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ON THE LIMIT PROPERTIES OF THE PICARD SINGULAR INTEGRAL

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Abstract. We present some direct and inverse approximation theorems for the Picard singular integral in the spaces $L^p(1 \leq p \leq \infty)$ and the generalized Holder spaces. Those theorems extend and improve the results for the Picard integral given in [**1**]–[**3**].

1. Notations **1.1.** Let $L^p \equiv L^p(R)$, $1 \leq p < \infty$ be the space of realvalued functions Lebesgue-integrable with p-th power over $R := (-\infty, +\infty)$ and let $L^\infty \equiv L^\infty(R)$ be the space of real-valued functions uniformly continuous and bounded on R. The norm in L^p we define as usual

$$\|f\|_{L^p} \equiv \|f(\cdot)\|_{L^p} := \begin{cases} \left(\int\limits_R |f(x)|^p dx\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup\limits_{x \in R} |f(x)| & \text{if } p = \infty. \end{cases} \tag{1}$$

Let X be one of the spaces L^p, $1 \leq p \leq \infty$, with the norm (1). For a given $f \in X$ denote by $\omega_2(\cdot; f, X)$ the modulus of smoothness of order 2, i.e.

$$\omega_2(t; f, X) := \sup_{|h| \leq t} \left\|\Delta_h^2 f\right\|_X \tag{2}$$

for $t \geq 0$, where

$$\Delta_h^2 f(x) := f(x+h) + f(x-h) - 2f(x). \tag{3}$$

Denote as in [**4**] by Ω^2 the set of functions of the type of modulus of smoothness of order 2 [**5**, p. 116], i.e. Ω^2 is the set of all functions ω satisfying the following conditions:

(a) ω is defined and continuos on $< 0, +\infty)$.

(b) ω is monotonically increasing and $\omega(0) = 0$.

(c) $\omega(h)h^{-2}$ is monotonically decreasing for $h > 0$.

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It is easy to verify that for every $\omega \in \Omega^2$, there exist positive constants M_1 and M_2 $(M_i = M_i(\omega), i = 1, 2)$ such that

$$M_1 t^2 \leq \omega(t) \leq M_2 t^2 \int_t^1 \omega(z) z^{-3} dz \tag{4}$$

for all $0 \leq t \leq 1/2$.

Similary as in [**4**], for a given $\omega \in \Omega^2$, denote by X^ω the class of all functions $f \in X$ for which

$$\|f\|^*_{X^\omega} := \sup_{h>0} \|\Delta_h^2 f\|_X / \omega(x) < +\infty. \tag{5}$$

In X^ω we define the norm

$$\|f\|_{X^\omega} := \|f\|_X + \|f\|^*_{X^\omega}. \tag{6}$$

Denote as in [**4**] by $\tilde{X}^\omega$, $\omega \in \Omega^2$, the class of all functions $f \in X$ such that $\lim_{h\to 0+} \|\Delta_h^2 f\|_X / \omega(h) = 0$. The norm in $\tilde{X}^\omega$ we define by (6). X^ω and $\tilde{X}^\omega$ with the norm (6) are called the generalized Hölder spaces. If $\omega, \mu \in \Omega^2$ and the function

$$q(h) := \omega(h)/\mu(h), \qquad h > 0, \tag{7}$$

is monotonically increasing, then

$$X^\omega \subset X^\mu \text{ and } \tilde{X}^\omega \subset \tilde{X}^\mu. \tag{8}$$

If $f \in X^\omega$, then

$$\omega_2(t; f, X) \leq \omega(t) \|f\|^*_{X^\omega}, \quad t > 0. \tag{9}$$

If $f \in \tilde{X}^\omega$, then

$$\omega_2(t; f, X) = o(\omega(t)) \text{ as } t \to 0+. \tag{10}$$

1.2. In the papers [**1**]–[**3**] are examined the limit properties of the Picard singular integral

$$P(x, r; f) := (2r)^{-1} \int_R f(x+t) \exp(-|t|/r) dt, \tag{11}$$

$x \in R$, $r \in I := (0, 1>$ and $r \to 0+$, for the functions f belonging to L^p $(1 \leq p \leq \infty)$ or the classical Hölder spaces $H^\alpha (0 < \alpha \leq 1)$ of continuous functions.

