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MULTIPLIERS OF MIXED-NORM SEQUENCE SPACES AND MEASURES OF NONCOMPACTNESS

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Abstract. Let $l^{p,q}$, $1 \le p$, $q \le \infty$, be the mixed-norm sequence space. We investigate the Hausdorff measure of noncompactness of the operator $T_\lambda : l^{r,s} \mapsto l^{u,v}$, defined by the multiplier $T_\lambda(a) = \{\lambda_n a_n\}$, $\lambda = \{\lambda_n\} \in l^\infty$, $a = \{a_n\} \in l^{r,s}$, and prove necessary and sufficient conditions for T_λ to be a compact.

1. Introduction and preliminaries

A complex sequence $\{\lambda_n\}$ is of class $l^{p,q}$, $0 < p, q \le \infty$, if

$$\sum_{m=0}^{\infty} \left(\sum_{n \in I(m)} |\lambda_n|^p \right)^{q/p} < \infty, \tag{1.0.1}$$

where $I(0) = \{0\}$ and $I(m) = \{n \in N : 2^{m-1} \le n < 2^m\}$, for $m > 0$. In the case where p or q is infinite, replace the corresponding sum by a supremum. It is known that $l^{p,q}$, $1 \le p, q \le \infty$, with norm

$$\|\lambda\| = \|\lambda\|_{p,q} = \left(\sum_{m=0}^{\infty} \left(\sum_{n \in I(m)} |\lambda_n|^p \right)^{q/p} \right)^{1/q}, \tag{1.0.2}$$

is a Banach space, usually called the mixed-norm space $l^{p,q}$. Note that $l^{p,p} = l^p$, and that if p or q is infinite, then the corresponding sum should be replaced by supremum: thus

$$\|\lambda\|_{p,\infty} = \sup_m \left(\sum_{n \in I(m)} |\lambda_n|^p \right)^{1/p}. \tag{1.0.3}$$

For any two subsets E and F of l^∞, the set of multipliers from E to F (denoted by (E, F)) is the set of all $\lambda = \{\lambda_n\} \in l^\infty$ such that $\lambda a = \{\lambda_n a_n\}$ is an element of

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F for all $a = \{a_n\} \in E$. Let $T_\lambda : E \mapsto F$ be the operator defined by $T_\lambda(a) = \lambda a$, $(a \in E)$. For the convenience of a reader, recall the following well known theorem of Kellog [**6**, Theorem 1]

THEOREM 1.1. *Let $1 \le r, s, u, v \le \infty$, and define p and q by*

$$\begin{aligned} 1/p = 1/u - 1/r \quad \textit{if} \quad r > u, \qquad p = \infty \quad \textit{if} \quad r \le u, \\ 1/q = 1/v - 1/s \quad \textit{if} \quad s > v, \qquad q = \infty \quad \textit{if} \quad s \le v. \end{aligned}$$

Then $(l^{r,s}, l^{u,v}) = l^{p,q}$.

Kellog proved that the operator $T_\lambda : l^{r,s} \mapsto l^{u,v}$ defined by $T_\lambda(x) = \lambda x$, $(x \in l^{r,s})$ is a bounded linear operator and that its operator norm $\|T_\lambda\|$ is equal to $\|\lambda\|$.

If Q is a bounded subset of a metric space X, then the Hausdorff measure of noncompactness of Q, is denoted by $\chi(Q)$, and

$$\chi(Q) = \inf\{\epsilon > 0 : Q \quad \text{has a finite } \epsilon\text{-net in } X\} \tag{1.0.4}$$

The function χ is called the Hausdorff measure of noncompactness, and for its properties see [**1**], [**2**], or [**8**]. Denote by $\overline{Q}$ the closure of Q. For the convenience of the reader, let us mention that: If Q, Q_1 and Q_2 are bounded subsets of a metric space (X, d), then

$$\chi(Q) = 0 \Longleftrightarrow Q \text{ is a totally bounded set,} \tag{1.0.5}$$

$$\chi(Q) = \chi(\overline{Q}), \tag{1.0.6}$$

$$Q_1 \subset Q_2 \Longleftrightarrow \chi(Q_1) \le \chi(Q_2), \tag{1.0.7}$$

$$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}, \tag{1.0.8}$$

$$\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}. \tag{1.0.9}$$

If our space X is a Banach space, then the function $\chi(Q)$ has some additional properties connected with the linear structure. We have e.g.

