{-1}(0)$. We have $ap(c'b)ap = apc'cp = ap$ and $ap \in \hat{A}$. Thus $ba - p \in T^{-1}(0)$ and $T(aba - ap) = 0$, so $a - ap \in T^{-1}(0)$. We get $a = ap + (a - ap) \in \hat{A} + T^{-1}(0)$. ■

THEOREM 4. *If $T(A) = B$ and T has the Riesz property, then*

$$\dot{A}\Phi_T(A) + T^{-1}(0) = T^{-1}(\dot{B}B^{-1}). \tag{1.3}$$

Proof. The inclusion $\subseteq$ in (1.3) is obvious. To prove the opposite inclusion, suppose that $T(a) \in \dot{B}B^{-1}$. By Lemma 2, there are $b \in \dot{A}, d \in T^{-1}(0)$ and $c \in \Phi_T(A)$ such that $T(a) = T(b+d)T(c)$. We get $a - bc \in T^{-1}(0)$ and $a = bc + (a - bc) \in \dot{A}\Phi_T(A) + T^{-1}(0)$. ■

THEOREM 5. *If B is a Banach algebra, $T(A) = B$, T is continuous, has the Riesz poroperty and the norm $\|.\|_B$ on B and the quotient norm $\|.\|_q$ are equivalent, then*

$$\hat{A} \cap cl(\Phi_T(A)) + T^{-1}(0) = T^{-1}(\hat{B} \cap clB^{-1}). \tag{1.4}$$

Proof. If $a = b + c, b \in \hat{A} \cap \mathrm{cl}(\Phi_T(A))$ and $T(c) = 0$, then $T(a) = T(b) \in \hat{B}$. Hence there is a sequence $(b_n), b_n \in \Phi_T(A)$, such that $\lim b_n = b$. Thus from $T(b_n) \in B^{-1}$ and $\lim T(b_n) = T(b)$ we get $T(a) \in \mathrm{cl}B^{-1}$.

To prove the opposite inclusion in (1.4), let $T(a) \in \hat{B} \cap \mathrm{cl}B^{-1}$. Then $a = b + c, b \in \hat{A}, T(c) = 0$ (Lemma 2) and $T(a) = T(b) \in \mathrm{cl}B^{-1}$. There is a sequence (b_n), $b_n \in \Phi_T(A)$, such that $\|T(b - b_n)\|_B \to 0, n \to \infty$. Hence $\|b - b_n + T^{-1}(0)\|_q \to 0$, $n \to \infty$. Let $\epsilon > 0$ and let n be a positive integer such that $1/n < \epsilon/2$ and $\|b - b_n + T^{-1}\|_q < \epsilon/2$. There is $t \in T^{-1}(0)$ such that

$$\|b - b_n + t\| \leq \|b - b_n + T^{-1}\|_q + 1/n$$

and $\|b - b_n + t\| < \epsilon$. From $b_n - t \in \Phi_T(A)$ we have $b \in \mathrm{cl}(\Phi_T(A))$. ■

Let us remark that from (0.1), Theorem 4 and Theorem 5 we have

$$\hat{A} \cap \mathrm{cl}(\Phi_T(A)) + T^{-1}(0) = \dot{A}\Phi_T(A) + T^{-1}(0).$$

THEOREM 6. *If B is a Banach algebra, $T(A) = B$ and*

$$\dot{A}\Phi_T(A) \subseteq \hat{A}, \tag{1.5}$$

then

$$\hat{A} \cap cl(\Phi_T(A)) = \dot{A}\Phi_T(A). \tag{1.6}$$

Proof. If $a \in \hat{A} \cap \mathrm{cl}(\Phi_T(A))$, there is $a' \in A$, such that $a = aa'a$ and $a' = a'aa'$. Let $b \in \Phi_T(A)$ and $\|b - a\|\|a'\| < 1$. Then

