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ON SOLVABILITY OF A NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract. The question of existence of solutions of a certain nonlinear elliptic boundary value problem is treated, by studying the existence of critical points of a suitable functional; the way in which this functional is related to the "standard" functional of the variational formulation of the problem is described, and regularity of solutions for the problem is also proved.

1. Introduction. In this note, the question of the solvability of the nonlinear elliptic boundary value problem

$$\begin{aligned} -\Delta v - \lambda_j v + f(v) = g, &\ \text{ in } \Omega \\ v = 0, &\ \text{ on } \partial\Omega \end{aligned} \tag{1.1}$$

is studied, where $\Omega \subseteq \mathbf{R}^N$ is bounded domain. The non-homogeneity term g is in $L^2(\Omega)$, while λ_j is a fixed eigenvalue of the Laplacian with Dirichlet boundary conditions. For the nonlinearity term f we make the following assumptions:

(F1) f is a potential operator, i.e. there is a C^1 function F, such that $\operatorname{grad} F = f$

(F2) $\Theta_1\|v_1 - v_2\|^2 \leq \langle f(v_1) - f(v_2), v_1 - v_2\rangle \leq \Theta_2\|v_1 - v_2\|^2, \quad \forall v_1, v_2 \in H_0^1(\Omega)$ where $\Theta_1, \Theta_2 \in \mathbf{R}$ are such that

$$\lambda_j - \lambda_{j+1} < \Theta_1 \leq \Theta_2 < \lambda_j - \lambda_{j-1}.$$

Let $f_j(v)$ be defined by

$$f_j(v) = f(v) - \lambda_j v \tag{1.2}$$

and F_j be such that

$$\operatorname{grad} F_j = f_j. \tag{1.3}$$

Then, the functional $I : H_0^1(\Omega) \to \mathbf{R}$, defined by

$$I(v) = \frac{1}{2} \mid \operatorname{grad} v \mid^2 + F_j(v) - \langle g, v\rangle \tag{1.4}$$

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satisfies the relation

$$\langle \operatorname{grad} I(v), \eta \rangle = \langle -\Delta v - \lambda_j v + f(v) - g, \eta \rangle \quad \forall \eta \in H_0^1(\Omega) \tag{1.5}$$

and, therefore, the solvability, or "more precisely" the $H_0^1(\Omega)$-weak solvabilty, of (1.1) is reduced to establishing the existence of critical points for $I(v)$.

The main object of this paper is to formulate and prove a necessary and sufficient condition for the existence of critical points of $I(v)$. It will be shown, that this existence is reduced to existence of critical points for another functional, defined on a suitable finite-demensional subspace of $H_0^1(\Omega)$, whereby, the desired necessary and sufficient condition will come out. The precise statement of this condition is left until Section 3, due to technicalities in its description. In Section 3, we provide, as well, the conditions under which f is a potential operator (i. e. such that $(F1)$ holds). The last section consists of a proof of a regularity result for the solutions of (1.1).

The results of this note are in a way related to those of the classical theory of [**9**]. The idea of using reduction methods, from functionals defined on infinite dimensional spaces to suitable others, defined on finite dimensional subspaces is due to Castro [**3**], [**4**], [**5**]. Conditions on the nonlinearity like $(F2)$, have been considered by many authors; see for example [**2**], [**5**], [**10**]. Of course there exists an extensive bibliography, with different approaches, for resonance at the first or higher eigenvalues; see the references in [**2**], [**3**] and [**5**].

2. Preliminaries. Let H be a real Hilbert space, and $F : H \to \mathbf{R}$ be a strictly convex function. Then, provided $F \in C^1(H; \mathbf{R})$, $\operatorname{grad} F$ is a strictly monotone operator, and F is weakly lower semicontinuous.

If F is strictly convex, it can have a minimum, at most at one point. A weakly lower semicontinuous function has a minimum iff it has a bounded minimizing sequence. All minimizing sequences of a coercive function are bounded.

Let X be a Banach space, with dual X^*. A mapping $\phi : X \to \bar{\mathbf{R}}$ is Gateaux differentiable at $x \in X$, if there exists an $f \in X^*$, such that

$$\lim_{t \to 0} \frac{1}{t} \{\phi(x + ty) - \phi(x)\} = \langle f, y \rangle \quad \forall y \in X.$$

In this case, f is said to be the Gateaux differential of ϕ at x, and is denoted by $\phi'(x)$.

