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CURVATURE PINCHING FOR ODD-DIMENSIONAL MINIMAL SUBMANIFOLDS IN A SPHERE

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Abstract. Using Gauchman's method, we have improved Simons' pinching constant (for codimension $p \geq 3 - 2/(n-1)$) and Ejiri's Ricci curvature pinching constant for odd-dimensional minimal submanifolds in a sphere.

0. Introduction Let M^n be an n-dimensional compact minimal subman ifold in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. Let h be the second fundamental form of M^n and $f(u) = \|h(u,u)\|^2$ for any $u \in UM$. In [**2,4,5**], Gauchman developed a method which is different from that of Ros [**10,11**], but influenced by Ros' method. By use of this method, Gauchman studied the $f(u)$-pinching problems for minimal submanifolds in S^{n+p} [**4**], totally real minimal submanifolds in $CP^{n+p}(c)$ [**5**], and totally real minimal submanifolds in $HP^{n+p}(1)$ [**2**], respectively. In this paper, we find that Gauchman's method can be used for a study of curvature pinching problems of minimal submanifolds. We apply Gauchman's method and some other techniques to curvature pinching problems of minimal submanifolds in a sphere S^{n+p}. For odd-dimensional minimal submanifolds in a sphere, we have improved Simons' pinching constant (for codimension $p \geq 3 - 2/(n-1)$) (Theorem 2.2) and we have improved Ejiri's Ricci curvature pinching constant (Theorem 3.2). We also obtained a Ricci curvature pinching theorem which generalizes Shen's result for 3-dimensional minimal submanifolds in a sphere (Theorem 3.3). This paper is a part of my Ph.D. thesis (see [**9**]), which includes various results on curvature pinching theorems for minimal submanifolds in a sphere S^{n+p}, totally real minimal submanifolds in a complex projective space $CP^{n+p}(c)$ and totally real minimal submanifolds in a quaternion projective space $HP^{n+p}(1)$, respectively (see also [**7,8**]).

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1. Preliminaries Let M be an n-dimensional compact Riemannian manifold which is immersed isometrically in an $(n+p)$-dimensional Riemannian manifold N^{n+p}. We choose a local field of orthonormal frames $e_1, \dots, e_{n+p}$ in N^{n+p} in such a way that, when restricted to M, vectors $e_1, \dots, e_n$ are tangent to M. The following conventions for the range of indices will be used

$$1 \le A, B, C, \dots \le n+p; \qquad 1 \le i, j, k, \dots \le n;$$
$$n+1 \le \alpha, \beta, \gamma, \dots \le n+p.$$

Let ω_A be the field of dual frames with respect to the frame field of N^{n+p} chosen above. Then, if they are restricted to M, we have

$$\omega_\alpha = 0, \quad \omega_{\alpha i} = \sum_j h^\alpha_{ij}\omega_j, \quad h^\alpha_{ij} = h^\alpha_{ji}.$$

The second fundamental form of M in N^{n+p} is

$$h(X,Y) = \sum_{\alpha,i,j} h^\alpha_{ij}\omega_i(X)\omega_j(Y)e_\alpha, \quad \text{for } X, Y \in TM. \tag{1.1}$$

Let $UM = \bigcup_{x\in M} UM_x$ and $UM_x = [u \in TM_x : \|u\| = 1]$. Thus $UM \to M$ is the unit tangent bundle over M. We define $f(u) = \|h(u,u)\|^2$ for $u \in UM$. Setting $u = \sum_i u^i e_i$, from (1.1) we have

$$f(u) = \sum_\alpha \left(\sum_{ij} h^\alpha_{ij} u^i u^j \right)^2. \tag{1.2}$$

$f(u)$ may be considered as a measure of the degree at which an immersion fails to be totally geodesic.