The purpose of this note is to generalize the resulte given in [**1**]–[**3**].

By $M_k(\cdot)$, $k = 1, 2, \ldots$, we shall denote suitable positive constants depending only on the indicated parameters.

2. Auxillary results. In this section we shall give some auxiliary inequalities.

Lemma 1. *If $f \in X$, then $\|P(\cdot,r;f)\|_X \le \|f\|_X$, $r \in I$.*

Proof. By (1) and (11) we have

$$\|P(\cdot,r;f)\|_X \le \|f\|_X (2r)^{-1} \int\limits_R \exp(-|t|/r)dt$$

for all $r \in I$. We also have

$$\frac{1}{2r}\int\limits_R \exp(-|t|/r)dt = \frac{1}{r}\int\limits_0^{+\infty} \exp(-t/r)dt = 1, \tag{12}$$

for all $r \in I$. Hence the proof is completed.

Lemma 2. *If $f \in X^\omega$, then $\|P(\cdot,r;f)\|^*_{X^\omega} \le \|f\|^*_{X^\omega}$ for $r \in I$, which proves that, for every fixed $r \in I$, the function $P(\cdot,r;f)$ also belongs to X^ω.*

Proof. By (5) we have

$$\|P(\cdot,r;f)\|^*_{X^\omega} = \sup_{h>0} \|\Delta_h^2 P(\cdot,r;f)\|_X/\omega(h), \qquad r \in I.$$

From (11) and (3) we get

$$\Delta_h^2 P(x,r;f) = P(x,r;\Delta_h^2 f) \ \ (x \in R, h \in R, r \in I), \tag{13}$$

which by Lemma 1 gives

$$\|\Delta_h^2 P(\cdot,r;f)\|_X \le \|\Delta_h^2 f\|_X,$$

$r \in I$, $h \in R$. From this and by (5) we obtain our assertion.

Lemma 1 and Lemma 2 imply the following

Corollary 1. *If $f \in X^\omega$, then $\|P(\cdot,r;f)\|_{X^\omega} \le \|f\|_{X^\omega}$ for all $r \in I$.*

Lemma 3. *If $f \in \tilde{X}^\omega$, then $P(\cdot,r;f) \in \tilde{X}^\omega$, for every fixed $r \in I$.*

Proof. By (13) and Lemma 1,

$$0 \le \frac{\|\Delta_h^2 P(\cdot,r;f)\|_X}{\omega(h)} = \frac{P(\cdot,r;\Delta_h^2 f)\|_X}{\omega(h)} \le \frac{\|\Delta_h^2 f\|_X}{\omega(h)}$$

$(h > 0, r \in I)$. Hence, by the assumption $f \in \tilde{X}^\omega$, we obtain

$$\lim_{h\to 0+} \|\Delta_h^2(\cdot,r;f)\|_X/\omega(h) = 0,$$

which proves that $P(\cdot,r;f) \in \tilde{X}^\omega$.

It is easy to verify that

LEMMA 4. *For $k = 0, 1, 2, \ldots$ and $r > 0$ we have*

$$\int_0^{+\infty} t^k \exp(-t/r)dt = k!r^{k+1}.$$

LEMMA 5. *If $f \in X$, then $\|\partial^2 P(\cdot, r; f)/\partial x^2\|_X \le \|f\|_X r^{-2}$ for every $r \in I$.*

Proof. From (11) we get

$$\begin{aligned}\frac{\partial^2}{\partial x^2} P(x, r; f) &= \frac{1}{2r} \int_R f(t) \left(\frac{\partial^2}{\partial x^2} \exp(-|x - t|/r) \right) dt \\ &= \left(2r^3\right)^{-1} \int_R f(x + t) \exp(-|t|/r)dt = r^{-2} P(x, r; f)\end{aligned}$$

for $x \in R$ and $r \in I$. From this and by Lemma 1 we obtain the desired inequality.

3. Approximation theorems Let

$$U(r, x; f) := P(x, r; f) - f(x) \tag{14}$$

for $x \in R$ and $r \in I$.