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2), \tag{1.0.10}$$

$$\chi(\lambda Q) = |\lambda|\chi(Q) \quad \text{for each} \quad \lambda \in C. \tag{1.0.11}$$

If X and Y are Banach spaces, then let us denote by $B(X, Y)$ the set of all bounded linear operators from X into Y. For $A \in B(X, Y)$ the Hausdorff measure of noncompactness of A, denoted by $\|A\|_\chi$, is defined by $\|A\|_\chi = \chi(AS)$, where $S = \{x \in X : \|x\| = 1\}$ is the unit sphere in X. It is known that $\|A\|_\chi = \chi(AK)$, where $K = \{x \in X : \|x\| \le 1\}$ is the unit ball in X. Further, A is compact if and only if $\|A\|_\chi = 0$, $\|\ \|_\chi$ is a seminorm on $B(X, Y)$, and $\|A\|_\chi \le \|A\|$.

In this paper, we investigate the Hausdorff measure of noncompactness of the operator T_λ.

2. Results

We start with the following auxiliary result.

LEMMA 2.1. *Let Q be a bounded subset of $l^{p,q}$, $p \in [1,\infty]$, $q \in [1,\infty)$, and let $P_n : l^{p,q} \mapsto l^{p,q}$, $n = 1,2,\dots,$ be the operator defined by*

$$P_n(x) = (x_1, \dots, x_n, 0, \dots), \quad x = (x_m) \in l^{p,q}.$$

Then

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.1.1}$$

Proof. It is clear that $Q \subset P_nQ + (I - P_n)Q$. Now, by the elementary properties of function χ (see [**1**], [**2**], or [**8**]) we have

$$\begin{aligned}\chi(Q) &\le \chi(P_nQ) + \chi((I - P_n)Q) = \chi((I - P_n)Q)\\ &\le \sup_{x\in Q} \|(I - P_n)x\|.\end{aligned} \tag{2.1.2}$$

Since the limit in (2.1.1) obviously exists, from (2.1.2) we get

$$\chi(Q) \le \lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\|. \tag{2.1.3}$$

Hence, it is enough to proove "$\ge$" in (2.1.1). Let $\epsilon > 0$ and $\{z_1, \dots, z_k\}$ be $[\chi(Q) + \epsilon]$-net of Q. If $K = \{x \in l^{p,q} : \|x\| \le 1\}$, then it is easy to see that

$$Q \subset \{z_1, \dots, z_k\} + [\chi(Q) + \epsilon]K. \tag{2.1.4}$$

By (2.1.4), for any $x \in Q$ there are $z \in \{z_1, \dots, z_k\}$ and $s \in K$ such that $x = z + [\chi(Q) + \epsilon]s$. Thus

$$\sup_{x\in Q} \|(I - P_n)x\| \le \sup_{1\le i\le k} \|(I - P_n)z_i\| + [\chi(Q) + \epsilon]. \tag{2.1.5}$$

Now, from the choice of p and q it follows that

$$\lim_{n\to\infty} \sup_{x\in Q} \|(I - P_n)x\| \le \chi(Q) + \epsilon.$$

The lemma is proved.

Let us mention that we have not been able to prove Lemma 2.1 for $q = \infty$. Also, we have not known any formula (similar to (2.1.1)) for $\chi(Q)$, $Q \subset l^\infty$, and set it as an open problem.

Now we prove the main result of the paper.

THEOREM 2.2. *Let r, s, u, v, p and q be as in Theorem 1.1. Then, for $\lambda \in (l^{r,s}, l^{u,v}) = l^{p,q}$, we have*

$$\|T_\lambda\|_\chi = 0, \quad \text{if } v < s, \tag{2.2.1}$$

$$\|T_\lambda\|_\chi = \limsup_{n\to\infty} |\lambda_n|, \quad \text{if } s \le v < \infty \text{ and } r \le u, \tag{2.2.2}$$

$$\|T_\lambda\|_\chi = \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p}, \quad \text{if } s \le v < \infty \text{ and } r > u, \tag{2.2.3}$$

$$\frac{1}{2} \limsup_{n\to\infty} |\lambda_n| \le \|T_\lambda\|_\chi \le \limsup_{n\to\infty} |\lambda_n|, \quad \text{if } v = \infty \text{ and } r \le u, \tag{2.2.4}$$

$$\frac{1}{2} \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p} \le \|T_\lambda\|_\chi \le \limsup_{m\to\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{1/p}, \text{ if } v = \infty \text{ and } r > u. \tag{2.2.5}$$

