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ESTIMATION OF PARAMETERS OF RCA WITH EXPONENTIAL MARGINALS

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Abstract. The estimation of parameters of time series whose marginal distribution is exponential with parameter μ, $\mu > 0$ is somewhat more complicated than the estimation of parameters of Gaussian time series. One possible approach using the method of least squares is given. Namely, the method of least squares is applied in two steps for estimating the parameters of generalized first order autoregressive time series with exponential marginals. A special case of estimating parameters of the model FAREX (1) is also given.

1. Introduction. The estimating of parameters of the random coefficient model instead of those first order autoregressions that are represented by mixtures of distributions has been inspired (Popović [1990]) by the fact that many of the well known first order autoregressions whose marginal distribution is exponential with parameter $\mu, \mu > 0$, $\{X_t,\ t = 0, \pm 1, \pm 2, \dots\}$ can be well represented by the random coefficient model

$$X_t = U_t X_{t-1} + V_t E_t,\ \ t \text{ is an integer.} \tag{1.1}$$

Necessary and sufficient conditions for such representation are:

I The sequence of random variables $\{X_t\}$ is semi-independent of the random sequences $\{U_t\}$, $\{V_t\}$ and $\{E_t\}$ (X_i and U_j, or V_j, or E_j are independent if and only if $i < j$).

II $\{E_t\}$ is the sequence of independent identically distributed (i.i.d.) random variables with exponential distribution with parameter μ and E_t is independent of U_i and V_j for every t, i and j.

III $\{U_t\}$, $\{V_t\}$ and $\{(U_t, V_t)\}$ are i.i.d. sequences of discrete random variables and vectors which satisfy the following special conditions:

$$P(0 \le U_t \le 1) = 1,\ \ P(0 \le V_t \le 1) = 1$$

$$0 < E(U_t^2),\ \ E(U_t) < 1$$

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$$E(V_t) = 1 - E(U_t), \;\; E(U_t^2) + E(V_t^2) = 1 - E(U_t V_t)$$

and, if

$$P(U_t = \alpha_i) = p_{u_i}, \;\; i = 1, 2, \dots, k, \;\; 0 \le \alpha_1 < \alpha_2 < \dots < \alpha_k \le 1, \;\; \sum_{i=1}^{k} p_{u_i} = 1$$

and

$$P(V_t = \beta_j) = p_{v_j}, \;\; j = 1, 2, \dots, r, \;\; 0 \le \beta_1 < \beta_2 < \dots < \beta_r \le 1, \;\; \sum_{j=1}^{r} p_{v_j} = 1$$

then

$$\sum_{i=1}^{k} \sum_{j=1}^{r} \frac{p_{u_i} p_{v_j}}{\alpha_i - \beta_j} \left(\frac{\alpha_i}{\mu + s\alpha_i} - \frac{\beta_j}{\mu + s\beta_j} \right) = \frac{1}{\mu + s}$$

for any real s.

The difference equation (1.1) has a unique, stationary (even strong stationary) and ergodic solution (Popović [1990]):

$$X_t = \sum_{i=1}^{\infty} \left(\bigcap_{j=0}^{i-1} U_{t-j} \right) V_{t-i} E_{t-i} + V_t E_t.$$

For convenience, further we shall use the following $a = E(U_t)$, $b = E(U_t^2)$, $c = E(V_t^2)$, $m = E(X_t) = 1/\mu$.

Nicholls and Quinn [1980] where the first who applied the least square estimation in two steps for parameters of autoregressive time series with random coefficients. But, their random coefficient model has the inovation sequence that is independent if the vector of random coefficients of the autoregression. We shall substitute the condition of independence by the set of conditions II and III and estimate parameters a, b and c supposing that the main parameter of the distribution, μ, is known. The strong consistency of the estimators and their asymptotic normal distribution will be proved below. In the Appendix, we shall use the procedure to estimate paramters of the autoregressive time series FAREX (1).

