PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 55 (69), 1994, 149–155

ON THE ESTIMATES OF THE CONVERGENCE RATE OF THE FINITE DIFFERENCE SCHEMES FOR THE APPROXIMATION OF SOLUTIONS OF HYPERBOLIC PROBLEMS, II part

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Abstract. Some new estimates of the convergence rate for hyperbolic initial–boundary value problems are obtained using the interpolation theory of the function spaces.

1. Introduction

For the finite difference arproximation to the solution of vibrating string equation, in [**1**] fractional order convergence rate estimates are obtained, e.g.

$$\|\overline{z}\|_{C_\tau(W^1_{2,h})} \leq C\,(h+\tau)^{\frac{2}{3}(s-1)}\,\|u_0\|_{W_2^s}\,, \qquad 1 \leq s \leq 4$$

(see also [**4**]). Here we show how the similar estimates in the other discrete norms can be obtained.

In the sequel we use the notations introduced in [**1**]. When we quote a formula from [**1**] we add roman I to the corresponding number, e.g. (I.3).

As the model problem we consider the first initial–boundary value problem (IBVP) for the vibrating string equation, in the domain $Q = (0,1) \times (0,T]$:

$$\begin{aligned} &\partial^2 u/\partial t^2 = \partial^2 u/\partial x^2, && (x,t) \in Q, \\ &u(0,t) = u(1,t) = 0, && t \in [0,t], \\ &u(x,0) = u_0(x), && \partial u(x,0)/\partial t = 0, \qquad x \in (0,1). \end{aligned} \tag{1}$$

In the sequel, we assume that $u_0(x) \in W_2^s(0,1)$, $s \geq 0$, and can be oddly extended for $x<0$ and $x>1$, preserving the class.

AMS Subject Classification (1985): Primary 65 N 10
Supported by the Scientific Fund of Serbia, grant number 0401F

Let us note that the formula (I.3) holds for $k=0$. Indeed, let us multiply the equation (1) by the function

$$w(x,t)=x\int_0^1\int_0^\xi u(\eta,t)\,d\eta\,d\xi-\int_0^x\int_0^\xi u(\eta,t)\,d\eta\,d\xi,$$

satisfying the conditions

$$\frac{\partial^2 w}{\partial x^2}=-\,u(x,t),\qquad w(0,t)=w(1,t)=0.$$

Integrating the obtained equation, after some simple transformations, we obtain:

$$\max_{t\in[0,1]}\Big(\Big\|\frac{\partial^2 w}{\partial t\,\partial x}\Big\|^2_{L_2}+\|u\|^2_{L_2}\Big)=\|u_0\|^2_{L_2}\,,$$

and

$$\|u\|_{C(L_2)}\le\|u_0\|_{L_2}\,. \tag{2}$$

Let $\overline{\omega}_h$ be a uniform mesh on $[0,1]$ with the step–size h, and $\overline{\omega}_\tau$ — a uniform mesh on $[-\tau/2,T]$ with the step–size τ. In addition to the norms defined in [**1**], let us set:

$$\|v\|_{W^{-1}_{2,h}}=\Big\{h\sum_{x\in\omega_h^-}\Big[h^2\sum_{\xi\in\omega_h}\sum_{\substack{\eta\in\omega_h\\ \eta\le\xi}}v(\eta)-h\sum_{\substack{\eta\in\omega_h\\ \eta\le x}}v(\eta)\Big]^2\Big\}^{1/2},$$

$$\|v\|_{W^2_{2,h}}=\big(\|v\|^2_h+|[v_x\|^2_h+\|v_{x\overline{x}}\|^2_h\big)^{1/2},\qquad \|v\|_{C_\tau(W^2_{2,h})}=\max_{t\in\omega_\tau^-}\|v(\,\cdot\,,t)\|_{W^2_{2,h}}\,,$$

$$\|v\|_{C_\tau(L_{2,h})}=\max_{t\in\omega_\tau^-}\|v(\,\cdot\,,t)\|_{L_{2,h}}\,,\qquad |[v\|_{L_{1,\tau}(L_{2,h})}=\tau\sum_{t\in\omega_\tau}|[v(\,\cdot\,,t)\|_{L_{2,h}}\,.$$

Finally, let C be the positive generic constant, independent of h and τ.