Let $x \in M$, suppose that $v \in UM_x$ satisfies $f(v) = \max_{u \in UM_x} f(u)$. We shall call v a maximal direction at x (see [**4**,**5**]). Assume that $e_1 = v$ is a maximal direction; we have at the point x, for any $t, x_2, \dots, x_n \in R$

$$\left\| h\big(e_1 + t\sum_{k\neq 1} x^k e_k,\ e_1 + t\sum_{k\neq 1} x^k e_k\big) \right\|^2 \le \left[1 + t^2 \sum_{k\neq 1}(x^k)^2\right]^2 \|h_{11}\|^2. \tag{1.3}$$

Expanding this in term of t, we obtain

$$4t \sum_{\alpha, k\neq 1} x^k h^\alpha_{11} h^\alpha_{1k} + O(t^2) \le 0.$$

It follows that

$$\sum_\alpha h^\alpha_{11} h^\alpha_{1k} = 0, \quad (k \neq 1)$$

which implies that $v = e_1$ is an eigenvector of the $(n\times n)$-matrix $(\sum_\alpha h^\alpha_{11} h^\alpha_{ij})$ at x. Hence, we can choose $e_2, \dots, e_n$ such that the matrix $(\sum_\alpha h^\alpha_{11} h^\alpha_{ij})$ is diagonalized at x. Therefore we have

$$\sum_\alpha h^\alpha_{11} h^\alpha_{ij} = 0, \quad (i \neq j). \tag{1.4}$$

Once more expanding (1.3) in terms of t, we obtain

$$2t^2\Big[\sum_{\alpha,k\neq 1}\big((h_{11}^\alpha)^2 - h_{11}^\alpha h_{kk}^\alpha - 2(h_{1k}^\alpha)^2\big)(x^k)^2 - 2\cdot\sum_{\alpha,i\neq j,i\neq 1,j\neq 1} h_{1i}^\alpha h_{1j}^\alpha x^i x^j\Big] + O(t^3)\geq 0. \tag{1.5}$$

Since (1.5) must hold for any real x^i, we obtain the following variational inequality

$$\sum_\alpha\big[(h_{11}^\alpha)^2 - h_{11}^\alpha h_{kk}^\alpha - 2(h_{1k}^\alpha)^2\big]\geq 0,\quad (k\neq 1). \tag{1.6}$$

Let M be a Riemannian manifold and L be a covariant tensor field on M of the type $(0,k)$. At any $x\in M$, L can be considered as a multilinear mapping $L:\ T_xM\times\ldots\times T_xM\to R$. Suppose that $v\in UM_x$ satisfies $L(v,\ldots,v)=\max_{u\in UM_x} L(u,\ldots,u)$. We shall call v a maximal direction at x with respect to L. For any $x\in M$, we set $f_L(x)=L(v,\ldots,v)$, where v is a maximal direction at x with respect to L. We have the following generalized Bochner's lemma.

Lemma 1.1 (Proposition 3.1 of [**5**]). *Let M be a compact Riemannian manifold and L be a covariant tensor field on M of the type $(0,k)$. If $(\Delta L)(v,\ldots,v)\geq 0$ for any maximal direction v with respect to L, where Δ denotes the Laplace operator, then $f_L=$ constant on M and $(\Delta L)(v,\ldots,v)=0$ for any maximal direction v.*

Let M be an n-dimensional compact submanifold in N^{n+p}. For any point $x\in M$, let $e_1,\ldots,e_{n+p}$ be a frame chosen above at x such that $e_1=v$ is a maximal direction at x, and $\sum_\alpha h_{11}^\alpha h_{ij}^\alpha=0$ for $i\neq j$. Let us define a 4-covariant tensor field L on M by the formula

$$L(X,Y,Z,W)=\langle h(X,Y),h(Z,W)\rangle, \tag{1.7}$$

where $X,Y,Z,W\in T_x(M)$, $x\in M$. It is clear that $f(u)=L(u,u,u,u)=\|h(u,u)\|^2$ for any $u\in UM$. We shall write $(\Delta L)_{ijkl}=(\Delta L)(e_i,e_j,e_k,e_l)$.

Therefore we have proved the following lemma ensuing from (1.2), (1.4), (1.6), (1.7) and Lemma 1.1.