2. Estimation procedure. Suppose that the sample $\{X_0, X_1, \dots, X_N\}$ is given. As the parameter μ is known, we shall translate that sample to the zero-expectation one, $\{Y_0, Y_1, \dots, Y_N\}$ in the following way

$$Y_t = X_t - m. \tag{2.1}$$

But, according to (1.1) it is easily verified that instead of the time series $\{X_t\}$, we can consider the time series $\{Y_t, t = 0, \pm 1, \pm 2, \dots\}$, where

$$Y_t = aY_{t-1} + R_t \tag{2.2}$$

for $R_t = B_t Y_{t-1} + \theta_t$, $B_t = U_t - a$, $\theta_t = m(U_t + V_t - 1) + V_t(E_t - m)$. This translation does not disturb the existence of the solution of difference equation

(2.1) comparing with difference equation (1.1), meaning that the solution exists and it is also unique, stationary, strong stationary and ergodic.

Let F_t be the σ-field generated by the set of random vectors $\{(U_s, V_s, E_s), s \leq t\}$; then the solution of the equation (1.1) is F_t-measurable.

Let G_t be the σ-field generated by the set of random vectors $\{(B_s, \theta_s), s \leq t\}$; then

$$\begin{aligned} E(R_t \mid G_{t-1}) &= Y_{t-1}E(B_t) + E(\theta_t) = 0 \\ E(R_t^2 \mid G_{t-1}) &= Y_{t-1}^2 E(B_t^2) + 2Y_{t-1}E(B_t\theta_t) + E(\theta_t^2) \\ &= (b - a^2)Y_{t-1}^2 + 2m(1-a-c)Y_{t-1} + m^2(1-b) \\ &= Z_{t-1}T_{t-1}b - W_{t-1}^2(a) + 2mT_{t-1} - 2mcY_{t-1} \end{aligned}$$

where $Z_{t-1} = Y_{t-1} - m$, $T_{t-1} = Y_{t-1} + m$ and $W_{t-1}(a) = aY_{t-1} + m$.

The representation (2.2) of the process $\{Y_t\}$ has also the following important properties: the remainders R_t are uncorrelated random variables

$$\mathrm{Cov}(R_t, R_{t+k}) = E(B_{t+k})E(B_tY_{t-1}Y_{t+k-1}) = 0 \quad (k \text{ is an integer})$$

and, Y_t represented by (2.2) is represented as the sum of two uncorrelated processes aY_{t-1} and R_t:

$$\mathrm{Cov}(aY_{t-1}, R_t) = aE(B_t)E(Y_{t-1}^2) + aE(Y_{t-1})E(\theta_t) = 0.$$

Now we can set the estimating procedure for parameters a, b and c by means of least squares.

Step 1. We shall represent remainders R_t as $R_t = Y_t - aY_{t-1}$. Then

$$\sum_{t=1}^{N} R_t^2 = \sum_{t=1}^{N} Y_t^2 - 2a\sum_{t=1}^{N} Y_tY_{t-1} + a^2\sum_{t=1}^{n} Y_{t-1}^2.$$

This quadratic function attains its minimum with respect to a for

$$\hat{a} = \mathbf{Y_1}^T\mathbf{Y}(\mathbf{Y}^T\mathbf{Y})^{-1} \tag{2.3}$$

where $\mathbf{Y} = (Y_0, Y_1, \dots, Y_{N-1})^T$ and $\mathbf{Y_1} = (Y_1, Y_2, \dots, Y_N)^T$.

THEOREM 2.1 *Under the assumptions* I - III, $\hat{a}$ *given by* (2.3) *is strongly consistent estimator for* a *and* $\sqrt{N}(\hat{a} - a)$ *has a distribution which converges to the normal distribution with expectation zero and covariance* $8b - 9a^2 - 4a - 4c + 5$.

The proof of the theorem will be given below.

