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SOME CLASSES OF LOCALLY CONVEX RIESZ SPACES

Stojan Radenović and **Slavko Simić**

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Abstract. We define and study some new classes of locally convex Riesz spaces (order b-barrelled; order C-quasi-barrelled; order D_b and order quasi-DF).

Let (E, C, t) be a locally convex Riesz space, where (E, C) is a real vector space which is partially ordered by a cone C, such that it is a Riesz space; (E, t) is a locally convex space which has a fundamental system of neighbourhoods of O consisting of solid sets in E. A subset $B \subset E$ is solid, if for all $a, b \in E$, $a \in B$ and $\mid b \mid \leq \mid a \mid$ implies $b \in B$. A subset $B \subset E$ is order bounded if B is contained in some order interval. Else, for the definitions concerning locally convex Riesz spaces, we follow [**2**], [**9**] and [**11**].

We begin with the following definitions:

Definition 1. A barrel V in l.c.R.s. (E, C, t) is an order b-barrel if V meets all solid convex order bounded subsets in neighbourhood of O.

Definition 2. A l.c.R.s. (E, C, t) is called order b-barrelled if each order b-barrel in it is a t-neighbourhood of O.

Definition 3. A l.c.R.s. (E, C, t) is said to be order C-quasibarrelled if for each sequence A_n of t-equicontinuous subset of E' which $\sigma_S(E', E)$-converges to zero, $\cup_{n\geq 1} A_n$ is t-equicontinuous (see [**3**] for the definition of C-quasibarrelled l.c. spaces).

Definition 4. A l.c.R.s. (E, C, t) is order D_b R.s. (resp. order quasi-DF R.s.) if it is order b-barrelled (resp. order C-quasibarrelled) with a fundamental sequence of order bounded sets.

The next theorems gives us a dual characterization of the order b-barrelled and order C-quasibarrelled R.s. in terms of the order structure, in a way similar to [**2**],

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[**7**] and [**9**] for bornological, quasibarrelled, order quasibarrelled, order countably quasibarrelled, ... R.s.

THEOREM 1. *For any l.c.R.s.* (E, C, t) *the following statements are equivalent:* (a) (E, C, t) *is order b-barrelled;* (b) *Each solid order b-barrel is a t-neighbourhood of* O*;* (c) *A subset* H *of* E' *is* t*-equicontinuous, if* H_A *(its restriction) is* t*-equicontinuous for each solid convex order bounded subset* A *of* E.

Proof. (b) $\Rightarrow$ (a): Let U be an order b-barrel in l.c.R.s. (E, C, t). Then there exists a solid convex t-neighbourhood V of O, such that $U \bigcap B \supset V \bigcap B$ for each solid convex order bounded subset B of E. It follows that $\operatorname{sk}(U \bigcap B) = \operatorname{sk}(U) \bigcap B \supset V \bigcap B = \operatorname{sk}\,(V \bigcap B)$, hence, the solid kernel $\operatorname{sk}(U)$ of U is a solid order b-barrel by [**11**, Proposition 11.3] that is, U is t-neighbourhood of O. (a) $\Leftrightarrow$ (c): This follows from the fact: H_A is t-equicontinuous if and only if H^O is an order b-barrel in l.c.R.s. (E, C, t). The implication (a) $\Rightarrow$ (b) is trivial.

THEOREM 2. *For any l.c.R.s.* (E, C, t) *the following statements are equivalent:* (a) (E, C, t) *is order* C*-quasibarrelled;* (b) *for each sequence* $\{U_n\}$ *of closed absolutely convex* t*-neighbourhoods of* O*, such that every solid convex order bounded subset of* E *is contained in* $\bigcap_{n \geq m} U_n$ *for some* m*, the set* $\bigcap_{m \geq 1} U_n$ *is a* t*-neighbourhood of* O*;* (c) *for each sequence* $\{U_n\}$ *of closed solid convex* t*-neighbourhoods of* O*, such that every order bounded subset of* E *is contained in* $\bigcap_{n \geq m} U_n$ *for some* m*, the set* $\bigcap_{n \geq 1} U_n$ *is a* t*-neighbourhood of* O.