Proof. Set $S = \{x \in l^{r,s} : \|x\| = 1\}$. To prove (2.2.1) suppose that $v < s$. If $1 \le u < r < \infty$ and $1 \le v < s < \infty$, then, by Theorem 1.1, p and q are real numbers. Now for $\lambda \in l^{p,q}$, by Lemma 2.1 we have

$$\|T_\lambda\|_\chi = \lim_{n\to\infty} \sup_{x\in S} \Big(\sum_{m=n}^{\infty} \Big(\sum_{k\in I(m)} |\lambda_k x_k|^u \Big)^{v/u} \Big)^{1/v}, \tag{2.2.6}$$

where $x = (x_1, x_2, \dots) \in S$. By the proof of [**6**, Theorem 1] we have

$$\Big(\sum_{m=n}^{\infty} \Big(\sum_{k\in I(m)} |\lambda_k x_k|^u \Big)^{v/u} \Big)^{1/v} \le \Big(\sum_{m=n}^{\infty} \Big(\sum_{n\in I(m)} |\lambda_n|^p \Big)^{q/p} \Big)^{1/q} \|x\|_{r,s}. \tag{2.2.7}$$

Now, (2.2.1) follows by (2.2.6) and (2.2.7).

Now, suppose that $1 \le u < r < \infty$ and $1 \le v < s = \infty$. Hence, $q = v$, and again by [**6**, Theorem 1] from (2.2.6) we get the inequality (2.2.7) (of course for $s = \infty$), and so (2.2.1) holds. Let us remark that all the other possibilities for r, s, u, v in the case $v < s$ could be proved in a similar way, and we omit the proof.

Let us prove (2.2.2). Now $p = q = \infty$. If L is a subset of integers, set $L(x) = L(x_i) = (x(L)_i)$, $x = (x_i) \in l^{r,s}$, where $x(L)_i = x_i$ if $i \in L$, and $x(L)_i = 0$ if $i \notin L$. Let $\epsilon > 0$. Then there is a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that

$$|\lambda_{n_k}| > \limsup_{n\to\infty} |\lambda_n| - \epsilon. \tag{2.2.8}$$

Set $M = \{n_k : k = 1, 2, \dots\}$, and let $e_i = \{\delta_{ij}\} \in l^\infty$, $i = 1, 2, \dots$. Now, for $K = \{x \in l^{r,s} : \|x\| \le 1\}$, by Lemma 2.1 we get

$$\begin{aligned} \|T_\lambda\|_\chi &= \chi(T_\lambda K) \ge \chi(T_{M(\lambda_i)} K) \ge \chi(\{M(\lambda_i) e_i : i \in N\}) \\ &\ge \limsup_{n\to\infty} |\lambda_n| - \epsilon. \end{aligned} \tag{2.2.9}$$

Hence

$$\|T_\lambda\|_\chi \ge \limsup_{n\to\infty} |\lambda_n|. \tag{2.2.10}$$

To prove the opposite inequality, suppose that $\epsilon > 0$. Then $L = \{n : |\lambda_n| > \limsup_{n\to\infty} |\lambda_n| + \epsilon\}$ is a finite set, and

$$T_\lambda(K) = T_{N\setminus L(\lambda_i)}(K) + T_{L(\lambda_i)}(K).$$

Hence

$$\chi(T_\lambda(K)) \le \chi(T_{N\setminus L(\lambda_i)}(K)) + \chi(T_{L(\lambda_i)}(K)) = \chi(T_{N\setminus L(\lambda_i)}(K)).$$

Now

$$\|T_{N\setminus L(\lambda_i)}\|_\chi = \chi(T_{N\setminus L(\lambda_i)}(K)) \le \|T_{N\setminus L(\lambda_i)}\| \le \limsup_{n\to\infty} |\lambda_n| + \epsilon,$$

and we get

$$\|T_\lambda\|_\chi \le \limsup_{n\to\infty} |\lambda_n|. \tag{2.2.11}$$

Clearly, now (2.2.2) follows from (2.2.10) and (2.2.11).

