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TRACE FORMULAS OF GELFAND–LEVITAN TYPE

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Abstract. We deduce some abstract formulas for the first and the second regularized trace of discrete operators under conditions that are easier for verification. Some known results are improved and completed as well.

Introduction. Gelfand and Levitan [**2**] proved that for the eigenvalues μ_n of the Sturm–Liouville operator

$$\begin{aligned} &-y'' + q(x)y = \lambda y \\ &y(0) = y(\pi) = 0, \qquad x \in [0,\pi],\ \ q \text{ real function},\ \ q \in C^1[0,\pi] \end{aligned}$$

the equality

$$\sum_{n\geq 1}\left[\mu_n - n^2 - \frac{1}{\pi}\int_0^\pi q(x)dx\right] = \frac{1}{2\pi}\int_0^\pi q(x)dx - \frac{1}{4}(q(0)+q(\pi)) \tag{1}$$

holds.

If we denote by T the differential operator generated by the differential expression $l(y) = -y''$ and by the boundary conditions $y(0) = y(\pi) = 0$, then $\lambda_n(T) = n^2$ and the corresponding eigenfunctions are $\varphi_n(x) = \sqrt{2/\pi}\,\sin nx$.

Denote by P the operator on $L^2(0,\pi)$ defined by $Pf(x) = q(x)f(x)$. In [**4**] it was proved that (1) can be rewritten as

$$\sum_{n\geq 1}[\mu_n - \lambda_n - (P\varphi_n,\varphi_n)] = 0 \tag{2}$$

The hypothesys was stated that formula (2) holds in the abstract case (i.e. when T is a discrete selfadjoint lower bounded operator on a separable Hilbert space $\mathcal{H}$ and P is some bounded operator).

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In [**7**] the relation (2) (in case $P = P^*$) was proved under some aditional conditions concerning the eigenvalue distribution function and under the condition that the sequence $\mu_n - \lambda_n - (P\varphi_n, \varphi_n)$ has special asymptotic behavior. But, to estimate the behavior of that sequence, it is necessary to know the asymptotic of eigenvalues of the perturbed operator $T + P$. In [**1**], [**5**] the relation (2) was proved under some conditions concerning the spectrum of the distribution function of T, but with parenthesis. Trace formulas for the powers of the Sturm-Liouville operator were obtained in [**3**].

In this paper we find the first and the second regularized trace (in the abstract case) assuming only that nonperturbed operator satisfies some conditions. Some results from [**5**] are upgraded. Also, we consider the case when the perturbation P is an unbounded operator (which is subordinated to some power of T).

In [**8**] the problem of regularized traces of higher orders was considered by the method of analytic extension, but in general case it is hard to find efectively that extension.

Main result. Let $T = T^*$ be a discrete lower bounded operator and let P be a bounded operator on a separable Hilbert space $\mathcal{H}$ ($P \in B(\mathcal{H})$). Denote by λ_k, φ_k the eigenvalues of T and the corresponding eigenvectors ($\|\varphi_k\| = 1$). Let (μ_k) denote the sequence of eigenvalues of the operator $T + P$ arranged according to growing moduli. By $a_n \asymp b_n$ we denote the fact that there exist constants $c_1 > 0$, $c_2 > 0$ such that $C_1 \leq a_n/b_n \leq C_2$

THEOREM 1. *If* $\lambda_{n+1}(T) - \lambda_n(T) \asymp n^{1/p-1}$ $(0 < p < 1)$ (λ_n *are the distinct eigenvalues of* T) *and* P *is* $B(\mathcal{H})$, *then*

$$\sum_{k\geq 1}{}' (\mu_k - \lambda_k - (P\varphi_k, \varphi_k)) = 0.$$

($\sum'$ *denotes that the terms, arising from the repeated eigenvalues* λ_n, *are grouped. If all the eigenvalues except finitely many are simple, then*

$$\sum_{k\geq 1} (\mu_k - \lambda_k - (P\varphi_k, \varphi_k)) = 0).$$

Before proving Theorem 1 we prove some Lemmas.

LEMMA 1. *Let* $\lambda_1 < \lambda_2 < \lambda_3 < \dots$, $\lambda_n \to +\infty$ *and* $0 < p < 1$. *Then* $\lambda_{n+1} - \lambda_n \asymp n^{1/p-1}$ *if and only if* $\lambda_{n+1} - \lambda_n \asymp \lambda_n^{1-p}$.