Step 2. We shall use now the estimator $\hat{a}$ instead of the real value a to find the estimators for the second moments b and c of the random coefficients U_t and V_t . So, we shall use $\hat{R}_t = Y_t - \hat{a}Y_{t-1}$ instead of R_t . Now the difference

$$D_t = R_t^2 - E(R_t^2 \mid G_{t-1}) = R_t^2 - bZ_{t-1}T_{t-1} + 2mcY_{t-1} + W_{t-1}^2(a) - 2mT_{t-1}$$

will give the quadratic function $\sum_{t=1}^{N} D_t^2$ which can be minimized with respect to b and to c. Hence, when we take $\hat{a}$ and $\hat{R}_t$ instead of a and R_t the minimums will be

$$\hat{b} = \frac{\mathbf{Y}^T[(\mathbf{Y}\hat{\mathbf{R}}^T\mathbf{A} - \mathbf{A}\hat{\mathbf{R}}^T\mathbf{Y}) + (\mathbf{Y}\hat{\mathbf{W}}^T\mathbf{A} - \mathbf{A}\hat{\mathbf{W}}^T\mathbf{Y}) - 2m(\mathbf{Y}\mathbf{T}^T\mathbf{A} - \mathbf{A}\mathbf{T}^T\mathbf{Y})]}{\mathbf{A}^T(\mathbf{A}\mathbf{Y}^T - \mathbf{Y}\mathbf{A}^T)\mathbf{Y}} \tag{2.4}$$

$$\hat{c} = \frac{(\hat{b}\mathbf{A}^T - \hat{\mathbf{R}}^T + \hat{\mathbf{W}}^T - 2m\mathbf{T}^T)\mathbf{Y}}{2m\mathbf{Y}^T\mathbf{Y}} \tag{2.5}$$

where $\mathbf{A} = (Z_0T_0, Z_1T_1, \dots, Z_{N-1}T_{N-1})^T$, $\mathbf{R} = (R_1^2, R_2^2, \dots, R_N^2)^T$,
$\mathbf{W} = (W_0^2(a), W_1^2(a), \dots, W_{N-1}^2(a))^T$.

THEOREM 2.2. *Under the conditions* I - III, *the vectors* $\mathbf{D} = (a, b, c)$ *and* $\hat{\mathbf{D}} = (\hat{a}, \hat{b}, \hat{c})$ *are such that the vector* $(\hat{\mathbf{D}} - \mathbf{D})$ *converges almost surely to zero-vector, and the vector* $\sqrt{N}(\hat{\mathbf{D}} - \mathbf{D})$ *has the distribution which converges to the normal distribution with mean value zero and covariance matrix* $\mathbf{K} = [K_{ij}]$ *where*

$$\begin{aligned}
&K_{11} = E((m^{-2}Y_{t-1}R_t)^2), \quad K_{12} = K_{21} = E((8m^6)^{-1}Y_{t-1}R_tZ_{t-1}T_{t-1}S_t),\\
&K_{13} = K_{31} = E(m^{-2}Y_{t-1}R_t\left(2mN^{-1}\sum_{j=1}^{N} Y_{j-1}^2\right)^{-1} Q_t^*S_t),\\
&K_{22} = E((64m^8)^{-1}Z_{t-1}^2T_{t-1}^2S_t^2),\\
&K_{23} = K_{32} = E((8m^4)^{-1}\left(2mN^{-1}\sum_{j=1}^{N} Y_{j-1}^2\right)^{-1} Z_{t-1}T_{t-1}Q_t^*S_t^2)
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
S_t &= R_t^2 + W_{t-1}^2(a) - 2mT_{t-1} + 2mcY_{t-1} - bZ_{t-1}T_{t-1}\\
Q_t^* &= (8m)^{-1}Z_{t-1}T_{t-1} - Y_{t-1}.
\end{aligned} \tag{2.7}$$

The proof is given below.

These two steps and two theorems complete the procedure.