2. Estimates in the second order norm

Let us aproximate the IBVP (1) with the standard symmetric weighted finite–difference scheme (FDS) [**2**]

$$v_{t\overline{t}}=[\sigma\,\hat v+(1-2\sigma)\,v+\sigma\,\check v]_{x\overline{x}},\qquad x\in\omega_h,\quad t\in\omega_\tau, \tag{3}$$

$$v(0,t)=v(1,t)=0,\qquad t\in\overline{\omega}_\tau\,, \tag{4}$$

$$v^0=v^1=u_0(x),\qquad x\in\omega_h\,. \tag{5}$$

Using the energy method we easilly obtain the equality:

$$\big(N_2(v)\big)^2\equiv|[v_{tx}\|^2_h+\tau^2\,(\sigma-0.25)\,\|v_{tx\overline{x}}\|^2_h+\|\overline{v}_{x\overline{x}}\|^2_h=\|v^0_{x\overline{x}}\|^2_h\,.$$

The expression $N_2(v)$ is the norm for $\sigma \geq 1/4$, while for $\sigma < 1/4$ it is the norm if

$$\tau \leq h \sqrt{\frac{1-c_0}{1-4\,\sigma}}\,, \qquad c_0 = \text{const} \in (0,1).$$

Further,

$$\|v^0_{x\overline{x}}\|_h = \|u_{0,x\overline{x}}\|_h = \Big\{ h \sum_{x\in\omega_h} \Big[\frac{1}{h} \int_{x-h}^{x+h} \Big(1 - \frac{|\xi - x|}{h} \Big) u_0''(\xi)\, d\xi \Big]^2 \Big\}^{1/2} \leq \frac{2}{\sqrt{3}} \|u_0''\|_{L_2}$$

and

$$\max_{\tau\in\omega_\tau^-} N_2(v) \leq C\, \|u_0\|_{W_2^2}\,. \tag{6}$$

The error $z = u - v$ satisfies the conditions

$$z_{t\overline{t}} = [\sigma\,\hat{z} + (1-2\sigma)\, z + \sigma \check{z}]_{x\overline{x}} + \psi, \qquad x \in \omega_h\,, \quad t \in \omega_\tau\,, \tag{7}$$

$$z(0,t) = z(1,t) = 0, \qquad t \in \overline{\omega}_\tau\,, \tag{8}$$

$$z^0 = z^1 = u(x, \tau/2) - u_0(x), \qquad x \in \omega_h\,, \tag{9}$$

where $\quad \psi = u_{t\overline{t}} - [\sigma\,\hat{u} + (1-2\sigma)\, u + \sigma\,\check{u}]_{x\overline{x}}\,.$

The a priori estimate

$$\max_{t\in\omega_\tau^-} N_2(z) \leq \|z^0_{x\overline{x}}\|_h + \frac{1}{\sqrt{c}}\, |[\psi_x\|_{L_{1,\tau}(L_{2,h})} \tag{10}$$

holds, where $c = 1$ for $\sigma \geq 1/4$, and $c = c_0$ for $\sigma < 1/4$.

Using the representations of z^0 and ψ obtained in [**1**] we easily obtain:

$$z^0_{x\overline{x}} = \frac{1}{h} \int_{x-h}^{x+h} \int_0^{\tau/2} \int_0^t \Big(1 - \frac{|x-\xi|}{h} \Big) \frac{\partial^4 u(\xi,\eta)}{\partial t^2\, \partial x^2}\, d\eta\, dt\, d\xi$$

and

$$\psi_x(x,t) =$$

$$- \frac{1}{h^2\tau} \int\limits_{x-h}^{x+h} \int\limits_{x}^{\xi} \int\limits_{\eta}^{\eta+h} \int\limits_{t-\tau}^{t+\tau} (\xi-\eta) \Big(1 - \frac{|\xi - x|}{h} \Big) \Big(1 - \frac{|\chi - t|}{\tau} \Big) \frac{\partial^5 u(\zeta,\chi)}{\partial x^3 \partial t^2}\, d\chi\, d\zeta\, d\eta\, d\xi$$