Proof. (c) $\Rightarrow$ (b): Let $\{U_n\}$ be a sequence of closed absolutely convex t-neighbourhoods of O which satisfies the condition: every solid convex order bounded subset of E is contained in $\bigcap_{n \geq m} U_n$ for some $m \in N$, that is, in $\operatorname{sk}\left(\bigcap_{n \geq m} U_n\right) = \bigcap_{n \geq m} \operatorname{sk}(U_n)$. From this it follows that $\bigcap_{n \geq 1} U_n$ is a t-neighbourhood of O, since $\bigcap_{n \geq 1} \operatorname{sk}(U_n)$ is a t-neighbourhood of O. (b) $\Rightarrow$ (a): Let $\{A_n\}$ be a sequence of t-equicontinuous subsets of E' which converges to O in the topology $\sigma_S(E', E)$, that is $A_n \subset W$ for $n \geq m(W)$, where $W = B^O$, for some closed solid convex order bounded subset B of E. From this it follows that $\bigcap_{n \geq m} A_n^O \supset B^{OO} = B$ i.e. $\bigcap_{n \geq 1} A_n^O$ is a t-neighbourhood of O. Similarly, we have that (a) $\Rightarrow$ (b). Since, the implication (b) $\Rightarrow$ (c) is trivial, the proof of the theorem is completed.

COROLLARY 1. *Using the definitions* 2 *and* 3*, Theorem* 2 (b) *or* (c) *and the definitions from* [**3**] *and* [**4**] *for the b-barrelled and* C*-quasibarrelled locally convex spaces, we easily deduce the following implications:*

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\begin{array}{ccccc}
\textit{order quasibarrelled} & \Rightarrow & \begin{array}{c}\textit{order countably}\\ \textit{quasibarrelled}\end{array} & \Rightarrow & \textit{order } \sigma\textit{-quasibarrelled} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\textit{order } b\textit{-barrelled} & \Rightarrow & \textit{order } C\textit{-quasibarrelled} & \Rightarrow & \begin{array}{c}\textit{order sequentially}\\ \textit{quasibarrelled}\end{array} \\
\Downarrow & & \Downarrow & & \Downarrow \\
b\textit{-barrelled} & \Rightarrow & C\textit{-quasibarrelled} & \Rightarrow & \textit{sequentially quasibarrelled}
\end{array}
$$

See also the table in [**7**].

In a way similar to [**4**] for the b-barrelled l.c.s, we have the following for the order b-barrelled spaces:

PROPOSITION 1. *If (E, C, t) is an order b-barrelled l.c.R.s. and A is a $\sigma_S(E', E)$-precompact subset of E', then A is t-equicontinuous.*

COROLLARY 2. *If (E, C, t) is an order b-barrelled l.c.R.s, then $(E', C', \sigma_S(E', E))$ is a semi-complete l.c.R.s.*

COROLLARY 3. *If (E, C, t) is an order b-barrelled l.c.R.s., then a subset A of E' is $\beta(E', E)$-bounded if and only if A is $\sigma_S(E', E)$-bounded.*

Proof. It is known [**11**] that if (E, C, t) is a l.c.R.s. with the topological dual E' and if $\sigma_S(E', E)$ is the locally solid topology on E' associated with $\sigma(E', E)$, then $\sigma_S(E', E)$ is coarser than the strong topology $\beta(E', E)$ (each order bounded subset of E is t-bounded). This shows that each $\beta(E', E)$-bounded subset of E' is $\sigma_S(E', E)$-bounded. Conversely, let A be a $\sigma_S(E', E)$-bounded subset of E'. By Corollary 2, A is $\sigma_S(E', E)$-strongly bounded, that is, A is $\sigma(E', E)$-strongly bounded, i.e. A is $\beta(E', E)$-bounded. (In a l.c.s. (E, t) a subset A is strongly bounded if and only if it absorbed by each t-barrel).

Remark 1. The previous corollary follows also from Corollary 1 (an order b-barrelled space is order sequentially quasibarrelled) and the Theorem 3 [**7**] but the proof is not the same (see also [**2**]).

