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THE ALGEBRAIC BETHE ANSATZ AND VACUUM VECTORS

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Abstract. We give a presentation of the Algebraic Bethe Ansatz, which applies uniformly to all 4×4 rank 1 solutions of the Yang equation and uses elementary linear algebra and algebraic geometry.

1. Introduction. The Algebraic Bethe Ansatz (ABA) is a method of formal construction of eigenvectors of the Hamiltonian of the Heisenberg ferromagnetic model:

$$H = -\sum_{i=1}^{N}(X\sigma_{i+1}^x\sigma_i^x + Y\sigma_{i+1}^y\sigma_i^y + Z\sigma_{i+1}^z\sigma_i^z).$$

The operator H maps $V^{\otimes N}$ to $V^{\otimes N}$, $V = \mathbf{C}^2$; σ_i^x denotes the operator acting on i-th V as Pauli matrix and as the identity on the others. The history of the method started in 1931 with Bethe's solution of the simplest case defined by the condition $X = Y = Z$. The next step was the so called XXZ model, obtained by putting $X = Y$, solved by Yang in 1967. Finally, the general problem, the XYZ model, was solved by Baxter in 1971.

Both Yang and Baxter exploited the connection with statistical mechanics on the plane lattice, the first of them with the six-vertex and the second with the eight-vertex model. Fundamental in their work was the relation now known as the Yang equation: $\Lambda_1 = \Lambda_2$, where

$$\Lambda_1 = \Lambda_{1pq\beta}^{ij\alpha} = L_{p\beta}^{k\gamma} L'^{l\alpha}_{q\gamma} R_{kl}^{ij}, \quad \Lambda_2 = \Lambda_{2pq\beta}^{ij\alpha} = R_{pq}^{kl} L'^{i\gamma}_{k\beta} L_{l\gamma}^{j\alpha}$$

are tensors in $W\otimes V\otimes V$, with local transition matrices L and L' acting on $W\otimes V$ and R on $V\otimes V$. The key role is played by the matrices R. We denote them by R_y in the Yang case and by R_b in the Baxter case. (They are solution of the Yang –Baxter equation.) The Yang equation implies

$$R(T\otimes T') = (T'\otimes T)R,$$

AMS Subject Classification (1991): Primary 58 F 07; Secondary 82 A 69 where $T = \prod_{n=1}^{N} L_n$, $L_n: W^{\otimes N} \otimes V \to W^{\otimes N} \otimes V$ acts as L on the n-th W and V, and as identity otherwise. Therefore, the operators $\mathrm{tr}_V T$ commute. The connection between the Heisenberg model and the vertex models mentioned above is in that the operator H commutes with all $\mathrm{tr}_V T$. It is well known that H and $\mathrm{tr}_V T$ have the same eigenvectors.

A beatiful explanation how the ABA finds these eigenvectors, as well as the history of the subject, is given in Faddeev and Tahtajan [**1**]. Starting with the matrices $R = R_y$, $R = R_b$, they find the vectors X, Y, U and V satisfying the relation $RX \otimes U = Y \otimes V$. This quadruple of vacuum vectors is computed in [**1**] in terms of theta functions, the main computational tool of that paper.

In this paper we give a sort of converse approach starting from vacuum vectors. This presentation of the ABA applies uniformly to all 4×4 rank 1 solutions of the Yang equation (the definition of the rank is given in [**2**]). It does not involve computations with theta functions, but uses elementary linear algebra and algebraic geometry. Our approach, however, depends on the classification of rank 1 solutions given by Krichever in [**2**] in general situation, and following his ideas of [**2**], in the remaining cases by Dragović in [**3–5**]. We note that one of these cases represents Cherednik's R-matrix, not considered in [**1**].

2. Vacuum vector and covector representation. Krichever's method is based on "the vacuum vector representation" of an arbitrary $2n \times 2n$ matrix L:

$$LX \otimes U = hY \otimes V$$

i.e., $L^{i\alpha}_{j\beta} X_i U_\alpha = hY_j V_\beta$, $i, j = 1, \dots, n$; $\alpha, \beta = 1, 2$. For the *vacuum vectors* X, Y, U, V we shall assume the following convention: $X_n = Y_n = U_2 = V_2 = 1$, $U_1 = u$, $V_1 = v$. The vacuum vectors are parametrized by *the spectral curve* Γ, defined by

$$P(u, v) = \det L(u, v) = \det \overline{V}^{\beta} L^{i\alpha}_{j\beta} U_\alpha = 0.$$

The spectral polynomial P is of degree 2 in each variable.