Proof of Theorem 2.1. Set the difference

$$\hat{a} - a = \frac{N^{-1}\mathbf{R}^T\mathbf{Y}}{N^{-1}\mathbf{Y}^T\mathbf{Y}.}$$

The sequences $\{Y_t^2\}$ and $\{Y_{t-1}R_t\}$ are strictly stationary and ergodic because of the properties of $\{Y_t\}$. This implies that the sequences $\{N^{-1}\mathbf{Y}'\mathbf{Y}\}$ and $\{N^{-1}\mathbf{R}^T\mathbf{Y}\}$ converge to m^2 and zero respectively. Hence, $(\hat{a} - a)$ converges almost surely to zero. It means that $\hat{a}$ is a strong consistent estimator for a.

According to the central limit theorem for martingales, each element of the random sequence $\{N^{-1/2}q\mathbf{R}^T\mathbf{Y}\}$ has a distribution which converges to the normal distribution with mean value zero and variance:

$$\begin{aligned} E(q^2Y_{t-1}^2R_t^2) &= E(E(q^2Y_{t-1}^2R_t^2 \mid G_{t-1})) = E(q^2Y_{t-1}^2E(R_t^2 \mid G_{t-1})) \\ &= q^2E(Y_{t-1}^4(b-a^2) + 2Y_{t-1}^3 m(1-a-c) + Y_{t-1}^2 m^2(1-b)) \\ &= q^2m^4(8b-9a^2-4a-4c+5), \end{aligned}$$

for any real q . From the other side,

$$\begin{aligned} &E(\mid N^{-1}\sum_{t=1}^{N} Y_{t-1}^2 - m^2 \mid^2) \\ &\quad = 2N^{-2}\sum_{\substack{t,j=1\\ j>t}}^{N} E\left(Y_{t-1}^2\left(a^{j-t}Y_{t-1} + \sum_{k=0}^{j-t-1} a^k R_{j-k}\right)^2\right) - m^4 \\ &\quad = 16m^4\left[\frac{N-1}{N^2}\frac{a^2}{1-a^2} - \frac{1}{N^2}\frac{1-a^{2N+2}}{(1-a^2)^2}\right] \end{aligned}$$

and the sequence $\{N^{-1}\mathbf{Y}^T\mathbf{Y}\}$ converges in probability to m^2. So,

$$\sqrt{N}(\hat{a}-a) = \frac{N^{-1/2}\mathbf{R}^T\mathbf{Y}}{N^{-1}\mathbf{Y}^T\mathbf{Y}}$$

has a distribution that converges to the normal distribution with mean value zero and variance $8b-9a^2-4a-4c+5$.

Proof of Theorem 2.2. We shall prove this theorem in two steps. First of all, we shall assume that a is a known parameter. Then, we shall set $\tilde{b}$ and $\tilde{c}$ to be the estimators of parameters b and c under this assumption. That means that $\tilde{b}$ and $\tilde{c}$ will be defined by the same fomulae as $\hat{b}$ and $\hat{c}$ with a and R_t instead of $\hat{a}$ and $\hat{R}_t$, (2.4) and (2.5). It seems reasonable to set $\tilde{\mathbf{D}} = (\hat{a}, \tilde{b}, \tilde{c})$. Then

$$\begin{aligned} \tilde{b} - \hat{b} &= (\mathbf{Y}^T[\mathbf{Y}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{A} - \mathbf{A}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{Y} + \mathbf{Y}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{A} \\ &\quad - \mathbf{A}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{Y}])/[\mathbf{A}^T(\mathbf{A}\mathbf{Y}^T - \mathbf{Y}\mathbf{A}^T)\mathbf{Y}] \\ \tilde{c} - \hat{c} &= \frac{[(\tilde{b}-\hat{b})\mathbf{A}^T - (\mathbf{R}^T - \hat{\mathbf{R}}^T) - (\mathbf{W}^T - \hat{\mathbf{W}}^T)]\mathbf{Y}}{2m\mathbf{Y}^T\mathbf{Y}}. \end{aligned}$$

The elements of the vectors $(\mathbf{R}^T - \hat{\mathbf{R}}^T)$ and $(\mathbf{W}^T - \hat{\mathbf{W}}^T)$ are

$$\begin{aligned} R_t^2 - \hat{R}_t^2 &= 2(a-\hat{a})Y_{t-1}R_t + (a-\hat{a})^2Y_{t-1}^2 \\ W_{t-1}^2(a) - W_{t-1}^2(\hat{a}) &= 2(a-\hat{a})Y_{t-1}W_{t-1}(a) - (a-\hat{a})^2Y_{t-1}^2 \end{aligned}$$

respectively.