Lemma 1.2 *Let M be a compact n-dimensional submanifold in an $(n+p)$--dimensional Riemannian manifold N^{n+p}. Let $b_{ij}=\sum_\alpha h_{11}^\alpha h_{ij}^\alpha$. With respect to the frame field chosen above, we have at any point $x\in M$*

$$f(v)=b_{11}=\sum_\alpha (h_{11}^\alpha)^2=\max_{u\in UM_x}[\|h(u,u)\|^2], \tag{1.8}$$

$$\frac{1}{2}(\Delta L)_{1111}=\sum_{\alpha,k}(h_{11k}^\alpha)^2+\sum_{\alpha,k}h_{11}^\alpha h_{11kk}^\alpha, \tag{1.9}$$

$$b_{ij}=0\quad (i\neq j), \tag{1.10}$$

$$2\sum_{\alpha}(h_{1k}^{\alpha})^2 + b_{kk} - f(v) \le 0, \quad (k \neq 1). \tag{1.11}$$

If $(\Delta L)_{1111} \ge 0$ *for any maximal direction* $e_1 = v$*, then* $f(v) = b_{11} =$ *constant on* M *and* $(\Delta L)_{1111} = 0$ *for any maximal direction* $e_1 = v$*.*

2. Scalar curvature pinching for odd-dimensional minimal submanifolds in $\mathbf{S}^{n+p}$. Now we let ambient space N^{n+p} be a unit sphere S^{n+p} of dimension $n+p$. Let M^n be an n-dimensional compact minimal submanifold in S^{n+p}. Gauss-Codazzi-Ricci equations of M^n are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_{\alpha}(h_{ik}^{\alpha}h_{jl}^{\alpha} - h_{il}^{\alpha}h_{jk}^{\alpha}), \tag{2.1}$$

$$h_{ijk}^{\alpha} = h_{ikj}^{\alpha}, \tag{2.2}$$

$$R_{\alpha\beta ij} = \sum_{k}(h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta}), \tag{2.3}$$

where R_{ijkl} and $R_{\alpha\beta ij}$ are the respective curvature tensors for tangent connection and the normal connection of M^n and h_{ijk}^{α} is the covariant derivative of h_{ij}^{α}.

By (2.1) the Ricci curvature and scalar curvature of M^n are

$$R_{ij} = (n-1)\delta_{ij} - \sum_{\alpha,k} h_{ik}^{\alpha}h_{kj}^{\alpha}, \tag{2.4}$$

$$R = n(n-1) - \|\sigma\|^2, \tag{2.5}$$

where $\|\sigma\|^2 = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2$.

It is well known [**1**,**13**] that if the lenght square $\|\sigma\|^2$ of the second fundamental form on M^n satisfies

$$\|\sigma\|^2 \le \frac{n}{2 - 1/p}$$

everywhere, then either $\|\sigma\|^2 = 0$ (i.e. M^n is totally geodesic) or

$$\|\sigma\|^2 = \frac{n}{2 - 1/p}.$$

In the latter case M^n is either a Clifford hypersurface or a Veronese surface in S^4. In [**8**], we have improved Simons' pinching constant for higher codimension. In fact, we have established

Theorem 2.1 [**8**]. *Let* M^n *be an* n*-dimensional* $(n \ge 2)$ *compact minimal submanifold in* S^{n+p}*. If*

$$\|\sigma\|^2 \le \frac{n(3n-2)}{5n-4}, \tag{2.6}$$

then M^n *is either a totally geodesic submanifold or a Veronese surface in* S^4*.*

In this section, we will improve the theorem above for odd-dimensional minimal subamnifolds in S^{n+p}. We will prove