Let us prove (2.2.3). Now $p < \infty$ and $q = \infty$. If L is a subset of integers, then set $L(x) = L(x_i) = (x(L)_i)$, $x = (x_i) \in l^{r,s}$, where $x(L)_i = x_i$ if $i \in L$, and $x(L)_i = 0$ if $i \notin L$. Let $\epsilon > 0$. Then there is a subsequence $\{I(m_k)\}$ of $\{I(m)\}$ such that

$$\left(\sum_{n\in I(m_k)} |\lambda_n|^p\right)^{1/p} > \limsup_{n\to\infty}\left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p} - \epsilon,\ \ k \in N. \tag{2.2.12}$$

Set $M = \{m_k : k = 1, 2, \ldots\}$, and $c_k = \left(\sum_{n\in I(m_k)} |\lambda_n|^p\right)^{-1/r}$, $k = 1, 2, \ldots$. For each k, define the sequence $x_k(n)$, by

$$x_k(n) = \begin{cases} c_k|\lambda_n|^{p/r}, & \text{if } n \in I(m_k) \\ 0, & \text{otherwise.} \end{cases} \tag{2.2.13}$$

Now $x_k(n) \in l^{r,s}$ and $\|x_k(n)\| = 1$, $k = 1, 2, \ldots$. Further, by Lemma 2.1 we get

$$\begin{aligned} \|T_\lambda\|_\chi &= \chi(T_\lambda K) \ge \chi(T_{M(\lambda_i)}K) \ge \chi(\{M(\lambda_i)x_k : k \in N\}) \\ &\ge \limsup_{k\to\infty}\left(\sum_{n\in I(m_k)} |\lambda_n|^p\right)^{1/p} - \epsilon. \end{aligned} \tag{2.2.14}$$

Hence

$$\|T_\lambda\|_\chi \ge \limsup_{n\to\infty}\left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p}. \tag{2.2.15}$$

To prove the opposite inequality, suppose that $\epsilon > 0$. Then

$$L \equiv \left\{m : \left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p} > \limsup_{m\to\infty}\left(\sum_{n\in I(m)} |\lambda_n|^p\right)^{1/p} + \epsilon\right\}$$

is a finite set, and

$$T_\lambda(K) = T_{N\setminus L(\lambda_i)}(K) + T_{L(\lambda_i)}(K).$$

Hence

$$\chi(T_\lambda(K)) \le \chi(T_{N\setminus L(\lambda_i)}(K)) + \chi(T_{L(\lambda_i)}(K)) = \chi(T_{N\setminus L(\lambda_i)}(K)).$$

Now

$$\|T_{N\setminus L(\lambda_i)}\|_\chi = \chi(T_{N\setminus L(\lambda_i)}(K)) \le \|T_{N\setminus L(\lambda_i)}\| \le \limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{1/p} + \epsilon,$$

and we get

$$\|T_\lambda\|_\chi \le \limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{1/p}. \tag{2.2.16}$$

Now (2.2.3) follows from (2.2.15) and (2.2.16).

Let us remark that from the proof of (2.2.11) ((2.2.16)) we get also the second inequalitity in (2.2.4) ((2.2.5)). To prove the first inequality in (2.2.4), similary as in the proof of "$\ge$" in (2.2.2) (we use the same notations) we have

$$\|T_\lambda\|_\chi = \chi(T_\lambda K) \ge \chi(T_{M(\lambda_i)}K) \ge \chi(\{M(\lambda_i)e_i : i \in N\}). \tag{2.2.17}$$

Now we can not invoke Lemma 2.1 (recall that $v = \infty$), but since

$$\|M(\lambda_i)e_i - M(\lambda_i)e_j\| \ge \limsup_{n\to\infty} |\lambda_n| - \epsilon, \qquad i \ne j,$$

by [**1**, Theorem 1.1.7 and Remark 1.3.2] we have

$$\chi(\{M(\lambda_i)e_i : i = 1, 2, \dots\}) \ge \frac{1}{2}\big(\limsup_{n\to\infty} |\lambda_n| - \epsilon\big). \tag{2.2.18}$$

Hence from (2.2.17) and (2.2.18) we have the first inequality in (2.2.4).

Finally, to prove the first inequality in (2.2.5), similary as in the proof of "$\ge$" in (2.2.3) (we use the same notations) we have

$$\|T_\lambda\|_\chi = \chi(T_\lambda K) \ge \chi(T_{M(\lambda_i)}K) \ge \chi(\{M(\lambda_i)x_k : k \in N\}). \tag{2.2.19}$$

Now, again, we can not invoke Lemma 2.1 (recall that $v = \infty$), but since

$$\|M(\lambda_i)x_i - M(\lambda_i)x_j\| \ge \limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{1/p} - \epsilon, \qquad i \ne j,$$

by [**1**, Theorem 1.1.7 and Remark 1.3.2] we have

$$\chi(\{M(\lambda_i)x_k : k \in N\}) \ge \frac{1}{2}\Bigg(\limsup_{m\to\infty}\Big(\sum_{n\in I(m)} |\lambda_n|^p\Big)^{1/p} - \epsilon\Bigg). \tag{2.2.20}$$

From (2.2.19) and (2.2.20) we have the first inequality in (2.2.5). This completes the proof of Theorem 2.2.