3.1. First we shall consider direct approximation problem in the norm of the space X.

THEOREM 1. *If $f \in X$, then $\|U(\cdot, r; f)\|_X \le \frac{5}{2}\omega_2(r; f, X)$ for all $r \in I$.*

Proof. By (11), (12), (14) and (3), we have

$$U(x, r; f) = (2r)^{-1} \int_0^{+\infty} \left(\Delta_t^2 f(x)\right) \exp(-t/r)dt$$

for all $x \in R$ and $r \in I$. From this and by (1), (2) and the properties of $\omega_2(\cdot; f, x)$ [**5**, p. 116] we get

$$\begin{aligned}\|U(\cdot, r; f)\|_X &\le (2r)^{-1} \int_0^{+\infty} \omega_2(t; f, X) \exp(-t/r)dt \\ &\le (2r)^{-1}\omega_2(r; f, X) \int_0^{+\infty} (\frac{t}{r} + 1)^2 \exp(-t/r)dt\end{aligned}$$

for all $r \in I$. Now using Lemma 4, we obtain our thesis.

From Theorem 1 and by (9) we obtain the following

Corollary 2. *If $f \in X^\omega$, then $\|U(\cdot,r;f)\|_X \le \frac{5}{2}\|f\|^*_{X^\omega}\omega(r)$ for all $r \in I$. In the case $\omega(t) = t^\alpha$, $0 < \alpha \le 2$, we have $\|U(\cdot,r;f)\|_X \le \frac{5}{2}\|f\|^*_{X^\omega} r^\alpha$ for all $r \in I$.*

3.2. In this part we shall give an inverse approximation theorem in the norm X. We shall use the notation (14).

Theorem 2. *Suppose that $f \in X$ and*

$$\|U(\cdot,r;f)\|_X \le \omega(r) \tag{15}$$

for all $r \in I$, where ω is given function belonging to Ω^2. Then

$$\omega_2(t;f,X) \le M_1^* t^2 \int_t^1 \omega(z) z^{-3} dz \tag{16}$$

for all $t \in (0,1/2)$, where $M_1^* = M_1(\|f\|_X,\omega)$.

Proof. We shall apply the Bernstein method [**5**, p.345]. For every fixed integer $m \ge 2$ we can write

$$\begin{aligned} f(x) = P(x,2^{-1};f) &+ \sum_{k=1}^{m-1} \left\{P(x,2^{-k-1};f) - P(x,2^{-k};f)\right\} \\ &+ f(x) - P(x,2^{-m};f), \qquad x \in R. \end{aligned} \tag{17}$$

Let

$$v_n(x;f) := P(x,2^{-n-1};f) - P(x,2^{-n};f) \qquad (n = 1,2,\dots). \tag{18}$$

From (17) and by (3), (14) and (18) we get

$$\begin{aligned} \Delta_h^2 f(x) &= \Delta_h^2 P(x,2^{-1};f) + \sum_{k=1}^{m-1} \Delta_h^2 v_k(x;f) + \Delta_h^2 U(x,2^{-m};f) \\ &=: A_1(x,h) + A_2(x,h) + A_3(x,h). \end{aligned}$$

for $x \in R$ and $h \in R$. We notice that

$$\Delta_h^2 P(x,r;f) = \int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} P(x+t_1+t_2,r;f)dt_1dt_2.$$

Hence, by Hölder-Minkowski inequality and by Lemma 5, we get

$$\begin{aligned} \|A_1(\cdot,h)\|_X &= \left\| \int_{-h/2}^{h/2}\int_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} P(x+t_1+t_2,1/2;f)dt_1dt_2 \right\|_X \\ &\le h^2 \|\frac{\partial^2}{\partial x^2} P(x,1/2;f)\|_X \le 4\|f\|_X h^2. \end{aligned} \tag{20}$$