Theorem 2.2. *Let M^n be a compact n-dimensional $(n \geq 3)$ minimal submanifold in S^{n+p}, and let n be odd. If*

$$\|\sigma\|^2 \leq \frac{n(3n-5)}{5n-9}, \tag{2.7}$$

then M^n is either a totally geodesic submanifold or $n = 3$ and $\|\sigma\|^2 = 2$ on M^3 and the second fundamental form is given by

$$\begin{gathered}(h^4_{ij}) = \begin{pmatrix} 1/\sqrt{2} & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (h^5_{ij}) = \begin{pmatrix} 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ (h^\alpha_{ij}) = 0, \;\; \alpha \geq 6. \end{gathered} \tag{2.8}$$

Remark 2.1. For odd-dimensional minimal submanifolds in S^{n+p}, our pinching constant $n(3n-5)/(5n-9)$ is independent of the codimension p of M^n and is not smaller than Simons' pinching constant $n/(2-1/p)$ in case of $p \geq 3 - 2/(n-1)$ (i.e. $n = 3$ and $p \geq 2$; $n \geq 5$ and $p \geq 3$).

Remark 2.2. Theorem 2.2 improves Theorem 2.1 for odd-dimensional minimal submanifolds in a sphere S^{n+p}.

Corollary 2.1 [**12**]. *Let M^3 be a compact 3-dimensional minimal submanifold in S^{3+p}. If*

$$\|\sigma\|^2 < 2, \tag{2.9}$$

then M^3 is a totally geodesic submanifold.

Remark 2.3. In [**4**], Gauchman obtained results (Theorem 3 and Theorem 4 of [**4**]) of kind described in Theorem 2.1 and Theorem 2.2 in which $f(u)$ was used instead of $\|\sigma\|^2$ for minimal submanifolds in a sphere, where $f(u) = \|h(u,u)\|^2$ for any $u \in UM$.

Proof of Theorem 2.2. We begin with Lemma 1.2. All the calculations below will be made at a point $x \in M$, unless otherwise stated. By Ricci identities, (2.2) and (1.10), from (1.9) we get

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq \sum_{\alpha,i} h^\alpha_{11} h^\alpha_{1i1i} \\ &= \sum_{\alpha,i}(h^\alpha_{11}h^\alpha_{ii}R_{i11i} + (h^\alpha_{11})^2 R_{1i1i}) + \sum_{\alpha,\beta,i} h^\alpha_{11}h^\beta_{1i}R_{\beta\alpha 1i}.\end{aligned} \tag{2.10}$$

Making use of (2.1), (1.10) and (2.3), one easily sees that

$$\begin{aligned}&\sum_{\alpha,i}(h^\alpha_{11}h^\alpha_{ii}R_{i11i} + (h^\alpha_{11})^2 R_{1i1i}) \\ &= nf(v) + \sum_{\alpha,k} b_{kk}(h^\alpha_{1k})^2 - \sum_k (b_{kk})^2 - f(v)\sum_{\alpha,k}(h^\alpha_{1k})^2,\end{aligned} \tag{2.11}$$

$$\sum_{\alpha,\beta,i} h^\alpha_{11} h^\beta_{1i} R_{\beta\alpha 1i} = \sum_{\alpha,k} b_{kk}(h^\alpha_{1k})^2 - f(v)\sum_{\alpha,k}(h^\alpha_{1k})^2.$$

Substituting (2.11) and (2.12) into (2.10), we obtain

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} \geq & nf(v) + 2\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 \\ & - \sum_{k\neq 1}(b_{kk})^2 - 2f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 - f(v)\sum_{\alpha}(h^\alpha_{11})^2.\end{aligned} \tag{2.13}$$

From (1.8) and (1.11) it follows that

$$2\sum_\alpha (h^\alpha_{1k})^2 \leq f(v) - b_{kk} \leq f(v) + \sqrt{\sum_\alpha (h^\alpha_{11})^2 \sum_\alpha (h^\alpha_{kk})^2} \leq 2f(v)$$

i.e. $\sum_\alpha (h^\alpha_{1k})^2 \leq f(v)$. Combining this with an elementary inequality, we find

$$\begin{aligned}2\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 &\geq -\frac{1}{a}\sum_{k\neq 1}(b_{kk})^2 - a\sum_{k\neq 1}\left(\sum_\alpha (h^\alpha_{1k})^2\right)^2 \\ &\geq -\frac{1}{a}f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{kk})^2 - af(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2,\end{aligned} \tag{2.14}$$