An operator $T : X \to X^*$ is called a potential operator, if it is the Gateaux differential, in X, of a function $\phi : X \to \bar{\mathbf{R}}$. Such an operator is also called the gradient of ϕ, and is denoted by $\operatorname{grad} \phi$.

Let Ω be a bounded domain in $\mathbf{R}^N$, with smooth boundary $\partial\Omega$. Then the eigenvalues of the boundary value problem

$$\begin{aligned} -\Delta u &= \lambda u \ \text{ in } \ \Omega \\ u &= 0 \ \text{ on } \ \partial\Omega \end{aligned} \tag{2.1}$$

form a non-decreasing sequence $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ with $\lim_{j\to\infty} \lambda_j = \infty$. If ψ_j is the unique eigenfunction corresponding to the eigenvalue λ_j, then the ψ_j can be so chosen as to be real and to form an orthonormal system, with $\psi_1 \geq 0$ in Ω.

In the sequel we shall work in the Sobolev space $H_0^1(\Omega)$ with inner product

$$\langle u, v\rangle_1 = \int_\Omega \langle \operatorname{grad} u(x), \operatorname{grad} v(x)\rangle dx$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $L^2(\Omega)$. Let $\|\cdot\|_1$ and $\|\cdot\|$ denote the corresponding norms.

Let λ_j be a fixed eigenvalue of (2.1), with corresponding eigenfunction ψ_j. Let

$$S_j^- = \operatorname{span}\{\psi_k : k = 0, 1, \dots, j-1\} \tag{2.2}$$

$$S_j^+ = \operatorname{span}\{\psi_k : k \geq j+1, k \in \mathbf{N}\} \tag{2.3}$$

$$K_j = \ker(-\Delta - \lambda_j I) = \operatorname{span}\{\psi_j\}. \tag{2.4}$$

Then

$$\|u\|_1^2 \leq \lambda_{j-1}\|u\|^2 \ \text{ for } \ u \in S_j^- \tag{2.5}$$

$$\|w\|_1^2 \geq \lambda_{j+1}\|w\|^2 \ \text{ for } \ w \in S_j^+ \tag{2.6}$$

$$H_0^1(\Omega) = S_j^- \oplus K_j \oplus S_j^+. \tag{2.7}$$

We shall abuse notation and drop the subscript 1 in inner products and norms, since it will be clear from the context to which of them we refer.

3. Existence of weak solutions. In this section, we establish a theorem that provides a necessary and sufficient condition for the (weak) solvability of (1.1). The proof will be accomplished in a sequence of lemmata.

LEMMA 3.1. *For each $y \in S_j^- \oplus K_j$, there exists a unique critical point $c_2(y, g)$ of the functional $I_y(w) := I(y + w)$, $w \in S_j^+$. Moreover $c_2(y, g)$ depends continuously on y.*

Proof. (i) I_y is strictly monotone. Indeed,

$$\begin{aligned}
&\langle \operatorname{grad} I_y(w_1) - \operatorname{grad} I_y(w_2), w_1 - w_2\rangle = \\
&\langle(-\Delta - \lambda_j I)(y + w_1) - (-\Delta - \lambda_j I)(y + w_2), w_1 - w_2\rangle \\
&\quad + \langle f(y + w_1) - f(y + w_2), w_1 - w_2\rangle = \\
&\langle -\Delta(w_1 - w_2), w_1 - w_2\rangle - \lambda_j\|w_1 - w_2\|^2 \\
&\quad + \langle f(y + w_1) - f(y + w_2), y + w_1 - (y + w_2)\rangle > \\
&\lambda_{j+1}\|w_1 - w_2\|^2 - \lambda_{j+1}\|w_1 - w_2\|^2 = 0.
\end{aligned}$$

(ii) I_y is coercive. Indeed,

$$\begin{aligned}
\langle \operatorname{grad} I_y(\tau w), w\rangle =& \langle -\Delta(\tau w) - \lambda_j \tau w + f(\tau w) - g, w\rangle \\
=& \frac{1}{\tau} \mid \operatorname{grad}(\tau w) \mid^2 - \tau\lambda_j\|w\|^2 - \langle g, w\rangle + \frac{1}{\tau}\langle f(\tau w), \ \tau w\rangle \\
\geq& \tau\lambda_{j+1}\|w\|^2 - \tau\lambda_j\|w\|^2 - \langle g, w\rangle + \tau\{\Theta_1\|w\|^2 + \langle f(0), w\rangle\} \\
\geq& \tau\{\lambda_{j+1} - \lambda_j + \Theta_1\}\|w\|^2 - \langle g, w\rangle + \tau\langle f(0), w\rangle.
\end{aligned}$$

But

$$I_y(w) = I_y(0) + \int_0^1 \langle \operatorname{grad} I_y(\tau w), w\rangle d\tau$$

and the above inequality implies that $\lim_{\|w\|\to\infty} I_y(w) = \infty$ as desired.