Now as a corollary of the above theorem we have

COROLLARY 2.3. *Let r, s, u, v, p and q be as in Theorem 1.1. Then, for $\lambda \in (l^{r,s}, l^{u,v}) = l^{p,q}$, we have:*

i) *T_λ is a compact, if $v < s$,*

ii) *T_λ is a compact $\leftrightarrow \limsup_{n\to\infty} |\lambda_n| = 0$, if $s \le v$ and $r \le u$,*

iii) *T_λ is a compact $\leftrightarrow \limsup_{m\to\infty} \left(\sum_{n\in I(m)} |\lambda_n|^p \right)^{1/p} = 0$, if $s \le v$ and $r > u$.*

Remark. Let us remark that it was observed (see [**4**, Lemma 2.4] or [**5**, Lemma 1.1.2]) that Kellog's theorem is true for $0 < r, s, u, v \le \infty$.

If X is an infinite-dimensional normed space and K is the unit ball in X, then it is known that $\chi(K) = 1$. In the next lemma we prove that it is also true in the spaces l^p, $0 < p < 1$. Recall that l^p, $0 < p < 1$ is a metric space with the metric $d(x, y) = \sum_{m=0}^{\infty} |x_n - y_n|^p$.

LEMMA 2.4. *Let Q, Q_1 and Q_2 be bounded subsets of l^p, $0 < p < 1$. Then*

$$\chi(Q) = \inf_{n\in N} \sup_{(x_k)\in Q} \sum_{i=n}^{\infty} |x_i|^p, \tag{2.4.1}$$

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2), \tag{2.4.2}$$

$$\chi(\alpha Q) = |\alpha|^p \chi(Q) \text{ for any scalar } \alpha, \tag{2.4.3}$$

$$\chi(K) = 1. \tag{2.4.4}$$

Proof. For (2.4.1) see [**7**, Theorem 4.1.] (let us remark thar this result also follows from Lemma 2.1). (2.4.2) follows from [**3**, p. 6], and (2.4.1) implies (2.4.3).

To prove (2.4.4) let us remark that clearly $\chi(K) \le 1$. If $\chi(K) = s < 1$, then we find $\epsilon > 0$ such that $s + \epsilon < 1$. Now, there is $(s+\epsilon)$-net of K, say $\{x_1, \ldots, x_k\}$. Hence

$$K \subset \bigcup_{i=1}^{k} \{x_i + (s+\epsilon)K\}, \tag{2.4.5}$$

and

$$s = \chi(K) \le \max_{1\le i\le k} \chi(\{x_i + (s+\epsilon)K\}) = (s+\epsilon)^p s. \tag{2.4.6}$$

Since $s + \epsilon < 1$, from (2.4.5) it follows $s = 0$, i.e. K is totally bounded. Hence we get a contradiction, and the proof is complete.

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REFERENCES

1. Р. Р. Ахмеров, М. И. Каменский, А. С. Потапов и др., *Меры некомпактности и уплотняющие операторы*, Наука, Новосибирск, 1986.
2. J. Banás and K. Goebl, *Measures of Noncompactness in Banach Spaces*, Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York and Basel, 1980.
3. D. Bugajewski, *Some remarks on Kuratowski's measure of noncompactness in vector space with metric*, Comment. Math. Prace Mat. **32** (1992), 5–9.
4. M. Jevtić and M. Pavlović, *On multipliers from* H^p *to* l^q, $0 < q < p < 1$, Arch. Math. **56** (1991), 174–180.
5. I. Jovanović, *Množitelji i dekompozicije prostora analitičkih funkcija sa mešovitim normama*, Doktorska disertacija, Univerzitet u Beogradu, Beograd, 1992.
6. C. N. Kellog, *An extension of the Hausdorff–Young theorem*, Michigan Math. J. **18** (1971), 121–127.
7. E. De Pascale, G. Trombetta and H. Weber, *Convexly totally bounded and strongly totally bounded sets. Solution of a problem of Idzik*, Ann. Scoula Norm. Sup. Pisa Cl. Sci. **20** (1993), 341–355.
8. V. Rakočević, *Funkcionalna analiza*, Naučna knjiga, Beograd, 1994.

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