In [**2–5**] it was shown that the 4×4 solutions of the Yang equation satisfy equations of the type

$$LX_l \otimes U_l = hX_{l+1} \otimes U_{l-1},$$

where X_l and $X_{l+n} = X_l \circ \Psi^n$ denote vector functions on the spectral curve Γ with certain analytical properties[1] such as that the order of poles is two. Ψ is an automorphism of Γ. In the general situation considered in [**2**] and represented by R_b the spectral curve is elliptical and Ψ is a translation on the elliptic curve (denoted in [**2**] by Krichever). As it was shown in [**3, 4**] the spectral curves might be also rational singular or reducible. Corresponding solutions are Cherednik's R-matrix and R_y; the automorphisms are some fractional-linear transformations. The automorphisms determine an additional parameter, the Planck constant, which all of these solutions contain.

[1]These properties guarantee that the matrix L is uniquely defined by the last relation

We will also consider *the covector representation* $(A^j B^\beta) L^{i\alpha}_{j\beta} = l(C^i D^\alpha)$. The next lemma gives a connection between the vector and covector representations. Let P^{ij}_L denote the determinant of the contraction of the matrix L over the i-th bottom and the j-th top index.

LEMMA 1. *If a* 4×4 *matrix* L *satisfies the condition*

$$L(X_l \otimes U_l) = h(X_{l+1} \otimes U_{l-1}),$$

then

$$(\overline{X}_{l+1} \otimes \overline{U}_{l+1})L = g(\overline{X}_{l+2} \otimes \overline{U}_l).$$

where $X = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$, $\overline{X} = [\,1 \quad -x\,]$.

Comment. We follow the convention that the second (resp. first) component of vacuum vector (covector) is normalized to unity, and the first (second) component is denoted by corresponding lower case letter.

Proof. Denoting the vacuum covectors of the matrix L by A, B, C, D we have

$$(A^j B^\beta) L^{i\alpha}_{j\beta} = l(C^i D^\alpha),$$

so $P^{12}_L(a, d) = 0$. By the condition of the lemma we have $P^{12}_L(x \circ \psi, u) = 0$. The last two equations make it possible to take $A = \overline{X} \circ \Psi$, $D = \overline{U}$. In the same way; considering the pairs (B, C) and $(U \circ \Psi, X)$ we obtain $B = \overline{U} \circ \Psi \circ \Phi$, $C = \overline{X} \circ \Phi$, where Φ is a transformation the spectral curve.

Now we compute Φ. We have $P^{21}_L(u \circ \psi, u) = 0$, $P^{21}_L(b, u) = 0$. There are two points on the curve Γ with the same second coordinate equal to $u(z)$. The corresponding first coordinates are $(u \circ \psi)(z)$, $(u \circ \psi \circ \tau_u)(z)$ where τ_u is the involution associated with u.

Thus, we have two possibilities for B:

$$B = \overline{U} \circ \Psi, \quad B = \overline{U} \circ \Psi \circ \tau_u.$$

In the first case the transformation would be identity, while in the second it would correspond to the shift $U_l \to U_{l+2}$, $X_l \to X_{l+2}$. Comparing divisors of matrix elements in the equation $A^j L^{i\alpha}_{j\beta} U_\alpha = k\overline{BX}$ we conclude that appropriate formula for B is the second one (the matrices on both sides are of demension 2, of rank 1 and with the same kernel and image).

We note that the proof of the Lemma generalizes to the case of arbitrary 4×4 matrices.

3. Local vacuum vectors. As suggested in [**1**], we change local transition matrices[2] i.e. solutions of the Yang equation $L(\lambda)$:

$$L^l_n(\lambda) = M^{-1}_{n+l}(\lambda) L_n(\lambda) M_{n+l-1} = \begin{bmatrix} \alpha^l_n(\lambda) & \beta^l_n(\lambda) \\ \gamma^l_n(\lambda) & \delta^l_n(\lambda) \end{bmatrix},$$

[2]The solutions of the Yang equation belong to one-parametar families 1–5. The parametar is usualy called spectral.

where M_l have vacuum vectors as the coloumns:

$$M_l = \begin{bmatrix} x_l & x_{l+1} \\ 1 & 1 \end{bmatrix}.$$

The monodromy matrix $T(\lambda) = \prod_{n=1}^{N} L_n(\lambda)$ then transforms into $T_N^l(\lambda) = M_{N+l}^{-1}(\lambda)\ T(\lambda)\ M_l(\lambda)$. We will denote the elements of $T_n^l(\lambda)$ by $A_N^l(\lambda)$, $B_N^l(\lambda)$, $C_N^l(\lambda)$, $D_N^l(\lambda)$.

Our goal is to find local vacuum vectors ω^l which are independent of λ and satisfy the conditions

$$\gamma_n^l(\lambda)\omega_n^l = 0, \quad \alpha_n^l(\lambda)\omega_n^l = g(\lambda)\omega_n^{l-1}, \quad \delta_n^l(\lambda)\omega_n^l = g'(\lambda)\omega_n^{l+1}.$$

Then $\Omega_N^l = \omega_1^l \otimes \ldots \otimes \omega_N^l$ would satisfy

$$A_n^l(\lambda)\Omega_n^l = g^N(\lambda)\Omega_n^{l-1}, \quad D_n^l(\lambda)\Omega_n^l = g^N(\lambda)\Omega_n^{l+1}, \quad \mathbf{C}_n^l(\lambda)\Omega_n^l = 0;$$

and therefore would form a family of generating vectors.