Proof. From the condition

$$C_1 n^{1/p-1} \leq \lambda_{n+1} - \lambda_n \leq C_2 n^{1/p-1} \qquad (C_1 > 0, C_2 > 0),$$

summing we get $\lambda_{n+1} - \lambda_1 \leq C_2 \int_0^n x^{1/p-1}dx$ and therefore $\lambda_n = 0(n^{1/p})$. Simlarly, we obtain $\lambda_n \geq \text{const} \cdot n^{1/p}$ (const does not depend on n). So, $\lambda_n \asymp n^{1/p}$. From

$\lambda_{n+1}-\lambda_n \asymp n^{1/p-1}$ it follows $\lambda_{n+1}-\lambda_n \asymp \lambda_n^{1-p}$. Suppose now $\lambda_{n+1}-\lambda_n \asymp \lambda_n^{1-p}$. Then we have $\lambda_{n+1}/\lambda_n \to 1$, $\lambda_{n+1}-\lambda_n \asymp \lambda_{n+1}^{1-p}$ and therefore

$$C_1(n-1) \le \sum_{\nu=1}^{n} \frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}^{1-p}} \le \int_{\lambda_1}^{\lambda_n} x^{p-1}dx \le \frac{\lambda_n^p}{p}$$

i.e. $\lambda_n \ge \text{const}\, n^{1/p}$.

Analogously we get $\lambda_n \le \text{const} \cdot n^{1/p}$. So $\lambda_n \asymp n^{1/p}$. From $\lambda_{n+1}-\lambda_n \asymp \lambda_n^{1-p}$ it follows $\lambda_{n+1}-\lambda_n \asymp n^{1/p-1}$.

Let $\Gamma_n = \{\lambda\colon |\lambda| = r_n = (\lambda_n+\lambda_{n+1})/2\}$ and $\phi(z) = \sum_{k\ge 1} |z-\lambda_k|^{-1}$. Observe that $|R_z|_1 = \phi(z)$. Here $R_z = (z-T)^{-1}$ and $|\cdot|_1$ is the nuclear norm.

Lemma 2. *If $z \in \Gamma_n$ then $|\phi(z)| \le C_3 \ln n / n^{1/p-1}$ $(0<p<1)$, where the constant C_3 does not depend on n.*

Proof. Since for $z\in\Gamma_n$, $\phi(z) \le \phi(r_n)$ and $1/(r_n-\lambda_n) \asymp n^{1/p-1}$ it is enough to prove that

$$\text{(a)} \qquad \sum_{k=n+2}^{\infty} \frac{1}{\lambda_k - \Gamma_n} \le \frac{C_4 \ln n}{n^{1/p-1}}; \qquad\qquad \text{(b)} \qquad \sum_{k=1}^{n-1} \frac{1}{r_n-\lambda_k} \le \frac{C_4 \ln n}{n^{1/p-1}};$$

where C_4 does not depend on n.

(a) We have

$$\begin{aligned} \sum_{k=n+2}^{\infty} \frac{1}{\lambda_k - r_n} &\le C_5 \sum_{k=n+2}^{\infty} \frac{\lambda_k-\lambda_{k-1}}{\lambda_k^{1-p}(\lambda_k - r_n)} \\ &\le C_5 \int_{\lambda_{n+1}}^{\infty} \frac{dx}{x^{1-p}(x-r_n)} = C_5 r_n^{p-1} h\left(\frac{\lambda_{n+1}}{r_n}\right) \end{aligned} \tag{4}$$

where

$$h(x) = \int_x^{+\infty} \frac{u^{p-1}}{u-1} du$$

and the constant C_5 does not depend on n.

From the asymptotic relation $h(x) \sim -\ln(x-1)$, $x \to 1+0$, $r_n \asymp n^{1/p}$, $\lim_{n\to\infty} \lambda_{n+1}/r_n = 1$ and (4) it follows that

$$\sum_{k=n+2}^{\infty} \frac{1}{\lambda_k - r_n} \le \text{const}\, \frac{\ln n}{n^{1/p-1}}.$$

The inequality (b) can be proved analogously. □

Lemma 3. *The operator $T+P$ is a discrete one.*

Proof. For $z \in \Gamma_n$ we have

$$\|P(z-T)^{-1}\| \le \|P\|/d(z,\sigma(T)) \le C_6/n^{1/p-1} \to 0 \qquad (n\to\infty).$$

Therefore the operator $I - P(z-T)^{-1}$ has a bounded inverse for $z \in \Gamma_n$ if n is large enough and

$$(z-T-P)^{-1} = (z-T)^{-1}\sum_{n\ge 0}\left(P(z-T)^{-1}\right)^n.$$

Since $(z-T)^{-1}$ is compact then $(z-T-P)^{-1}$ is also compact and so $T+P$ is a discrete operator.