From the assumption (15) we have

$$\|A_3(\cdot,h)\|_X \leq 4\omega(2^{-m}). \tag{21}$$

From the Fubini theorem if follows that

$$P\left(x,2^{-n-1};P(\cdot;2^{-n};f)\right) = P\left(x,2^{-n};P(\cdot,2^{-n-1};f)\right)$$

for $n = 1,2,\dots$ and $x \in R$. Hence and by (18) we get

$$v_n(x;f) = P\left(x,2^{-n-1};f(\cdot) - P(\cdot,2^{-n};f)\right) - P\left(x,2^{-n};f(\cdot) - P(\cdot,2^{-n-1};f)\right).$$

Further, we find

$$\Delta_h^2 v_n(x;f) = \int\limits_{-h/2}^{h/2}\int\limits_{-h/2}^{h/2} \frac{\partial^2}{\partial x^2} v_n(x+t_1+t_2;f)dt_1dt_2,$$

Hence, by (14) and Lemma 5, we get

$$\begin{aligned}\|A_2(\cdot,h)\|_X &\leq \sum_{k=1}^{m-1}\|\Delta_h^2 v_k(\cdot;f)\|_X \\ &\leq h^2\sum_{k=1}^{m-1}\left\{\left\|P\left(\cdot,2^{-k-1};U(\cdot,2^{-k};f)\right)\right\|_X + \left\|P\left(\cdot,2^{-k};U(\cdot,2^{-k-1};f)\right)\right\|_X\right\} \\ &\leq h^2\sum_{k=1}^{m-1}\left\{\left\|U(\cdot,2^{-k};f)\right\|_X\cdot 2^{2k+2} + \left\|U(\cdot,2^{-k-1};f)\right\|_X\cdot 2^{2k}\right\}.\end{aligned}$$

Now applying the assumptions (14) and (15), and by properties of the function $\omega \in \Omega^2$, we obtain

$$\begin{aligned}\|A_2(\cdot,h)\|_X &\leq h^2\sum_{k=1}^{m-1}\left\{2^{2k+2}\omega(2^{-k}) + 2^{2k}\omega\left(2^{-k-1}\right)\right\} \\ &\leq 5h^2\sum_{k=1}^{m-1}2^{2k}\omega\left(2^{-k}\right) \leq 10h^2\int\limits_{2^{-m}}^{1/2}\omega(z)z^{-3}dz.\end{aligned} \tag{22}$$

Using (20)–(22) and (19), we get

$$\left\|\Delta_h^2 f\right\|_X \leq 4\|f\|_X h^2 + 4\omega\left(2^{-m}\right) + 10h^2\int\limits_{2^{-m}}^{1/2}\omega(z)z^{-3}dz \tag{23}$$

for every integer $m \geq 2$ and $r \in I$. Now let $0 < t < 1/2$, $|h| \leq t$ and let m be an integer such that $2^{-m} \leq t < 2^{-m+1}$. Then, by properties of $\omega \in \Omega$, from (23) and (2) it follows that

$$\omega_2(t;f,X) \leq 4\|f\|_X t^2 + 4\omega(t) + \frac{5}{2}t^2\int\limits_t^1\omega(z)z^{-3}dz.$$

Now using (4), we obtain (16).

In the case $\omega(h) = M_5(\alpha)h^\alpha$, $0 < \alpha \leq 2$, from Theorem 2 we get

COROLLARY 3. *If $f \in X$ and $\|U(\cdot, r; f)\|_X \leq M_5 r^\alpha$, $r \in I$, where $0 < \alpha \leq 2$, then*

$$\omega(t; f, X) \leq M_6(\alpha)\begin{cases} t^\alpha & \text{if } 0 < \alpha < 2 \\ t^2|\log t| & \text{if } \alpha = 2, \end{cases}$$

for all $0 < t < 1/2$.

3.3. Now we shall examine the limit properties of $P(\cdot, r; f)$ in the Hölder norms.

THEOREM 3. *Suppose that $\omega, \mu \in \Omega^2$ and the function $q(\cdot)$ defined by (7) is monotonically increasing for $h > 0$. If $f \in X^\omega$, then*

$$\|U(\cdot, r; f)\|_{X^\mu} \leq M_6(\mu)\|f\|^*_{X^\omega} q(r) \tag{24}$$

for all $r \in I$, where $M_6(\mu) = \frac{5}{2}\mu(1) + 12$.