Theorem. *The following relations are valid:*

$$\gamma_n^l(\lambda)U_l = 0, \quad \alpha_n^l(\lambda)U_l = g(\lambda)U_{l-1}, \quad \delta_n^l(\lambda)U_l = g'(\lambda)U_{l+1}.$$

Proof. It can be easily seen that $\gamma^l(\lambda) = \overline{X}_{l+1}L(\lambda)X_l$, so

$$\gamma^l(\lambda)U_l = \overline{X}_{l+1}L(\lambda)X_lU_l = h\overline{X}_{l+1}X_{l+1}U_{l-1} = 0.$$

Similarly, $\alpha^l(\lambda)U_l = \overline{X}_{l+2}L(\lambda)X_lU_l = gU_{l-1}$. Lemma 1 gives the relation $\overline{X}_{l+1}\overline{U}_{l+1}LU_l = 0$, which implies that

$$\delta^l(\lambda)U_l = \overline{X}_{l+1}L(\lambda)X_{l+1}U_l = g'U_{l+1},$$

finishing the proof.

We also need the images of shifted vacuum vectors.

Lemma 2. *The image of a shifted vacuum vector is a combination of two shifted vacuum vectors, i.e.:*

$$LX_{l+1} \otimes U_l = hX_{l+1} \otimes U_l + gX_l \otimes U_{l+1}.$$

The proof follows from Lemma 1 by repeating arguments from the end of the proof of the Theorem.

To apply ABA one needs not only generating vectors, but also commuting relations between elements of the transfer matrices. The relations from Lemma 1 and 2 suffice to prove the following.

Proposition. *The elements of the transfer matrices commute according to the rules:*

$$B^k_{l+1}(\lambda)B^{k+1}_l(\mu) = B^k_{l+1}(\mu)B^{k+1}_l(\lambda);$$
$$B^k_{l-2}(\lambda)A^{k+1}_{l-1}(\mu) = h'A^k_l(\mu)B^{k+1}_{l-1}(\lambda) + h''B^k_{l-2}(\mu)A^{k+1}_{l-1}(\lambda);$$
$$B^{k+2}_l(\mu)D^{k+1}_{l-1}(\lambda) = k'D^k_l(\lambda)B^{k+1}_{l-1}(\mu) + k''B^{k+2}_l(\lambda)D^{k+1}_{l-1}(\mu).$$

Using the Proposition, it is not difficult to find the relations between λ_i necessary for the sum of vectors

$$\Psi_l(\lambda_1, \dots, \lambda_n) = B^{l-1}_{l+1}(\lambda_1)\dots B^{l-n}_{l+n}(\lambda_n)\Omega^{l-n}_N$$

to be an eigenvector for the operator $\mathrm{tr}T(\lambda) = A^l_l(\lambda) + D^l_l(\lambda)$ (see [**1**] for details).

4. Concluding remarks. Krichever construction gives a link between the Yang–Baxter equation and *integrable mapping* and *integrable systems with discrete time* (see [**6, 7**]). We explain it briefly.

The spectral curve defines a 2–2 relation $P_L(u,v) = 0$ in $P^1 \times P^1$. It commutes with the relation $P_{L'}(u,v) = 0$ and that is the reason why it is so powerful in the theory of the Yang–Baxter equation. The same relation arises in billiard systems within a plain quadric (see [**8**–**10**]). The involution we use in the proof of Lemma 1, induced by that relation, corresponds to the passage from x_{i-1} to x_{i+1} in the billiard, where $x_{i-1}x_i$, x_ix_{i+1} are successive billiard segments. As is well known the segments $x_{i-1}x_i$, x_ix_{i+1} lie on tangents from x_i to a particular quadric confocal with the billiard border.

The objects we consider have pretty well understood Lie-algebraic nature:

1. R-matrices we use are of A_1 type. There are R-matrices constructed for each type of Dynkin diagram (see [**11, 12**]).

2. Commuting polynomials which are integrable mappings of A_1 are Tchebishev polynomials. Commuting polynomials were recently been constructed for every root system [**13**]. It is an interesting question if they are connected with R-matrices from 1.

3. Heisenberg Hamiltonian is obviously of A_1 type. There are some generalizations for the other root systems and applications of ABA in these cases [**14**].

4. Convex polihedra generating integrable billiard systems are classified as affine Weil cells (see [**15**]). What are generalized billiards within quadrics?

The connections among all of these problems is clear on the basic A_1 level. It seems that similar connections should exist on higher levels. One of the difficulties in the generalization is that among matrices which are not of even order there are many without vacuum vectors.

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