We shall now investigate the convergence of $(\tilde{b}-\hat{b})$. The elements of the random sequence $\{N^{-2}\mathbf{Y}^T\mathbf{Y}(\hat{\mathbf{R}}^T - \mathbf{R}^T)\mathbf{A}\}$ can be written as

$$\begin{aligned} N^{-2}\mathbf{Y}^T\mathbf{Y}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{A} =& 2(a-\hat{a})\left(N^{-1}\sum_{t=1}^{N} Y_{t-1}Z_{t-1}T_{t-1}\right)\left(N^{-1}\sum_{j=1}^{N} R_jY_{j-1}^2\right) \\ &+ (a-\hat{a})^2\left(N^{-1}\sum_{t=1}^{N} Y_{t-1}Z_{t-1}T_{t-1}\right)\left(N^{-1}\sum_{j=1}^{N} Y_{j-1}^3\right). \end{aligned}$$

According to the ergodic theorem, as Y_t has moments of arbitrary finite order, we have

$$N^{-1}\sum_{t=1}^{N} Y_{t-1}Z_{t-1}T_{t-1} \xrightarrow{a.s} m^3 \ , \ N \to \infty$$
$$N^{-1}\sum_{j=1}^{N} R_j Y_{j-1}^2 \xrightarrow{a.s} 0 \ , \ N \to \infty$$
$$N^{-1}\sum_{j=1}^{N} Y_{j-1}^3 \xrightarrow{a.s} 2m^3 \ , \ N \to \infty.$$

According to Theorem 2.1, $(a-\hat{a})$ converges almost surely (a.s.) to zero, so that

$$(a-\hat{a})\left(N^{-1}\sum_{t=1}^{N} Y_{t-1}Z_{t-1}T_{t-1}\right)\left(N^{-1}\sum_{j=1}^{N} R_j Y_{j-1}^2\right) \xrightarrow{a.s} 0 \ , \ N \to \infty.$$

According to the same theorem,

$$\sqrt{N}(a-\hat{a})\left(N^{-1}\sum_{t=1}^{N} Y_{t-1}Z_{t-1}T_{t-1}\right)\left(N^{-1}\sum_{j=1}^{N} R_j Y_{j-1}^2\right) \xrightarrow{P} 0 \ , N \to \infty.$$

Further we shall use also the fact that if $\sqrt{N}(a-\hat{a})$ converges in distribution, then $N^{1/4}(a-\hat{a})$ converges in probability (P) to the mean of the random variable $\sqrt{N}(a-\hat{a})$, i.e. to zero.

The following convergences can be verified

$$\sqrt{N}(a-\hat{a})^2\left(n^{-1}\sum_{t=1}^{n} Y_{t-1}Z_{t-1}T_{t-1}\right)\left(N^{-1}\sum_{j=1}^{N} Y_{j-1}^3\right) \xrightarrow{P} 0 \ , \ N \to \infty,$$
$$N^{-2}\mathbf{Y}^T\mathbf{Y}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{A} \xrightarrow{a.s} 0 \ , \ N \to \infty,$$
$$N^{-3/2}\mathbf{Y}^T\mathbf{Y}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{A} \xrightarrow{P} 0 \ , \ N \to \infty.$$