where $a > 0$ is an arbitrary real number. On the other hand $(b_{kk})^2 \leq f(v)\sum_\alpha (h^\alpha_{kk})^2 \leq f(v)^2$, $(f(v)+b_{kk})(f(v)-b_{kk}) \geq 0$. Combining this with (1.11), we have $b_{kk} \geq -f(v)$, therefore we get the following estimate

$$2\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 \geq -2f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2. \tag{2.15}$$

Combining (2.14) with (2.15), we obtain the following estimate

$$\begin{aligned}2\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 &= b\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 + (2-b)\sum_{\alpha,k\neq 1} b_{kk}(h^\alpha_{1k})^2 \\ &\geq -\frac{bf(v)}{2a}\sum_{\alpha,k\neq 1}(h^\alpha_{kk})^2 - (2-b+\frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2,\end{aligned} \tag{2.16}$$

where $a > 0$ and $2 \geq b \geq 0$ are arbitrary real numbers.

By (2.13) and (2.16), we have

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} \geq & nf(v) - (4-b+\frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 \\ & - \frac{b}{2a}f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{kk})^2 - \sum_{k\neq 1}(b_{kk})^2 - f(v)^2.\end{aligned} \tag{2.17}$$

We can write $b_k = b_{kk} = \sum_\alpha h_{11}^\alpha h_{kk}^\alpha$. By (1.8) and minimality of the immersion, we have

$$-f(v) \le b_k \le f(v), \quad (k \ne 1). \tag{2.18}$$

$$\sum_{k=2}^n b_k = \sum_{k=2}^n b_{kk} = -f(v). \tag{2.19}$$

Because we assume that n is an odd number, it can easily be seen that the convex function $f(b_2, \dots, b_n) = \sum_{k=2}^n (b_k)^2$ of $(n-1)$ variables $b_2, \dots, b_n$ subject to the linear constraints (2.18) and (2.19) attains its maximal value when (after suitable renumbering of $e_1, \dots, e_n$) (see [**5**])

$$b_2 = \ldots = b_m = -b_{m+1} = \ldots = -b_{2m} = f(v); \;\; b_{2m+1} = 0,$$

where $n = 2m+1$. Therefore, we have

$$\sum_{k \ne 1} (b_{kk})^2 \le (n-2) f(v)^2. \tag{2.20}$$

We also know, by the Cauchy inequality, that

$$\sum_{k \ne 1} (b_{kk})^2 \le f(v) \sum_{\alpha, k \ne 1} (h_{kk}^\alpha)^2. \tag{2.21}$$

Combining (2.20) with (2.21), we have

$$\begin{aligned} -\sum_{k \ne 1} (b_{kk})^2 &= -(1 - \frac{b}{2(n-1)a}) \sum_{k \ne 1} (b_{kk})^2 - \frac{b}{2(n-1)a} \sum_{k \ne 1} (b_{kk})^2 \\ &\ge -(1 - \frac{b}{2(n-1)a}) f(v) \sum_{\alpha, k \ne 1} (h_{kk}^\alpha)^2 - \frac{(n-2)b}{2(n-1)a} f(v)^2. \end{aligned} \tag{2.22}$$

Substituing (2.22) into (2.17), we obtain

$$\begin{aligned} &\frac{1}{2}(\Delta L)_{1111} \\ &\ge f(v) \Big[n - (4 - b + \frac{ab}{2}) \sum_{\alpha, k \ne 1} (h_{1k}^\alpha)^2 - (1 + \frac{(n-2)b}{2a(n-1)}) \sum_{\alpha, k} (h_{kk}^\alpha)^2 \Big]. \end{aligned} \tag{2.23}$$

Let

$$4 - b + \frac{ab}{2} = 2 \frac{1 + (n-2)b}{2(n-1)a}, \quad \text{i.e.} \quad b = \frac{4(n-1)a}{3n - 5 - (n-1)(a-1)^2}.$$