$$+ \frac{1}{h^2\tau} \int\limits_{x-h}^{x+h} \int\limits_{\xi}^{\xi+h} \int\limits_{t-\tau}^{t+\tau} \int\limits_{t}^{\zeta} (\zeta-\chi) \Big(1 - \frac{|\xi - x|}{h} \Big) \Big(1 - \frac{|\zeta - t|}{\tau} \Big) \frac{\partial^5 u(\eta,\chi)}{\partial x^3 \partial t^2}\, d\chi\, d\zeta\, d\eta\, d\xi$$

$$- \frac{\sigma\tau}{h^2} \int\limits_{x-h}^{x+h} \int\limits_{\xi}^{\xi+h} \int\limits_{t-\tau}^{t+\tau} \Big(1 - \frac{|\xi - x|}{h} \Big) \Big(1 - \frac{|\zeta - t|}{\tau} \Big) \frac{\partial^5 u(\eta,\zeta)}{\partial x^3 \partial t^2}\, d\zeta\, d\eta\, d\xi\,.$$

Consequently

$$\|z^0_{x\overline{x}}\|_h \le C\,\tau^2 \max_{t\in[0,T]} \Big\|\frac{\partial^4 u}{\partial t^2 \partial x^2}\Big\|_{L_2} \tag{11}$$

and

$$|[\psi_x\|_{L_{1,\tau}(L_{2,h})} \le C\,(h+\tau)^2 \max_{t\in[0,T]} \Big\|\frac{\partial^5 u}{\partial x^3\,\partial t^2}\Big\|_{L_2(0,1)}\,. \tag{12}$$

From the inequalities (10)–(12) and (I.3) we immediatly obtain the following convergence rate estimate:

$$\max_{t\in\omega_\tau^-} N_2(z) \le C\,(h+\tau)^2\,\|u_0\|_{W_2^5}\,. \tag{13}$$

On the other hand, from

$$\max_{t\in\omega_\tau^-} N_2(z) \le \max_{t\in\omega_\tau^-} N_2(u) + \max_{t\in\omega_\tau^-} N_2(v),$$

$$\max_{t\in\omega_\tau^-} N_2(u) \le C\left(\Big\|\frac{\partial^2 u}{\partial t\,\partial x}\Big\|_{C(L_2)} + \Big\|\frac{\partial^2 u}{\partial x^2}\Big\|_{C(L_2)}\right) \le C\,\|u_0\|_{W_2^2}\,,$$

and (6) follows

$$\max_{t\in\omega_\tau^-} N_2(z) \le C\,\|u_0\|_{W_2^2}\,. \tag{14}$$

Let D denote the space of functions defined on the mesh $\overline{\omega}_h \times \overline{\omega}_\tau$, with the norm $\max_{t\in\omega_\tau^-} N_2(\cdot)$, and let R be the linear mapping $u_0 \mapsto z$. From the relations (13) and (14) it follows that R is a bounded operator from W_2^5 into D, as well as from W_2^2 into D. Using the interpolation theory of function spaces [**3**] we conclude that R is a bounded linear operator from $(W_2^2,\,W_2^5)_{\theta,2}$ into D $(0<\theta<1)$, and

$$\max_{t\in\omega_\tau^-} N_2(z) \le C\,(h+\tau)^{2\,\theta}\,\|u_0\|_{(W_2^2,\,W_2^5)_{\theta,2}}\,.$$

Further [**3**], we have

$$(W_2^2,\,W_2^5)_{\theta,2} = W_2^{2\,(1-\theta)+5\,\theta} = W_2^{3\,\theta+2}.$$

Setting $3\,\theta+2=s$ we finally obtain the required convergence rate estimate for the FDS (3)–(5):

$$\max_{t\in\omega_\tau^-} N_2(z) \le C\,(h+\tau)^{\frac{2}{3}(s-2)}\,\|u_0\|_{W_2^s}\,, \qquad 2\le s\le 5. \tag{15}$$

From (15) we also obtain

$$\|z\|_{C_\tau(W_{2,h}^2)} \le C\,(h+\tau)^{\frac{2}{3}(s-2)}\,\|u_0\|_{W_2^s}\,, \qquad 2\le s\le 5. \tag{16}$$

3. Estimate in $\mathbf{L_2}$-norm

To obtain the error estimate in the case of less smooth solutions let us approximate the initial conditions with the average values:

$$v^0 = v^1 = S_x u_0, \tag{17}$$

where S_x is the Steklov smoothing operator [**1**]. The FDS (3), (4), (17) satisfies the relation

$$\big(N_0(v)\big)^2 \equiv \|v_t\|^2_{W^{-1}_{2,h}} + \tau^2\,(\sigma - 0.25)\,\|v_t\|^2_h + \|\overline{v}\|^2_h = \|v^0\|^2_h \le \|u_0\|^2_{L_2}\,. \tag{18}$$

The expression $N_0(v)$ is a norm if σ satisfies the previous conditions (in section 2).