Using the same arguments, we have

$$N^{-2}\mathbf{Y}^T\mathbf{A}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{Y} \xrightarrow{a.s} 0 \text{ and } N^{-3/2}\mathbf{Y}^T\mathbf{A}(\mathbf{R}^T - \hat{\mathbf{R}}^T)\mathbf{Y} \xrightarrow{P} 0 \ ,$$
$$N^{-2}\mathbf{Y}^T\mathbf{Y}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{A} \xrightarrow{a.s} 0 \text{ and } N^{-3/2}\mathbf{Y}^T\mathbf{Y}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{A} \xrightarrow{P} 0 \ ,$$
$$N^{-2}\mathbf{Y}^T\mathbf{A}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{Y} \xrightarrow{a.s} 0 \text{ and } N^{-3/2}\mathbf{Y}^T\mathbf{A}(\mathbf{W}^T - \hat{\mathbf{W}}^T)\mathbf{Y} \xrightarrow{P} 0$$

when $N \to \infty$. Finally $N^{-2}\mathbf{A}^T(\mathbf{A}\mathbf{Y}^T - \mathbf{Y}\mathbf{A}^T)\mathbf{Y} \xrightarrow{a.s} 8m^6 \ , \ N \to \infty$.

So, $(\tilde{b}-\hat{b})$ converges to zero almost surely and $\sqrt{N}(\tilde{b}-\hat{b})$ converges to zero in probability.

The same adequate arguments will lead us to the convergence of the difference $(\tilde{c}-\hat{c})$ almost surely to zero and $\sqrt{N}(\tilde{c}-\hat{c})$ in probability to zero when $N \to$

∞. Hence, the vector $(\tilde{\mathbf{D}} - \hat{\mathbf{D}})$ converges almost surely to the zero-vector, and $\sqrt{N}(\tilde{\mathbf{D}} - \hat{\mathbf{D}})$ in probability to the same vector.

Next, we shall consider the differences

$$\tilde{b} - b = \mathbf{A}^T\mathbf{S}(\mathbf{A}^T\mathbf{A})^{-1} \text{ and } \tilde{c} - c = \mathbf{Q}^T\mathbf{S}(2m\mathbf{Y}^T\mathbf{Y})^{-1}$$

where $\mathbf{S} = (S_1, S_2, \dots, S_{N-1})^T$ and $\mathbf{Q} = (Q_1, Q_2, \dots, Q_{N-1})^T$ with S_t defined by (2.7) and

$$Q_t = Z_{t-1}T_{t-1}\left(N^{-1}\sum_{k=1}^{N} Z_{k-1}^2 T_{k-1}^2\right)^{-1}\left(N^{-1}\sum_{j=1}^{N} Y_{j-1}Z_{j-1}T_{j-1}\right) - Y_{t-1}.$$

Let us assign $\mathbf{D}^* = (a^*, b^*, c^*)^T$, where

$$\begin{aligned} a^* &= m^{-2}N^{-1}\mathbf{Y}_{\mathbf{1}}^T\mathbf{Y} \\ b^* &= (8m^4)^{-1}N^{-1}\mathbf{A}^T\mathbf{S} + b \\ c^* &= (2mN^{-1}\mathbf{Y}^T\mathbf{Y})^{-1}N^{-1}\mathbf{Q}^{*T}\mathbf{S} + c. \end{aligned}$$

Now we have $a^* - \hat{a} = [m^{-2} - (N^{-1}\mathbf{Y}^T\mathbf{Y})^{-1}]N^{-1}\mathbf{Y}_{\mathbf{1}}\mathbf{Y}$ and

$$N^{-1}\mathbf{Y}_{\mathbf{1}}\mathbf{Y} \xrightarrow{a.s} am^2 \ , \ N^{-1}\mathbf{Y}^T\mathbf{Y} \xrightarrow{a.s} m^2 \ , \ N \to \infty$$

so

$$a^* - \hat{a} \xrightarrow{a.s} 0 \ , \ \sqrt{N}(a^* - \hat{a}) \xrightarrow{P} 0 \ , \ N \to \infty.$$

Because of the ergodicity of the sequence $\{Z_{t-1}^2 T_{t-1}^2\}$ and the facts that

$$\begin{aligned} (\tilde{b} - b)N^{-1}\mathbf{A}^T\mathbf{A} - 8m^4(b^* - b) &= N^{-1}\mathbf{A}^T\mathbf{S} - N^{-1}\mathbf{A}^T\mathbf{S} = 0 \\ \tilde{b} - b^* &= (\tilde{b} - b) - (b^* - b), \end{aligned}$$

we will have that $\tilde{b} - b^* \xrightarrow{a.s} 0$ and $\sqrt{N}(\tilde{b} - b^*) \xrightarrow{P} 0$, $N \to \infty$.