From (i) and (ii) it follows that $I_y(w)$ has a minimum, $c_2(y,g)$ say. Its uniqueness follows from the strict monotonicity of $I_y(w)$.

To prove the continuity of $c_2(y,g)$ with respect to y, let us suppose, arguing as in [**4**], that $c_2(y,g)$ is not continuous. Then, there exists a sequence (y_n) such that

$\lim_{n\to\infty} y_n = y \in S_j^- \oplus K_j$ and $| c_2(y_n,g) - c_2(y,g) | \geq 2\delta$, for some $\delta > 0$.

Let Π be the projection of $H_0^1(\Omega)$ onto S_j^+ defined by $\Pi(y+w) = w$ and let Π^* be its adjoint. Π^* is a projection itself. We can see that

$$w = c_2(y,g) \Longleftrightarrow \Pi^*(\operatorname{grad} I(y+w)) = 0,$$

since $\langle \operatorname{grad} I(y + c_2(y,g)), \xi\rangle = 0$ if $\xi \in S_j^+$. By the continuiuty of $\operatorname{grad} I$ and the above relationships, it follows that

$$\|\Pi^*(\operatorname{grad} I(y_n + c_2(y,g)))\| < (\lambda_{j+1} - \lambda_j + \Theta_1)\delta^2 \tag{3.1}$$

for sufficiently large n. Therefore,

$$\begin{aligned}
&\|\Pi^*(\operatorname{grad} I(y_n + c_2(y,g)))\|\|c_2(y_n,g) - c_2(y,g)\| \geq \\
&\quad \geq \langle -\operatorname{grad} I(y_n + c_2(y,g)), c_2(y_n,g) - c_2(y,g)\rangle \\
&\quad \geq \langle \operatorname{grad} I(y_n + c_2(y_n,g)) - \operatorname{grad} I(y_n + c_2(y,g)), c_2(y_n,g) - c_2(y,g)\rangle \\
&\quad \geq (\lambda_{j+1} - \lambda_j + \Theta_1)\|c_2(y_n,g) - c_2(y,g)\|^2 \geq 4(\lambda_{j+1} - \lambda_j + \Theta_1)\delta^2,
\end{aligned}$$

contradicting (3.1).

Remark 3.1. If we consider the decomposition $g = g_0 + \hat{g}$, $g_0 \in K_j \oplus S_j^-$, $\hat{g} \in S_j^+$ we can easily see that $c_2(y,g)$ actually depends only on $\hat{g}$.

Proof. For $p \in K_j \oplus S_j^-$ consider the equation

$$-\Delta v - \lambda_j v + f(v) = g + p$$

and let T be such that

$$\langle \operatorname{grad} T(v), q\rangle = \langle -\Delta v - \lambda_j v + f(v) - g - p, q\rangle.$$

Then, for each $w \in S_j^+$, we have

$$\begin{aligned}
&\langle \operatorname{grad} T_y(c_2(y,g)), w\rangle \\
&\quad = \langle -\Delta(y + c_2(y,g)) - \lambda_j(y + c_2(y,g)) + f(y + c_2(y,g)) - g, w\rangle - \langle p, w\rangle \\
&\quad = \langle \operatorname{grad} T_y(c_2(y,g)), w\rangle = 0
\end{aligned}$$

and hence, from the definition of $c_2(y, g+p)$, we conclude that

$$c_2(y, g+p) = c_2(y,g), \quad \text{for each } p \in K_j \oplus S_j^-.$$

LEMMA 3.2. *The functional* $R: K_j \oplus S_j^- \to \mathbf{R}: y \to I_y(c_2(y,g))$ *is of class* C^1, *and satisfies the relation*

$$\langle \operatorname{grad} R(y), \zeta\rangle = \langle \operatorname{grad} I(y + c_2(y,g)), \zeta\rangle, \quad \forall \zeta \in K_j \oplus S_j^-. \tag{3.2}$$