For $s < 1/2$ the solution of the IBVP (1) may not be continuous, so we define the error by:

$$z = S_x u - v.$$

Such defined error satisfies the conditions (7) and (8), with

$$\psi = S_x\big\{u_{t\bar t} - [\sigma\,\hat u + (1 - 2\,\sigma)\,u + \sigma\,\check u]_{x\bar x}\big\},$$

and the initial conditions

$$z^0 = z^1 = S_x u(\,\cdot\,,\tau/2) - S_x u_0. \tag{19}$$

The a priori estimate

$$\max_{t\in\omega_\tau^-} N_0(z) \le \|z^0\|_h + \frac{1}{\sqrt{c}}\,\tau \sum_{t\in\omega_\tau} \|\psi_x\|_{W^{-1}_{2,h}}\,. \tag{20}$$

holds, where, as before, $c = 1$ for $\sigma \ge 1/4$, and $c = c_0$ for $\sigma < 1/4$.

From the equality (19) we easily obtain

$$z^0 = \frac{1}{h}\int_{x-h/2}^{x+h/2}\int_0^{\tau/2}\int_0^{\eta} \frac{\partial^2 u(\xi,\zeta)}{\partial t^2}\,d\zeta\,d\eta\,d\xi$$

wherefrom we get

$$\|z^0\|_h \le C\,\tau^2\,\Big\|\frac{\partial^2 u}{\partial t^2}\Big\|_{C(L_2)} \le C\,\tau^2\,\|u_0\|_{W_2^2}\,. \tag{21}$$

The expression $\|\psi_x\|_{W^{-1}_{2,h}}$ can be represented as

$$\|\psi_x\|_{W^{-1}_{2,h}} = |[M + \varphi\|_h \le |M| + |[\varphi\|_h\,,$$

where

$$M = M(t) = S_t^2\Big(h \sum_{x\in\omega_h^-} \frac{\partial u(x+h/2,\,t)}{\partial x}\Big)$$

$$= \frac{1}{\tau}\sum_{x\in\omega_h^-}\int_{t-\tau}^{t+\tau}\int_x^{x+h}\int_\xi^{x+h/2}\int_{x+h/2}^{\eta}\Big(1 - \frac{|t-\chi|}{\tau}\Big)\,\frac{\partial^3 u(\zeta,\,\chi)}{\partial x^3}\,d\zeta\,d\eta\,d\xi\,d\chi$$

and

$$
\begin{aligned}
\varphi =& S_x \big[\sigma\,\hat u + (1-2\,\sigma)\,u + \sigma\,\check u\big]_x - S_t^2\Big[\frac{\partial u(x+h/2,\,t)}{\partial x}\Big] \\
=& \frac{\sigma\tau}{h^2}\int_{x-h/2}^{x+h/2}\int_{\xi}^{\xi+h}\int_{t-\tau}^{t+\tau}\Big(1-\frac{|t-\zeta|}{\tau}\Big)\,\frac{\partial^3 u(\eta,\zeta)}{\partial t^2\,\partial x}\,d\zeta\,d\eta\,d\xi \\
&+ \frac{1}{h^2}\int_{x-h/2}^{x+h/2}\int_{x}^{\xi}\int_{x}^{\eta}\int_{\zeta}^{\zeta+h}\frac{\partial^3 u(\chi,t)}{\partial x^3}\,d\chi\,d\zeta\,d\eta\,d\xi \\
&+ \frac{1}{h\tau}\int_{x}^{x+h}\int_{t-\tau}^{t+\tau}\int_{\eta}^{t}\int_{t}^{\zeta}\Big(1-\frac{|t-\eta|}{\tau}\Big)\,\frac{\partial^3 u(\xi,\chi)}{\partial t^2\,\partial x}\,d\chi\,d\zeta\,d\eta\,d\xi \\
&+ \frac{1}{h\tau}\int_{x}^{x+h}\int_{x+h/2}^{\xi}\int_{x+h/2}^{\eta}\int_{t-\tau}^{t+\tau}\Big(1-\frac{|t-\chi|}{\tau}\Big)\,\frac{\partial^3 u(\zeta,\chi)}{\partial x^3}\,d\chi\,d\zeta\,d\eta\,d\xi\,.
\end{aligned}
$$