Noting that $\|\sigma\|^2 = \sum_{\alpha, i, j} (h_{ij}^\alpha)^2 \ge \sum_\alpha (h_{kk}^\alpha)^2 + 2 \sum_{\alpha, k \ne 1} (h_{1k}^\alpha)^2$, choosing $a = 1$, we obtain from (2.23)

$$\frac{1}{2}(\Delta L)_{1111} \ge f(v) \big[n - \frac{5n-9}{3n-5} \|\sigma\|^2(x) \big]. \tag{2.24}$$

By (2.7), $(\Delta L)_{1111} \geq 0$. We obtain $(\Delta L)_{1111} = 0$ from Lemma 1.2. Thus, if $f(v) = 0$, then $\|h(u,u)\|^2 = 0$ for any $u \in UM$, so that M^n is totally geodesic. If $f(v) \neq 0$, then $\|\sigma\|^2(x) = n(3n-5)/(5n-9)$, so that (2.13) - (2.24) all are equalities with $a = 1$ and $b = 4(n-1)/(3n-5)$. We easily get $n = 3$, and we have $h^\alpha_{11} = -h^\alpha_{22}$, $h^\alpha_{33} = 0$, $h^\alpha_{13} = h^\alpha_{23} = 0$, $\sum_\alpha (h^\alpha_{12})^2 = f(v)$ and $\|\sigma\|^2 = 2$ on M^3. By (1.10), we can choose $e_4 = h(e_1,e_1)/\sqrt{f(v)}$ and $e_5 = h(e_1,e_2)/\sqrt{f(v)}$. Therefore we have (2.8) and that completes the proof.

3. Ricci curvature pinching for odd-dimensional minimal submanifolds in $\mathbf{S}^{n+p}$. Ejiri [**3**] obtained the following well known Ricci curvature pinching theorem

THEOREM 3.1. *Let M^n be a compact n-dimensional $(n \geq 4)$ minimal submanifold in S^{n+p}. If the Ricci curvature of M^n satisfies*

$$\mathrm{Ric}(M^n) \geq n-2, \tag{3.1}$$

then M^n is totally geodesic, or $n = 2m$ and $M^n = S^m(\sqrt{1/2}) \times S^m(\sqrt{1/2})$ or $n = 4$ and $M^4 = CP^2(4/3) \to S^7$.

It is generally considered that the above theorem is the best possible result, but, in fact, Ejiri's theorem above is only the possible best result for even-dimensional minimal submanifolds in S^{n+p}. In this section we establish the following best possible Ricci curvature pinching theorem for odd-dimensional minimal submanifolds in S^{n+p}

THEOREM 3.2. *Let M^n be a compact n-dimensional $(n \geq 5)$ minimal submanifold in S^{n+p}. Assume that n is odd. If the Ricci curvature of M^n satisfies*

$$\mathrm{Ric}(M^n) \geq n - 2 - 1/(n-1), \tag{3.2}$$

then M^n is either a totally geodesic submanifold or $n = 5$ and $R_{11} = R_{22} = R_{33} = R_{44} = 3 - 1/4$, $R_{55} = 4$ and $\|\sigma\|^2 = 5$ on M^5.

Remark 3.1. Our Ricci curvature pinching constant $(n-2-1/(n-1))$ is better than Ejiri's $(n-2)$ for odd-dimensional minimal submanifold M^n in S^{n+p}.