Finally, let us discuss the difference $(c^* - \tilde{c})$:

$$c^* - \tilde{c} = (c^* - c) + (c - \tilde{c}) = (2mN^{-1}\mathbf{Y}^T\mathbf{Y})^{-1}[N^{-1}(\mathbf{Q}^* - \mathbf{Q})^T\mathbf{S}]$$

where the elements of $\mathbf{Q}^*$ are

$$Q_t^* = Z_{t-1}T_{t-1}\left[N^{-1}\sum_{k=1}^{N} E(Z_{k-1}^2 T_{k-1}^2)\right]^{-1}\left[N^{-1}\sum_{j=1}^{N} E(Y_{j-1}Z_{j-1}T_{j-1})\right] - Y_{t-1}.$$

It is obvious that $Q_t^* - Q_t \xrightarrow{a.s} 0$, $N \to \infty$. Simple computations will lead us to the result

$$c^* - \tilde{c} \xrightarrow{a.s} 0 \text{ and } \sqrt{N}(c^* - \tilde{c}) \xrightarrow{P} 0 \ , \ N \to \infty.$$

It is clear now that

$$\mathbf{D}^* - \tilde{\mathbf{D}} \xrightarrow{a.s} 0 \text{ and } \sqrt{N}(\mathbf{D}^* - \tilde{\mathbf{D}}) \xrightarrow{a.s} 0 \ , \ N \to \infty.$$

For any real vector $\mathbf{v} = (v_1, v_2, v_3)^T$ we have

$$\mathbf{v}^T(\mathbf{D}^* - \mathbf{D}) = N^{-1}\sum_{t=1}^{N} H_t(\mathbf{v})$$

where

$$H_t(\mathbf{v}) = v_1(a_t^* - a_t) + v_2(b_t^* - b_t) + v_3(c_t^* - c_t)$$
$$a_t^* - a_t = m^{-2}Y_{t-1}R_t,\;\; b_t^* - b_t = (8m^4)^{-1}Z_{t-1}T_{t-1}S_t,$$
$$c_t^* - c_t = \left(2mN^{-1}\sum_{j=1}^{N} Y_{j-1}^2\right)^{-1} Q_t^* S_t$$

and consequently

$$\begin{aligned} H_t(\mathbf{v}) =& v_1 m^{-2} Y_{t-1} R_t \\ &+ \left[v_2(8m^4)^{-1}Z_{t-1}T_{t-1} + v_3\left(2mN^{-1}\sum_{j=1}^{N} Y_{j-1}^2\right)^{-1} Q_t^* \right] S_t. \end{aligned}$$

It is easily verified that $E(R_t \mid G_{t-1}) = 0$ and $E(S_t \mid G_{t-1}) = 0$. According to this $E(H_t(\mathbf{v}) \mid G_{t-1}) = 0$. On the other hand, ergodicity and strong stationarity of the sequence $\{H_t(\mathbf{v})\}$ imply the convergence $\mathbf{v}^T(\mathbf{D}^* - \mathbf{D}) \xrightarrow{a.s.} 0$, $N \to \infty$. Besides, the properties of the sequence $\{H_t(\mathbf{v})\}$ mentioned above, enable us to apply the central limit theorem for martingales for this sequence. Hence, according to the central limit theorem, $\mathbf{v}^T\sqrt{N}(\mathbf{D}^* - \mathbf{D})$ converges in distribution to a random variable with zero-mean normal distribution whose variance is $E(H_t^2(\mathbf{v})) = \mathbf{v}^T\mathbf{K}\mathbf{v}$. The elements of the matrix $\mathbf{K}$ are defined by the formulae (2.6). As the conclusion is valid for any three-dimensional vector $\mathbf{v}$, the assertion of this theorem is proved.