Proof. Let $\rho > 0$ and $\zeta \in K_j \oplus S_j^-$. Then

$$\frac{R(y+\rho\zeta) - R(y)}{\rho} = \frac{I(y+\rho\zeta + c_2(y+\rho\zeta, g)) - I(y + c_2(y,g))}{\rho}$$

and by the definition of $c_2(y,g)$

$$\begin{aligned} &\le \frac{I(y+\rho\zeta + c_2(y,g)) - I(y + c_2(y,g))}{\rho} \\ &= \int_0^1 \langle \operatorname{grad} I(y + c_2(y,g) + \tau\rho\zeta), \zeta\rangle d\tau. \end{aligned} \tag{3.3}$$

Similarly, we get that

$$\frac{R(y+\rho\zeta) - R(y)}{\rho} \ge \int_0^1 \langle \operatorname{grad} I(y + c_2(y+\rho\zeta, g) + \tau\rho\zeta), \zeta\rangle d\tau. \tag{3.4}$$

From (3.3), (3.4) and the continuity of $\operatorname{grad} I$ and $c_2(y,g)$, we have

$$\lim_{\rho\to 0^+} \frac{R(y+\rho\zeta) - R(y)}{\rho} = \langle \operatorname{grad} I(y + c_2(y,g)), \zeta\rangle$$

and hence R, having a continuous Gateaux derivative, is a C^1 -functional. But, by the definition,

$$\langle \operatorname{grad} R(y), \zeta\rangle = \lim_{\rho\to 0^+} \frac{R(y+\rho\zeta) - R(y)}{\rho}$$

and, therefore, $\langle \operatorname{grad} R(y), \zeta\rangle = \langle \operatorname{grad} I(y + c_2(y,g)), \zeta\rangle$.

LEMMA 3.3. *For each* $u \in K_j$, *there exists a unique critical point* $c_1(u,g)$ *of the functional* $R_u(\phi) := R(u+\phi), \phi \in S_j^-$. *Moreover,* $c_1(u,g)$ *depends continuously on* u. *Finally, the functional* $J: K_j \to \mathbf{R}: u \to R_u(c_1(u,g))$ *is of class* C^1 *and satisfies the relation:*

$$\langle \operatorname{grad} J(u), \xi\rangle = \langle \operatorname{grad} R(u + c_1(u,g)), \xi\rangle, \quad \forall \xi \in K_j. \tag{3.5}$$

Proof. It has already been shown, that there exists a function $c_2(\cdot, g)$ such that

$$I(u + z + c_2(u+z, g)) = \min\{I(u+z+w) : w \in S_j^+\}, \text{ for } u \in K_j \text{ and } z \in S_j^-.$$

Moreover, $R(u+\phi) = I(u + \phi + c_2(u+\phi, g))$. Now,

$$\begin{aligned} R(u+\phi) =& I(u + \phi + c_2(u+\phi, g)) \le I(u+\phi) \\ =& I(u) + \langle \operatorname{grad} I(u), \phi\rangle + \int_0^1 \langle \operatorname{grad} I(u+\tau\phi) - \operatorname{grad} I(u), \phi\rangle d\tau. \end{aligned} \tag{3.6}$$

From (3.6) it follows, that

$$R(u+\phi) \leq I(u) + \|\operatorname{grad} I(u)\| \|\phi\| + \frac{1}{2}\{\Theta_2 + \lambda_{j-1} - \lambda_j\}\|\phi\|^2$$

and hence

$$\lim_{\|\phi\|\to\infty} R(u+\phi) = -\infty. \tag{3.7}$$

Now, for fixed $u \in K_j$, and for a sequence $\{\phi_k\}$ in S_j^-, such that

$$\lim_{k\to\infty} R(\phi_k) = \sup\{R(u+\phi) : \phi \in S_j^-\}$$

we can see that, since $\{\phi_k\}$ turns to be bouned, $\phi_k \rightharpoonup \hat{\phi}, \hat{\phi} \in K_j$. But then

$$R(u+\phi_k) \leq I(u+\phi_k + c_2(u+\phi_k, g)) \leq I(u+\phi_k + c_2(u+\hat{\phi}, g)).$$

So, the function $\phi \to I(u+\phi+c_2(u+\phi,g))$ is concave, and, therefore,

$$\limsup I(u+\phi_k + c_2(u+\hat{\phi}, g)) \leq I(u+\hat{\phi} + c_2(u+\hat{\phi}, g)) = R(u+\phi).$$

Hence R has a critical point at $\hat{\phi}$. We denote this critical point by $c_1(u,g)$.