Proof. By our assumptions and (8) we have $P(\cdot, r; f) \in X^\mu$ for every $r \in I$. Hence the function $U(\cdot, r; f)$ ($r \in I$) defined by (14), belongs to X^μ and by (6) we have

$$\|U(\cdot, r; f)\|_{X^\mu} = \|U(\cdot, r; f)\|_X + \|U(\cdot, r; f)\|^*_{X^\mu}$$

for every $r \in I$. Using Corollary 2, we get

$$\|U(\cdot, r; f)\|_X \leq \frac{5}{2}\|f\|^*_{X^\omega}\mu(1)q(r), \quad r \in I.$$

By (5), for every $r \in I$, we have

$$\|U(\cdot, r; f)\|^*_{X^\mu} \leq \sup_{h>r} \frac{\left\|\Delta^2_h U(\cdot, r; f)\right\|_X}{\mu(h)} + \sup_{0<h\leq r} \frac{\left\|\Delta^2_h U(\cdot, r; f)\right\|_X}{\mu(h)} =: S_1(r) + S_2(r).$$

Since $\left\|\Delta^2_h U(\cdot, r; f)\right\|_X \leq 4\,\|U(\cdot, r; f)\|_X$ and by Corollary 2, we get

$$S_1(r) \leq 4(\mu(r))^{-1}\|U(\cdot, r; f)\|_X \leq 10\|f\|^*_{X^\omega} \cdot q(r), \quad r \in I.$$

From (3), (14) and (13) it follows that $\Delta^2_h U(x, r; f) = P(x, r; \Delta^2_h f) - \Delta^2_h f(x)$ for $x \in R$, $h \in R$ and $r \in I$. Hence, by Lemma 1, we get

$$\left\|\Delta^2_h U(\cdot, r; f)\right\|_X \leq \left\|P(\cdot, r; \Delta^2_h f)\right\|_X + \left\|\Delta^2_h f\right\|_X \leq 2\left\|\Delta^2_h f\right\|_X, \quad r \in I.$$

Consequently,

$$\begin{aligned} S_2(r) &\leq 2 \sup_{0<h\leq r} \frac{\left\|\Delta^2_h f\right\|_X}{\mu(h)} = 2 \sup_{0<h\leq r} q(h)\frac{\left\|\Delta^2_h f\right\|_X}{\omega(h)} \\ &\leq 2\|f\|^*_{X^\omega} q(r) \quad \text{for } r \in I. \end{aligned}$$

Summing up, we obtain (24).

Similarly, we obtain the following

THEOREM 4. *Suppose that the functions* ω, μ, q *satisfy the assumptions of Theorem* 3. *If* $f \in \tilde{X}^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{\tilde{X}^{\mu}} = o(q(r)) \text{ as } r \to 0+.$$

Theorem 3 and Theorem 4 imply

COROLLARY 4. *Let the assumptions of Theorem* 3 *be satisfied and let* $q(h) \leq M_7(\gamma)h^{\gamma}$, $h > 0$, $0 < \gamma < 2$. *If* $f \in X^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{X^{\mu}} \leq M_8(\mu, \gamma)\|f\|^{*}_{X^{\omega}} r^{\gamma}$$

for all $r \in I$, *where* $M_8(\mu, \gamma) = \left(\frac{5}{2}\mu(1) + 12\right) M_7(\gamma)$. *If* $f \in \tilde{X}^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{\tilde{X}^{\mu}} = o\left(r^{\gamma}\right) \text{ as } r \to 0+.$$

In the case $\omega(h) = h^{\alpha}$, $\mu(h) = h^{\beta}$, $0 < \beta < \alpha \leq 2$ *and* $f \in X^{\omega}$ *we have*

$$\|U(\cdot, r; f)\|_{X^{\mu}} \leq \frac{29}{2} r^{\alpha - \beta}$$

for all $r \in I$. *If moreover* $f \in \tilde{X}^{\omega}$, *then*

$$\|U(\cdot, r; f)\|_{\tilde{X}^{\mu}} = o\left(r^{\alpha-\beta}\right) \text{ as } r \to 0+.$$

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