Remark. Similarly as in [**9**] we can prove that the multiplicities of eigenvalues of T and $T+P$ are equal if the absolute value of the eigenvalues are large enough. Also, the operators T and $T+P$ have equal number of eigenvalues (with their multiplicity) in $D_n = \{\lambda \colon |\lambda| = r_n\}$ for n large enough.

Proof of Theorem 1. Let $R_\lambda{}' = (\lambda - T - P)^{-1}$, $R_\lambda = (\lambda - T)^{-1}$, $\lambda \in \Gamma_n$. Then $R_\lambda{}' - R_\lambda - R_\lambda P R_\lambda = \sum_{k\ge 2} R_\lambda (PR_\lambda)^k$. The operator R_λ is a nuclear one and hence

$$\operatorname{tr}(R_\lambda{}' - R_\lambda) - \operatorname{tr} R_\lambda P R_\lambda = \sum_{k\ge 2}\operatorname{tr} R_\lambda (PR_\lambda)^k$$

Since [**8**]

$$\operatorname{tr} R_\lambda (PR_\lambda)^k = \frac{1}{k}\frac{d}{d\lambda}\operatorname{tr}(PR_\lambda)^k; \qquad k\ge 1$$

from (5) it follows

$$\frac{1}{2\pi i}\int_{\Gamma_n}\operatorname{tr}(R_\lambda{}' - R_\lambda)d\lambda - \frac{1}{2\pi i}\int_{\Gamma_n}\operatorname{tr} PR_\lambda d\lambda = \sum_{k\ge 2}\frac{1}{k}\cdot\frac{1}{2\pi i}\int_{\Gamma_n}\operatorname{tr}(PR_\lambda)^k d\lambda. \tag{6}$$

Having in mind that

$$\operatorname{tr} PR_\lambda = \sum_{\nu\ge 1}\frac{(P\varphi_\nu,\varphi_\nu)}{\lambda - \lambda_\nu}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda \operatorname{tr}(R_\lambda{}' - R_\lambda) = {\sum_{k=1}^{n}}{}'(\mu_k - \lambda_k) \ \text{(the property of Riesz projectors)}$$

from (6) it follows

$${\sum_{k=1}^{n}}{}'(\mu_k - \lambda_k - (P\varphi_k,\varphi_k)) = \sum_{k\ge 2}\frac{1}{k}\frac{1}{2\pi i}\int_{\Gamma_n}\operatorname{tr}(PR_\lambda)^k d\lambda. \tag{7}$$

Now, we estimate

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n} \operatorname{tr}(PR_\lambda)^k d\lambda.$$

From $|\operatorname{tr}(PR_\lambda)^k| \le |PR_\lambda|_1 X \|P\|^{k-1}|R_\lambda|_1\|R_\lambda\|^{k-1}$, by Lemma 2 it follows

$$|\operatorname{tr}(PR_\lambda)^k| \le C_3\|P\|^k \frac{\ln n}{n^{1-1/p}}\|R_\lambda\|^{k-1}. \tag{8}$$

Let $\Gamma_n^{'} = \{\lambda\colon |\lambda| = r_n, 0 \le \arg\lambda \le \pi/2\}$, $\Gamma_n^{''} = \{\lambda\colon |\lambda| = r_n, -\pi/2 \le \arg\lambda \le 0\}$ $d_n = (\lambda_{n+1} - \lambda_n)/2$ and $\varphi_n = d_n r_n^{-1}$. Clearly, $d_n \asymp n^{1/p-1}$ and $\varphi_n \to 0$ $(n \to \infty)$. Now, we estimate

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n'} \operatorname{tr}(PR_\lambda)^k d\lambda.$$

From (8) it follows

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n'} \operatorname{tr}(PR_\lambda)^k d\lambda\right| \le \frac{C_3}{2\pi}\|P\|^k \frac{\ln n}{n^{1/p-1}}\int\limits_{\Gamma_n'} \|R_\lambda\|^{k-1}|d\lambda|. \tag{9}$$

Since

$$\int\limits_{\Gamma_n'} \|R_\lambda\|^{k-1}|d\lambda| = \int\limits_0^{\varphi_n} \|R_{r_n e^{i\theta}}\|^{k-1} r_n d\theta + \int\limits_{\varphi_n}^{\pi/2} d\theta + \int\limits_{\varphi_n}^{\pi/2} \|R_{r_n e^{i\theta}}\|^{k-1} r_n d\theta,$$

$\|R_{r_n e^{i\theta}}\| \le d_n^{-1}$ $(0 \le \theta \le \varphi_n)$ and $\|R_{r_n e^{i\theta}}\| \le r_n^{-1}\sin\theta$ $(\varphi_n \le \theta \le \pi/2)$ from (9) it follows