We know from [**2**] that a l.c.R.s. (E, C, t) is a order DF space if it is order countably quasibarrelled with a fundamental sequence of order bounded subsets. Also, from [**2**] it follows that an order DF l.c.R.s. is DF-space. In the sequel we give the following results:

THEOREM 3. *Let (E, C, t) be an order D_b Riesz space. Then $(E', C', \sigma_S(E', E))$ is a Frećhet lattice and $\sigma_S(E', E) = \beta(E', E)$, i.e. each t-bounded subset of E is order-bounded.*

Proof. Since (E, C, t) contains a countable fundamental system of order bounded sets, it follows that the l.c.R.s. $(E', C', \sigma_S(E', E))$ is metrisable, i.e. bornological. By Corollary 2 it is a Frećhet lattice. It remains to show that the topology $\beta(E', E)$ is coarser than $\sigma_S(E', E)$. From [**9**, exercise 17, p. 70] it follows that if (x'_n) is a null sequence in $(E', C', \sigma_S(E', E))$, it is t-equicontinuous by Proposition 1, hence (x'_n) is $\beta(E', E)$-bounded. This shows that $\sigma_S(E', E) = \beta(E', E)$, i.e. the proof is complete.

COROLLARY 4. *If (E, C, t) is an order D_b l.c.R.s. then it is D_b l.c.R.s. i.e. (E, t) is D_b l.c.s. in sense of* [**5**].

From [**3**] it follows that a locally convex space (E, t) is a sequentially-DF space if it is sequentially quasibarrelled with fundamental sequence of t-bounded sets. We say that a l.c.R.s. (E, C, t) is an order sequentially-DF space if it is order sequentially quasibarrelled [**7**] with a fundamental sequence of order bounded sets.

THEOREM 4. *Let (E, C, t) be an order quasi-DF (resp. order sequentially-DF) l.c.R.s. Then (E, t) is quasi-DF (resp. sequentially-DF) l.c.s. in sense of* [**3**].

Proof. Similarly, as in the previous theorem, by definition 3, it folows that $\sigma_S(E', E) = \beta(E', E)$. According to the remark 1.7 [**3**] the space $(E', C', \sigma_S(E', E))$ $= (E', C', \beta(E', E))$ need not be a Frećhet lattice as in the case of order D_b spaces.

A l.c.R.s. (E, C, t) is C-quasibarrelled if (E, t) is a C-quasibarrelled l.c.s. [**3**].

PROPOSITION 2. *For any l.c.R.s. (E, C, t) the following statements are equivalent:* (a) *(E, C, t) is C-quasibarrelled;* (b) *for each sequence $\{U_n\}$ of closed absolutely convex t-neighbourhoods of O, such that every t-bounded subset of E is contained in $\bigcap_{n\geq m} U_n$ for some m, the set $\bigcap_{n\geq 1} U_n$ is a t-neighbourhood of O;* (c) *for each sequence $\{U_n\}$ of closed solid convex t-neighbourhoods of O, such that every t-bounded subset of E is contained in $\bigcap_{n\geq m} U_n$ for some m, the set $\bigcap_{n\geq 1} U_n$ is a t-neighbourhood of O.*

Proof. The proof is the same as in [**8**] for sequentially quasibarrelled Riesz spaces, only using t-neighbourhoods instead of $\sigma_S(E', E)$-neighbourhoods of O.

By the previous proposition (c) the following result follows.

THEOREM 5. *Any l-ideal in a quasi-DF (resp. C-quasibarrelled) l.c.R.s. (E, C, t) is a space of the same type, with respect to the relative topology.*

Proof. See [**7**, Theorem 8, Corollaries 4 and 5] and [**8**, Theorem 2].

Remark 2. From the theorems 3 and 4 i.e. by the Corollary 2 [**6**], it follows that the class of order D_b (resp. order quasi-DF; order sequentially-DF) l.c.R.s. is stable with respect to any l-ideal.

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Prirodno-matematički fakultet
34001 Kragujevac, p.p. 60
Yugoslavia

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