From these representations and (I.3) follows

$$\max_{t\in\omega_\tau^-}|M| \le C\,h^2\,\Big\|\frac{\partial^3 u}{\partial x^3}\Big\|_{C(L_2)} \le C\,h^2\,\|u_0\|_{W_2^3}\,,$$

$$\max_{t\in\omega_\tau^-}\|[\varphi\|_h \le C\,(h+\tau)^2\,\Big(\Big\|\frac{\partial^3 u}{\partial x^3}\Big\|_{C(L_2)} + \Big\|\frac{\partial^3 u}{\partial t^2\,\partial x}\Big\|_{C(L_2)}\Big) \le C\,(h+\tau)^2\,\|u_0\|_{W_2^3}\,,$$

and

$$\max_{t\in\omega_\tau^-}\|\psi\|_{W_{2,h}^{-1}} \le C\,(h+\tau)^2\,\|u_0\|_{W_2^3}\,. \tag{22}$$

Finally, from (20)–(22) we obtain the following estimate of the convergence rate for the FDS (3), (4), (17):

$$\max_{t\in\omega_\tau^-}\|\overline{z}\|_h \le \max_{t\in\omega_\tau^-} N_0(z) \le C\,(h+\tau)^2\,\|u_0\|_{W_2^3}\,. \tag{23}$$

On the other hand, from evident inequality

$$\max_{t\in\omega_\tau^-}\|\overline{z}\|_h \le \max_{t\in\omega_\tau^-}\|\overline{v}\|_h + \max_{t\in\omega_\tau^-}\|\overline{S_x u}\|_h\,,$$

and estimates (18) and (2) we obtain

$$\max_{t\in\omega_\tau^-}\|\overline{v}\|_h \le \max_{t\in\omega_\tau^-} N_0(v) = \|v^0\|_h = \|S_x u_0\|_h \le \|u_0\|_{L_2}\,,$$

and

$$\max_{t\in\omega_\tau^-}\|\overline{S_x u}\|_h \le \max_{t\in\overline{\omega}_\tau}\|S_x u\|_h \le \max_{t\in[0,\,T]}\|u\|_{L_2} \le \|u_0\|_{L_2}\,.$$

Consequently

$$\max_{t\in\omega_\tau^-}\|\overline{z}\|_h \le 2\,\|u_0\|_{L_2}\,. \tag{24}$$

From (23) and (24) by interpolation we obtain the following convergence rate estimate for the FDS (3), (4), (17):

$$\|\overline{z}\|_{C_\tau(L_{2,h})} = \max_{t \in \omega_\tau^-} \|\overline{z}\|_h \le C\,(h+\tau)^{\frac{2}{3}s}\,\|u_0\|_{W_2^s}\,, \qquad 0 \le s \le 3. \tag{25}$$

4. Fourth-order scheme

For $\tau < h$, in the same way for the fourth–order difference scheme (I.21), (I.5), (I.22) we obtain the following convergence rate estimates:

$$\max_{t \in \omega_\tau^-} N_2(z) \le C\,h^{\frac{4}{5}(s-2)}\,\|u_0\|_{W_2^s}\,, \qquad 2 \le s \le 7,$$

$$\|z\|_{C_\tau(W_{2,h}^2)} \le C\,h^{\frac{4}{5}(s-2)}\,\|u_0\|_{W_2^s}\,, \qquad 2 \le s \le 7,$$

and

$$\|\overline{z}\|_{C_\tau(L_{2,h})} \le C\,h^{\frac{4}{5}s}\,\|u_0\|_{W_2^s}\,, \qquad 0 \le s \le 5.$$

In the last case, initial condition (I.22) should be replaced by

$$v^0 = v^1 = S_x u_0 + \frac{\tau^2}{8}\,(S_x u_0)_{x\overline{x}}\,.$$

The obtained results could be extended, without difficulties, to the case of IBVP with nonhomogeneous second initial condition.

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(Received 09 02 1993)