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n'} \operatorname{tr}(PR_\lambda)^k d\lambda\right| \le \frac{C_3}{2\pi}\|P\|^k \frac{\ln n}{n^{1/p-1}}\left(\frac{r_n\varphi_n}{d_n^{k-1}} + \frac{1}{r_n^{k-2}}\int\limits_{\varphi_n} \frac{d\theta}{(\sin\theta)^{k-1}}\right).$$

Since the function

$$x \to x^{k-2}\int\limits_x^{\pi/2} \frac{d\theta}{(\sin\theta)^{k-1}}$$

is bounded on $[0, \pi/2]$ (for $k \ge 3$) from previous inequality we get

$$\begin{aligned}\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n'} \operatorname{tr}(PR_\lambda)^k d\lambda\right| &\le \frac{C_3}{2\pi}\|P\|^k \frac{\ln n}{n^{1/p-1}}\left(\frac{1}{d_n^{k-2}} + \frac{1}{r_n^{k-2}} O\left(\frac{1}{\varphi_n^{k-2}}\right)\right) \\ &\le C_7^k \frac{\ln n}{n^{(1/p-1)(k-1)}}\end{aligned}$$

(C_7 does not depend on n and k). Similarly, we can prove that

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n''}\operatorname{tr}(PR_\lambda)^k d\lambda\right| \le C_7^k \frac{\ln n}{n^{(1/p-1)(k-1)}}.$$

Because T is semibounded, we have (for $k \ge 3$)

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n}\operatorname{tr}(PR_\lambda)^k d\lambda\right| \le C_8^k \frac{\ln n}{n^{(1/p-1)(k-1)}}. \tag{10}$$

Analogously we prove

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n}\operatorname{tr}(PR_\lambda)^2 d\lambda\right| \le \text{const}\,\frac{\ln^2 n}{n^{1/p-1}}$$

where conts does not depend on n. Since $0 < p < 1$, from (7), (10) and (11) it follows

$$\left|\sum_{k=1}^{n}{}'(\mu_k - \lambda_k - (P\varphi_k,\varphi_k))\right| \le C_9 \frac{\ln^2 n}{n^{1/p-1}}$$

where constant C_9 does not depend on n, and finally

$$\sum_{k\ge 1}{}'(\mu_k - \lambda_k - (P\varphi_k,\varphi_k)) = 0.\ \square$$

Example. Let T be the linear operator defined by the differential expresion $l(y) = -y''$ and the boundary conditions $y(0) = y(\pi) = 0$. It is easy to verify that $\lambda_n(T) = n^2$ and $\varphi_n(x) = \sqrt{2/\pi}\ \sin nx$. Let $P\colon L^2(0,\pi) \to L^2(0,\pi)$ be a bounded operator defined by $Pf(x) = q(x)f(x)$ (q is a real smooth function on $[0,\pi]$.) Here we have $p = 1/2$, $\lambda_{n+1} - \lambda_n = 2n+1 \asymp (n^2)^{1/2}$. By Theorem 1, applying simple transformations [**7**], we get (1).

Theorem 2. *If $\mathcal{N}_T(\lambda) = \sum_{\lambda_n(T)<\lambda} \sim C\lambda^p$ $(\lambda \to +\infty)$, $0 < p < 2/3$ and $\sum_{\nu=1}^{n} \frac{1}{\lambda_{n+1}-\lambda_\nu} = o(1)$, then there exists a sequence of positive integers n_m such that*

$$\lim_{m\to+\infty}\sum_{k=1}^{n_m}(\mu_k - \lambda_k - (P\varphi_k,\varphi_k)) = 0$$

(Here the eigenvalues are counted according to their multiplicity).

From the asymptotic relation $\mathcal{N}_T(\lambda) \sim C\lambda^p$ $(0 < p < 1)$ it follows that there exists a sequence $n_1 < n_2 < n_3 < \dots$ such that $\lambda_{n_k+1} - \lambda n_k \ge C_0 n_k^{1/p-1}$ (for some constant $C_0 > 0$). Let $\Pi_k = \{\lambda\colon |\lambda| = r_k = (\lambda_{n_k+1} + \lambda n_k)/2\}$.