$$1 + (b - a)a' \in A^{-1}, \qquad T(1 + (b - a)a') \in B^{-1}.$$

There is a $b' \in \Phi_T(A)$ such that $T(b)T(b') = 1$ and $bb' = 1 + t$ for some $t \in T^{-1}(0)$. If $a'' = a' + (1 - a'a)b'(1 - aa')$, then $a = aa''a$. Let us remark that from the proof of (0.1) we get $T(a'') \in B^{-1}$, hence $a'' \in \Phi_T(A)$. Thus $\hat{A} \cap \mathrm{cl}(\Phi_T(A)) \subseteq \{a \in A : a \in a\Phi_T(A)a\}$. Now if $a = asa$ and $s \in \Phi_T(A)$, then there are $s_1 \in \Phi_T(A)$ and $t_1 \in T^{-1}(0)$ such that $ss_1 = 1 + t_1$. Now $a = a(ss_1 - t_1) = as(s_1 - at_1)$, $asas = as \in \dot{A}$, $T(s_1 - at_1) = T(s_1) \in B^{-1}$ and $a \in \dot{A}\Phi_T(A)$.

To prove the opposite inclusion in (1.6), let $a = bc, b \in \dot{A}$ and $T(c) \in B^{-1}$. For $a_n = (b + (1-b)/n)c$ we have $\lim a_n = a, (b + (1-b)/n)(b + n(1-b)) = 1$ and $a_n \in \Phi_T(A)$. Hence $a \in \mathrm{cl}(\Phi_T(A))$. ■

We can not say anything about the implication

$$T \text{ has the Riesz property} \Rightarrow \dot{A}\Phi_T(A) \subseteq \hat{A}.$$

Let $S : C \to D$ be an additive functor from an additive category C to an additive category D. S is finitely regular [**3**, Definition 2.4], if $S^{-1}(0) \subseteq \hat{C}$.

Lemma 7. *If $S(C) = D$ and S is finitely regular, then $\dot{C}\Phi_S(C) \subseteq \hat{C}$.*

Proof. If $a = bc, b \in \dot{C}$ and $S(c) \in D^{-1}$, there is $c' \in \Phi_S(C)$ such that $S(c)S(c') = 1$ Hence $S(b)S(c)S(c')S(b)S(c) = S(b)S(c) \in \hat{D}$. By [**3**, Theorem 2.5], we have $a = bc \in S^{-1}(\hat{D}) = \hat{C}$. ■

Corollary 8. *If B is a Banach algebra, $T(A) = B$ and T is finitely regular, then $\hat{A} \cap cl(\Phi_T(A)) = \dot{A}\Phi_T(A)$.*

Proof follows from Lemma 7 and Theorem 6. ■

We can not conclude that [**5**, Theorem 3] follows from our Corollary 8.

Corollary 9. *If $T(A) = B$ and T is finitely regular, then $\dot{A}\Phi_T(A) \subseteq \hat{A} \cap cl(\Phi_T(A))$.*

Proof follows from Lemma 7 and Theorem 6. ■

Theorem 10. *Suppose that the inessential ideals I and J of A have the same sets of idempotents and J is closed. Let $P_0 : A \to A/I$ and $P : A \to A/J$ respectively, be the natural homomorphisms of A onto A/I and A/J. If P_0 is finitelu regular, then*

$$\hat{A} \cap \mathrm{cl}(\Phi_P(A)) = \dot{A}\Phi_P(A).$$

Proof. From Theorem 6 it follows $\hat{A} \cap \mathrm{cl}(\Phi_P(A)) \subseteq \dot{A}\Phi_P(A)$. From [**1**, Proposition 2.2] we get $\Phi_{P_0}(A) = \Phi_P(A)$ and by Lemma 7 we have $\dot{A}\Phi_{P_0}(A) \subseteq \hat{A}$. Hence $\dot{A}\Phi_P(A) \subseteq \hat{A}$ and the proof follows from Theorem 6. ■

Now, as a Corrolary, we get [**5**, Theorem 3]. Let X be an infinite-dimensional complex Banach space. We shall use $B(X)$, $F(X)$ and $K(X)$ respectively, to denote the set of all bounded, finite-rank and compact linear operators on X.

Corollary 11. *If X is a Banach space then $\widehat{B(X)} \cap$ cl $(\Phi(X)) = B(X)^{\cdot}\Phi(X)$.*

Proof. Set $I = F(X)$ and $J = K(X)$. It is well known that $F(X)$ and $K(X)$ have the same sets of idempotents and $F(X) \subseteq \widehat{B(X)}$. The proof follows by Theorem 10. ■

Acknowledgement. I am grateful to Prof. Vladimir Rakočević for helpful suggestions and conversations.

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(Received 15 03 1994)