Appendix. The exponential autoregressive model FAREX (1) was introduced by Mališić [1987].

Let $\{X_t\}$ be a stationary sequence of random variables whose marginal distribution is exponential with parameter μ. Then FAREX (1) is defined as follows:

$$X_t = \begin{cases} \alpha X_{t-1}, & \text{w.p.}\quad p \\ \beta X_{t-1} + \delta_t, & \text{w.p.}\quad 1-p \end{cases}$$

where $\{\delta_t\}$ is a sequence of i.i.d. random variables

$$\delta_t = \begin{cases} 0, & \text{w.p.}\quad (\alpha - p)\beta/[(1-p)\alpha] \\ E_t, & \text{w.p.}\quad (1-\beta)/(1-p) \\ \alpha E_t, & \text{w.p.}\quad p(\beta-\alpha)/[(1-p)\alpha]. \end{cases}$$

The necesary and sufficient condition for the existence of FAREX(1) is

$$0 < p \le \alpha \le \beta < 1.$$

For $p = \alpha = \beta$ we have well known EAR(1). Other special cases are also valuable. Further, we shall consider only the case $0 < p < \alpha < \beta < 1$, while the other special cases can be estimated in the same way. The random coefficient representation of this model is

$$\begin{aligned}
&P(U_t = \alpha) = 1 - P(U_t = \beta) = p\\
&P(V_t = 0) = \beta - p(\beta - \alpha)/\alpha, \qquad P(V_t = \alpha) = p(\beta-\alpha)/\alpha, \;\; P(V_t = 1) = 1 - \beta\\
&P((U_t, V_t) = (\alpha, 0)) = p, \qquad P((U_t, V_t) = (\beta, 0)) = (\alpha - p)\beta/\alpha,\\
&P((U_t, V_t) = (\beta, 1)) = 1 - \beta, \qquad P((U_t, V_t) = (\beta, \alpha)) = p(\beta - \alpha)/\alpha
\end{aligned}$$

and then

$$\begin{aligned}
a &= E(U_t) = \beta - p(\beta - \alpha)\\
b &= E(U_t^2) = \beta^2 - p(\beta^2 - \alpha^2)\\
c &= E(V_t^2) = \alpha p(\beta - \alpha) + 1 - \beta.
\end{aligned}$$

So, if we estimate a, b and c in the way that has been proposed above, we can take the estimator $(\hat{p}, \hat{\alpha}, \hat{\beta})$ for parameter vector (p, α, β) as the unique solution of the system above, by replacing a, b, c, p, α and β with $\hat{a}, \hat{b}, \hat{c}, \hat{p}, \hat{\alpha}$ and $\hat{\beta}$, which satisfies the condition $0 < p < \alpha < \beta < 1$. The solution will be:

$$(\hat{p}, \hat{\alpha}, \hat{\beta}) = \left(\frac{(\hat{a} + \hat{b} + \hat{c} - 1 - \hat{a}^2)^2}{(\hat{a} - \hat{a}^2 + \hat{b} + \hat{c} - 1)^2 + (1 - \hat{a})^2(\hat{b} - \hat{a}^2)}, \; \frac{\hat{b} - \hat{a} + \hat{a}\hat{c}}{\hat{a} + \hat{b} + \hat{c} - 1 - \hat{a}^2}, \; \frac{1 - \hat{b} - \hat{c}}{1 - \hat{a}} \right).$$

It is clear that the procedure that has been applied for estimating of parameters of EAR (p) by Billard and Mohamed (1991) will be disturbed in the case of FAREX (1) by the fact that X_t does not depend on δ_t with probablity $p > 0$.

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