LEMMA 4. $\phi(r_k) = O(r_k^{2p-1})$

Proof. Since

$$\frac{1}{r_k - \lambda_{nk}} = O(r_k^{2p-1}) \text{ and } \sum_{\nu=1}^{n_{k-1}} \frac{1}{r_k - \lambda_\nu} \leq \frac{n_k}{r_k - \lambda_{n_k}} = O(r_k^{2p-1})$$

it is enough to prove that

$$\sum_{\nu=n_{k+2}}^{\infty} \frac{1}{\lambda_\nu - r_k} = O(r_k^{2p-1}). \tag{12}$$

Since

$$\sum_{\nu=n_{k+2}}^{\infty} \frac{1}{\lambda_\nu - r_k} \leq \int\limits_{\lambda_{n_k+1}}^{\infty} \frac{dN(\lambda)}{\lambda - r_k} = \frac{N(\lambda)}{\lambda - r_k}\Big|_{\lambda_{n_k+1}}^{\infty} + \int\limits_{\lambda_{n_k+1}}^{\infty} \frac{N(\lambda)}{(\lambda - r_k)^2} d\lambda$$

$$= \frac{-n_k}{\lambda_{n_k+1} - r_k} + O\left(\int\limits_{\lambda_{n_k+1}}^{\infty} \frac{\lambda^p}{(\lambda - r_k)^2} d\lambda \right)$$

and $n_k/(\lambda_{n_k} - r_k) = O(r_k^{2p-1})$, we obtain

$$\sum_{\nu=n_k+2}^{\infty} \frac{1}{\lambda_\nu - r_k} = O(r_k^{2p-1}) + O\left(\int\limits_{\lambda_{n_k+1}}^{\infty} \frac{\lambda^p}{(\lambda - r_k)^2} d\lambda \right).$$

Now, we estimate $\int_{\lambda_{n_k+1}}^{\infty} \lambda^p (\lambda - r_k)^{-2} d\lambda$. Having in mind that

$$\int\limits_{x}^{\infty} t^p (t-1)^{-2} dt \sim \frac{1}{x-1} \qquad (x \to 1+0)$$

and

$$\int\limits_{\lambda_{n_k+1}}^{\infty} \lambda^p (\lambda - r_k)^{-2} d\lambda = r_k^{p-1} \int\limits_{\lambda_{n_k+1/r_k}}^{\infty} t^p (t-1)^{-2} dt \sim \frac{r_k^p}{\lambda_{n_{k+1}} - r_k} = O(r_k^{2p-1})$$

we get (12). □

Proof of Theorem 2. is obtained similarly as for Theorem 1 using Lemma 4. Integration is performed over the sequence of contours Π_k. One obtains

$$\left| \sum_{\nu=1}^{n_k} (\mu_\nu - \lambda_\nu - (P\varphi_\nu, \varphi_\nu)) \right| \leq \text{const}\, n_k^{3-2/p} + \frac{1}{2} \left| \frac{1}{2\pi i} \int\limits_{\Pi_k} \text{tr}\,(PR_\lambda)^2 d\lambda \right|$$

It has to be proved that

$$\frac{1}{2\pi i}\int\limits_{\Pi_k} \operatorname{tr}(PR_\lambda)^2 d\lambda \to 0 \qquad (k\to+\infty). \tag{13}$$

Since

$$\operatorname{tr}(PR_\lambda)^2 = \sum_{k,\nu\geq 1} \frac{(P\varphi_\nu,\varphi_k)(P\varphi_k,\varphi_\nu)}{(\lambda-\lambda_\nu)(\lambda-\lambda_k)},$$

we get

$$\frac{1}{2\pi i}\int\limits_{\Pi_k} \operatorname{tr}(PR_\lambda)^2 d\lambda = 2\sum_{\nu=1}^{n_k}\sum_{l=n_k+1}^{\infty} \frac{(P\varphi_\nu,\varphi_l)(P\varphi_l,\varphi_\nu)}{\lambda_\nu-\lambda_l}.$$

Applying Abel transformation and Bessel inequality we get

$$\left|2\sum_{\nu=1}^{n_k}\sum_{l=n_k+1}^{\infty} \frac{(P\varphi_\nu,\varphi_l)(P\varphi_l,\varphi_\nu)}{\lambda_\nu-\lambda_l}\right| \leq \text{const}\sum_{\nu=1}^{n_k}\frac{1}{\lambda_{n_k+1}-\lambda_\nu} \to 0 \ \ (k\to+\infty)$$

(const does not depend on k), which proves (13). $\square$

Remark. Theorem 2 is proved in [**5**] in case $0<p\leq 1/2$. If the additional condition $\sum_{\nu=1}^{n}(\lambda_{n+1}-\lambda_\nu)^{-1}=o(1)$ is satisfied, then Theorem 2 is valid also for $1/2<p<2/3$.