The proof of the remaining assertions of the lemma, can be carried out at the same lines as in Lemmata 3.1 and 3.2.

Lemma 3.4. *The point $v = u+z+w$, $u \in K_j$, $z \in S_j^-$, $w \in S_j^+$ is a critical point of functional I if and only if $w = c_2(u+z+g)$, $z = c_1(u,g)$ and u is a critical point of the functional J.*

For the proof, it suffices to observe that, for each $h \in K_j$ we have:

$$\langle \operatorname{grad} J(u), h\rangle = \langle \operatorname{grad} I(u + c_1(u,g) + c_2(u+c_1(u,g),g)), h\rangle.$$

The rest is a direct consequence of the previous lemmata.

Now we are in a position to state and prove the main result of this section:

Theorem 3.1. *A necessary and sufficient condition for the (weak) solvability of the boundary value problem*

$$-\Delta v - \lambda_j v + f(v) = g \ \text{ in } \ \Omega \tag{3.8}$$

$$v = 0, \ \text{ on } \ \partial\Omega$$

where λ_j, f, g are as in the Introduction, is the following: there exists a $u \in K_j$, such that

$$\langle f(u + c_1(u,\hat{g}) + c_2(u + c_1(u,\hat{g}),\hat{g})), h\rangle = \langle g_0, h\rangle \ \ \forall h \in K_j. \tag{3.9}$$

Proof. As in Remark 3.1, we consider the decomposition

$$g = g_0 + \hat{g}, \ \ g_0 \in K_j \oplus S_j^-, \ \ \hat{g} \in S_j^+.$$

By Lemma 3.4, we have that $v = u + z + w$, $u \in K_j$, $z \in S_j^-$, $w \in S_j^+$, is a critical point of the functional J if and only if

$$\begin{aligned}\langle -\Delta(u + c_1(u,g) + c_2(u + c_1(u,g),g)) - \lambda_j(u + c_1(u,g) + c_2(u + c_1(u,g),g)) \\ + f(u + c_1(u,g) + c_2(u + c_1(u,g),g)) - g, h\rangle = 0, \ \forall h \in K_j. \qquad (3.10)\end{aligned}$$

Since $\langle g,h\rangle = \langle g_0,h\rangle + \langle \hat{g},h\rangle = \langle g_0,h\rangle$ and using Remark 3.1 and the definition of $c_1(u,g)$, (3.10) becomes

$$\begin{aligned}\langle g_0,h\rangle =& \langle f(u + c_1(u,\hat{g}) + c_2(u + c_1(u,\hat{g}),\hat{g})),h\rangle \\ &+ \langle(-\Delta - \lambda_j\tilde{I})(u + c_1(u,\hat{g}) + c_2(u + c_1(u,\hat{g}),\hat{g})),h\rangle,\end{aligned} \qquad (3.11)$$

where $\tilde{I}$ denotes the identity mapping. Since the operator $-\Delta - \lambda_j\tilde{I}$ is selfadjoint (3.11) gives (3.9).

Remark 3.2. By standard arguments, the result of Theorem 3.1 holds, if, instead the Laplacian, a general second order elliptic operator is considered in (3.8).

We close this Section by providing conditions under which $(F1)$ holds, i. e. such that f is a potential operator.

PROPOSITION 3.1. *Consider the function $\hat{f}(x,v)\colon = f(v) - \lambda_j v - g$ and let $N\colon L^2(\Omega) \to L^2(\Omega)$ denote its Nemitskiĭ operator, i.e. $Nv = \hat{f}(x,v(x))$, $x \in \Omega$, $v \in \mathbf{R}$. If N is continuous, then N is a potential operator, and we can find the functional φ whose gradient is N:*

$$\varphi(v) = \varphi_0 + \int_\Omega dx \int_0^{v(x)} \hat{f}(x,\xi)d\xi. \qquad (3.12)$$

Proof. Let $v(x),\xi(x) \in L^2(\Omega)$, and consider the curvilinear integral along the interval $v(x) + \rho\xi(x)$, $\rho \in [0,1]$

$$\begin{aligned}i\colon &= \int_0^1 (N(v+\rho\xi), d(v+\rho\xi)) = \int_0^1 (\hat{f}(x,v(x)+\rho\xi(x)), d(v(x)+\rho\xi(x))) \\ &= \int_0^1 d\rho \int_\Omega \hat{f}(x,v(x)+\rho\xi(x))\xi(x)dx = \int_\Omega dx \int_0^1 \hat{f}(x,v(x)+\rho\xi(x))\xi(x)d\rho \\ &= \int_\Omega dx \int_{v(x)}^{v(x)+\xi(x)} \hat{f}(x,z)dz = \Psi(v+\xi) - \Psi(v), \qquad (3.13)\end{aligned}$$

where $z = v(x) + \rho\xi(x)$, and, for any $\varphi(x) \in L^2(\Omega)$

$$\Psi(\varphi) = \int_\Omega dx \int_0^{\varphi(x)} \hat{f}(x,z)dz.$$

From [**11**, Cor. 2.1] it follows that (3.13) implies that i is independent of path, and then from Gavurin's Theorem [**11**, Th. 6.2], N is a potential operator.

Remark 3.2. The notation $\int_0^1 (N(v+\rho\xi), d(v+\rho\xi))$ is used to emphasize that the values of N are functionals.

Remark 3.3. As it is known, the continuity of N is equivalent to

$$|\hat{f}(x,v)| \leq \alpha(x) + \beta|v|, \quad \beta \geq 0, \quad \alpha(x) \in L^2(\Omega).$$

Such a condition is satisfied if f is, for instance, sublinear in the sense that $\lim_{v\to\infty} f(v)/v = 0$.

4. Regularity. In this section, it is proved that under suitable hypotheses, any weak ($H_0^1(\Omega)$) solution of our problem, is a classical solution. This is performed by a bootstrap argument, as follows:

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbf{R}^N$, $N \geq 2$. Let $\partial\Omega$ be of class $C^{2+\alpha}, \alpha \in (0,1)$. Let $g \in C^\alpha(\bar{\Omega})$. If the problem*

$$\begin{aligned} -\Delta u - \lambda_j u + f(u) &= g \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{4.1}$$

with f satisfying $(F2)$ has a weak $H_0^1(\Omega)$-solution u, then $u \in C^{2+\alpha}(\bar{\Omega})$.

Proof. Since u is a $H_0^1(\Omega)$-solution, it is in $L^2(\Omega)$. Assume that $u \in L^P(\Omega)$, $p \geq 2$. The conditions on f and g imply that $H \in L^P(\Omega)$, where

$$H(x) = g(x) + \lambda_j u(x) - f(u(x)). \tag{4.2}$$

Consider now the problem

$$\begin{aligned} -\Delta u &= H, \text{ in } (\Omega) \\ u &= 0, \text{ on } \partial\Omega. \end{aligned} \tag{4.3}$$

By [**1**, Th. 15.2], $u \in W^{2,P}(\Omega)$. By Sobolev's Embedding Theorem [**7**, Cor. 7.11], we get that

$$\begin{aligned} &\text{if } 2p \geq N, \quad \text{then } u \in L^q(\Omega), \quad \text{for } q \in [1,\infty) \text{ while} \\ &\text{if } 2p < N, \quad \text{then } u \in L^q(\Omega), \quad \text{for } q = Np/(N-2p). \end{aligned}$$

Repeating this argument a sufficient number of times, we arrive at

$$u \in W^{2,q}(\Omega) \text{ for } q \text{ such that } 0 < \alpha < 1 - N/q.$$

But $W^{2,q}(\Omega) \subseteq W^{1,q}(\Omega)$ and by [**8**, Th. 5.7.8 (i)], $u \in C^{1-N/q}$. Now $C^{1-N/q} \subseteq C^\alpha$ and hence $H \in C^\alpha$. By [**7**, Th. 6.14], there exists a unique solution $w \in C^{2+\alpha}$ of

$$\begin{aligned} -\Delta w &= H, \quad \text{in } \Omega \\ w &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{4.4}$$

Now u and w are $H_0^1(\Omega)$-solutions of (4.4), and since (4.4) has a unique solution, $u = w$. Hence u is a classical solution of (4.1).

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