THEOREM 3. *If* $\lambda_{n+1}(T)-\lambda_n(T)\asymp n^{1/p-1}$ $(0<p<1/2)$ *then*

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}(\mu_k^2-\lambda_k^2-2\lambda_k(P\varphi_k,\varphi_k))-A_n\right)=0$$

where

$$A_n=\sum_{k=1}^{n}\left(\sum_{\nu=1}^{\infty}(P\varphi_\nu,\varphi_k)(P\varphi_k,\varphi_\nu)-\sum_{\nu=n+1}^{\infty}(P\varphi_\nu,\varphi_k)(P\varphi_k,\varphi_\nu)\frac{\lambda_\nu+\lambda_k}{\lambda_\nu-\lambda_k}\right).$$

Proof. Starting from

$$R_\lambda{}'-R_\lambda=\sum_{k\geq 1}R_\lambda(PR_\lambda)^k,$$

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda^2\operatorname{tr}(R_\lambda{}'-R_\lambda)d\lambda=\sum_{k=1}^{n}{}'(\mu_k^2-\lambda_k^2)$$

$$\int\limits_{\Gamma_n}\lambda^2\operatorname{tr}R_\lambda(PR_\lambda)^k d\lambda=\frac{2}{k}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^k d\lambda$$

$$(\Gamma_n=\{\lambda\colon|\lambda|=r_n=(\lambda_n+\lambda_{n+1})/2\}$$

we get

$$\sum_{k=1}^{n}{}'(\mu_n^2-\lambda_k^2)-2\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}PR_\lambda d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^2d\lambda$$
$$=\sum_{k\geq 3}\frac{2}{k}\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^kd\lambda. \tag{14}$$

Since

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)d\lambda=\sum_{k=1}^{n}\lambda_k(P\varphi_k,\varphi_k)$$

and

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^2d\lambda=A_n$$

from (14) we obtain

$$\sum_{k=1}^{n}{}'(\mu_k^2-\lambda_k^2-2\lambda_k(P\varphi_k,\varphi_k))-A_n=\sum_{k\geq 3}\frac{2}{k}\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^kd\lambda. \tag{15}$$

Similarly as in Theorem 1, we prove that

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda\operatorname{tr}(PR_\lambda)^kd\lambda\right|\leq C_{10}^k\frac{\ln n}{n^{-1+(k-2)(1/2-1)}};\quad k\geq 3$$

where the constant C_{10} does not depend on n and k. From (15) and the previous inequality it follows that

$$\left|\sum_{k=1}^{n}{}'(\mu_k^2-\lambda_k^2-2\lambda_k(P\varphi_k,\varphi_k))-A_n\right|\leq C_{11}\frac{\ln n}{n^{1/p-2}}$$

(C_{11} does not depend on n). Since $0<p<1/2$, this inequality implies the statement of Theorem 3. □

THEOREM 4. *Let $T=T^*$ be a discrete lower bounded operator (on $\mathcal{H}$) with the eigenvalues λ_k and with the orthonormal system of eigenvectors φ_k; $P\in B(\mathcal{H})$ and $\mathcal{N}_T(\lambda)\sim C\lambda^p$ where $0<p<1/3$. If μ_k are the eigenvalues of $T+P$, then there exists a sequence of positive integers n_k such that*

$$\lim_{k\to\infty}\left(\sum_{\nu=1}^{n_k}(\mu_\nu^2-\lambda_\nu^2-2\lambda_\nu(P\varphi_\nu,\varphi_\nu))-A_{n_k}\right)=0$$

Proof. We follow the proofs of Theorems 1 and 3. Crucial step is the estimation of the integral

$$\frac{1}{2\pi i}\int\limits_{\Pi_k}\lambda\operatorname{tr}(PR_\lambda)^md\lambda$$

(Π_k and n_k are previously defined). Using Lemma 4 we obtain

$$\left|\frac{1}{2\pi i}\int\limits_{\Pi_k}\lambda \operatorname{tr}(PR_\lambda)^m d\lambda\right| \le C_{12}^m \frac{1}{n_k^{(m-2)(1/p-1)-2}} \quad (m\ge 3) \tag{16}$$

where contant C_{12} does not depend on k and m. The equality

$$\sum_{\nu=1}^{n_k}(\mu_\nu^2-\lambda_\nu^2-2\lambda_\nu(P\varphi_\nu,\varphi_\nu))-A_{n_k}=\sum_{m\ge 3}\frac{2}{m}\frac{1}{2\pi i}\int\limits_{\Pi_k}\lambda \operatorname{tr}(PR_\lambda)^m d\lambda,$$

combined with (16) gives

$$\left|\sum_{\nu=1}^{n_k}(\mu_\nu^2-\lambda_\nu^2-2\lambda_\nu(P\varphi_\nu,\varphi_\nu))-A_{n_k}\right|\le C_{13}\frac{1}{n_k^{1/p-3}}.$$

This implies the statement of Theorem 4 (because $0<p<1/3$). □

Consider now the case when perturbation P is an unbounded operator which is “subordinated” to some power of T.

Theorem 5. *Let $T=T^*$ be a discrete, positive operator on the Hilbert space $\mathcal{H}$, $\ker T=\{0\}$. Let (λ_k) be the distinct eigenvalues of T, let $\{\varphi_k\}$ be the corresponding system of eigenvectors and $\lambda_{n+1}(T)-\lambda_n(T)\asymp n^{1/\alpha-1}$ $(0<\alpha<1/2)$. If P is a closed operator, $\mathcal{D}(T)\subset\mathcal{D}(P)$ ($\mathcal{D}(T)$, $\mathcal{D}(P)$ are the domains of T and P) such that $\|Px\|\le A_0\|T^\beta x\|$, $\forall x\in\mathcal{D}(T)$, $0\le\beta<1/2-\alpha$, $A_0=\text{const}$ then the operator $T+P$ is also discrete and the formula*

$$\sum_{n\ge 1}{}'(\mu_n-\lambda_n-(p\varphi_n,\varphi_n))=0$$

holds.

(μ_n are the eigenvalues of $T+P$. $\sum'$ denotes that the terms arising from the repeated eigenvalues λ_n are grouped). Before proving this Theorem, we prove a few lemmas.

Lemma 5. *If $\phi_1(\lambda)=\sum_{n\ge1}\lambda_n^\beta|\lambda-\lambda_n|^{-1}$, $\phi_2(\lambda)=\sum_{n\ge1}\lambda_n^{2\beta}|\lambda-\lambda_n|^{-2}$, then for $\lambda\in\Gamma_n=\{\lambda\colon|\lambda|=r_n=(\lambda_n+\lambda_{n+1})/2\}$ the following inequalities*

$$\phi_2(\lambda)\le\phi_2(r_n)=O(n^{-2(1-\alpha-\beta)/2}),\qquad \phi_1(\lambda)\le\phi_1(r_n)=O(n^{-1(1-\alpha-\beta)/2}\ln n)$$

hold.

Proof of this Lemma is obtained similarly as the one of Lemma 2. □

From Lemma 5 it follows

$$\max_{\lambda\in\Gamma_n}\|PR_\lambda\|=O(n^{-(1-\alpha-\beta)/2}),\ \ \max_{\lambda\in\Gamma_n}|PR_\lambda|_1=O(n^{-(1-\alpha-\beta)/\alpha}\ln n). \tag{17}$$

LEMMA 6. *Under the assumptions of Theorem 5 the equlity*

$$\operatorname{tr} R_\lambda (PR_\lambda)^k = -k^{-1} \operatorname{tr} (PR_\lambda)^k, \qquad k \in N \tag{18}$$

holds.

Proof. From the assumptions of Theorem 5 it follows that the operator $P_1 = PT^{-\beta}$ can be extended to a bounded operator on $\mathcal{H}$. Let $G_\lambda = T^\beta R_\lambda$. Then $PR_\lambda = P_1 G_\lambda$. The operator G_λ is bounded and

$$G_\lambda = \sum_{n \geq 1} \lambda_n^{-\beta} (\lambda - \lambda_n)^{-1} (\cdot, \varphi_n) \varphi_n$$

The relation (18) can be rewritten as

$$\operatorname{tr} R_\lambda (PR_\lambda)^k = -k^{-1} d(P_1 G_\lambda)^k / d\lambda. \tag{19}$$

Since P_1 and G_λ are bounded operators (therefore G_λ is nuclear) then

$$\begin{aligned}
\frac{d}{d\lambda} \operatorname{tr} (P_1 G_\lambda) &= \operatorname{tr} \frac{d}{d\lambda} (P_1 G_\lambda)^k \\
&= \operatorname{tr} (P_1 G_\lambda' \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} \qquad\qquad (20) \\
&\quad + P_1 G_\lambda' \ldots P_1 G_\lambda + \ldots + \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} P_1 G_\lambda{}') \ \ (G_\lambda' = dG_\lambda / d\lambda) \\
&= \operatorname{tr} P_1 G_\lambda' \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} \\
&\quad + \operatorname{tr} P_1 G_\lambda P_1 G_\lambda' \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-2} + \ldots + \operatorname{tr} \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} P_1 G_\lambda'
\end{aligned}$$

Having in mind that $dG_\lambda / d\lambda = -T^\beta R_\lambda^2$ we obtain

$$\begin{aligned}
\operatorname{tr} P_1 G_\lambda' \underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} &= -\operatorname{tr} (P_1 T^\beta R_\lambda^2 \underbrace{PR_\lambda \ldots PR_\lambda}_{k-1}) \\
&= -\operatorname{tr} (PR_\lambda R_\lambda \underbrace{PR_\lambda \ldots PR_\lambda}_{k-1}) = -\operatorname{tr} R_\lambda (PR_\lambda)^k.
\end{aligned}$$

Similary

$$\begin{aligned}
&\operatorname{tr} (P_1 G_\lambda P_1 G_\lambda' \ldots P_1 G_\lambda) = -\operatorname{tr} R_\lambda (PR_\lambda)^k \\
&\quad \vdots \\
&\operatorname{tr} (\underbrace{P_1 G_\lambda \ldots P_1 G_\lambda}_{k-1} P_1 G_\lambda') = -\operatorname{tr} R_\lambda (PR_\lambda)^k.
\end{aligned}$$

From the previous equalities we get (20), then (19) and finaly (18). □

Proof of Theorem 5. From (17) and Lemma 6 it follows that for n large enough we have

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda \mathrm{tr}\,(R_\lambda^{'}-R_\lambda)d\lambda-\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,PR_\lambda d\lambda-\frac{1}{4\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,(PR_\lambda)^2d\lambda$$
$$=\sum_{k\geq 3}\frac{1}{k}\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,(PR_\lambda)^k d\lambda$$

where $R_\lambda^{'}=(\lambda-T-P)^{-1}$, $R_\lambda=(\lambda-T)^{-1}$. Since

$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\lambda \mathrm{tr}\,(R_\lambda^{'}-R_\lambda)d\lambda={\sum_{k=1}^{n}}'(\mu_k-\lambda_k),$$
$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,PR_\lambda d\lambda={\sum_{k=1}^{n}}'(P\varphi_k,\varphi_k),$$
$$\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,(PR\lambda)^2 d\lambda\to 0\quad (n\to\infty)\quad (\text{because }\ \alpha+\beta<1/2),$$

we obtain

$$
{\sum_{k=1}^{n}}'(\mu_k-\lambda_k-(P\varphi_k,\varphi_k))=\sum_{k\geq 3}\frac{1}{k}\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,(PR_\lambda)^k d\lambda+o(1). \tag{21}
$$

From (17) it follows

$$\left|\frac{1}{2\pi i}\int\limits_{\Gamma_n}\mathrm{tr}\,(PR_\lambda)^k d\lambda\right|\leq C_{14}^k\frac{\ln n}{n^{(k(1-\alpha-\beta)-1)/\alpha}}\qquad (k\geq 3)$$

where C_{14} does not depend on k and n. Combining this inequality and (21) we get

$$\left|{\sum_{k=1}^{n}}'(\mu_k-\lambda_k-(P\varphi_k,\varphi_k))\right|\leq C_{15}\frac{\ln n}{n^{(2-3\alpha-3\beta)/\alpha}}+o(1)$$

where the constant C_{15} does not depend on n. Since $\alpha+\beta<1/2$ the statement of Theorem 5 follows immediately.

Example. Let T be the differential operator generated by the differantial expresion $l(y)=-y^{(6)}$ and the boundary condition

$$
y(0)=y''(0)=y^{(4)}(0)=y(\pi)=y''(\pi)=y^{(4)}(\pi)=0 \tag{22}
$$

Let P be an operator defined by $Py(x)=r(x)y(x_0)$ where $x_0\in(0,\pi)$, $r\in C^1[0,\pi]$ and $r(0)=r(\pi)=0$. It is easy to verify that $\lambda_n(T)=n^6$, $\mathcal{N}_T(\lambda)\sim\lambda^{1/6}$,

$\varphi_n(x) = \sqrt{2/\pi}\ \sin nx$ and $\|Py\| \le \mathrm{const}\,\|T^{1/6}y\|$, where the function y satisfies the conditions (22). So, in this case $\alpha = \beta = 1/6$. Applying Theorem 5 we get

$$\sum_{n\ge 1}(\lambda_n(T+P) - n^6 - (P\varphi_n,\ \varphi_n)) = 0.$$

Since the series

$$\sum_{n\ge 1}(P\varphi_n, \varphi_n) = \sum_{n\ge 1}\sqrt{\frac{2}{\pi}}\ \sin nx_0 \int_0^\pi r(x)\sqrt{\frac{2}{\pi}}\ \sin nx\, dx$$

is absolutely convergent and its sum is $r(x_0)$, we obtain

$$\sum_{n\ge 1}(\lambda_n(T+P) - n^6) = r(x_0).$$

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