Proof of Theorem 3.2 By (2.13), (2.15) and (2.20), we get

$$\frac{1}{2}(\Delta L)_{1111} \geq nf(v) - 4f(v)\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 - (n-1)f(v)^2. \tag{3.3}$$

From (2.4), our assumption (3.2) and from: $R_{11} = (n-1) - f(v) - \sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2$, we have

$$\sum_{\alpha,k\neq 1}(h^\alpha_{1k})^2 \leq \frac{n}{n-1} - f(v). \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq nf(v) - 4f(v)\Big(\frac{n}{n-1} - f(v)\Big) - (n-1)f(v)^2 \\ &= (n-5)f(v)\Big(\frac{n}{n-1} - f(v)\Big).\end{aligned} \tag{3.5}$$

By (3.4) we know that $n/(n-1) - f(v) \geq 0$. Thus $(\Delta L)_{1111} \geq 0$. By Lemma 1.2, $(\Delta L)_{1111} = 0$ and $f(v) =$ constant on M^n. Therefore it follows that $f(v) = 0$, or $f(v) = n/(n-1)$, or $n = 5$.

(1) Case $f(v) = 0$. M^n is totally geodesic.

(2) Case $f(v) = n/(n-1)$. In this case (2.20) is an equality. Thus for all α we get (after suitable renumbering of $e_1, \dots, e_n$)

$$h_{11}^{\alpha} = \dots = h_{mm}^{\alpha} = -h_{m+1\ m+1}^{\alpha} = \dots = -h_{2m\ 2m}^{\alpha},\ h_{nn}^{\alpha} = 0. \tag{3.6}$$

On the other hand, by (3.4), we have $h_{1k}^{\alpha} = 0,\ k \neq 1,\ \alpha = n+1, \dots, n+p$. Since by (3.6), directions $e_1, \dots, e_{2m}$ all are maximal, it follows that

$$h_{ij}^{\alpha} = 0, \quad i \neq n,\ j \neq i,\ \alpha = n+1, \dots, n+p. \tag{3.7}$$

This implies $h_{ij}^{\alpha} = 0,\ i \neq j,\ \alpha = n+1, \dots, n+p$, i.e., M^n is a submanifold with a flat normal connection. From (3.6) and (3.7), we have

$$\|\sigma\|^2 = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2 = \sum_{\alpha,k} (h_{kk}^{\alpha})^2 = n. \tag{3.8}$$

By Kenmotsu's theorem [**6**], we have $M^n = S^k(\sqrt{k/n})\ S^{n-k}(\sqrt{(n-k)/n})$ and $p = 1$. But it contradicts the following

$$h_{11} = \dots = h_{mm} = -h_{m+1\ m+1} = \dots = -h_{2m\ 2m} = \sqrt{n/(n-1)},\ h_{nn} = 0. \tag{3.9}$$

Thus $f(v) = n/(n-1)$ is false. We have $f(v) = 0$, i.e. M^n is totally geodesic.

(3) Case $n = 5$ and $f(v) \neq n/(n-1)$. By Lemma 1.2, $f(v) =$ constant on M^5 and (3.5) is an equality. Thus, (2.13), (2.15), (2.20), (3.3) - (3.5) all are identities and $R_{11} = 3 - 1/4$. By (2.20), we have for all α

$$h_{11}^{\alpha} = h_{22}^{\alpha} = -h_{33}^{\alpha} = -h_{44}^{\alpha},\ h_{55}^{\alpha} = 0. \tag{3.10}$$

By (2.4) (in this case), for all α we have $h_{15}^{\alpha} = 0$. Because (3.10) implies that the directions e_1, e_2, e_3 and e_4 are all maximal, we have $h_{k5}^{\alpha} = 0$ and

$$R_{11} = R_{22} = R_{33} = R_{44} = 3 - 1/4,\ R_{55} = 4. \tag{3.11}$$

Thus $R = 15$ and $\|\sigma\|^2 = 5$ on M^5. By (1.11) and (3.4), we find that $5/12 \leq f(v) < 5/4$. From (2.15), we also know that $h_{12}^{\alpha} = h_{34}^{\alpha} = 0$ and the proof is completed.

Neither Theorem 3.1 nor Theorem 3.2 yields any results for 3-dimensional minimal submanifolds in a sphere. For that case we establish the following theorem

THEOREM 3.3. *Let M^3 be a 3-dimensional compact minimal submanifold in S^{3+p}. If the Ricci curvature of M^3 satisfies*

$$\mathrm{Ric}(M^3) \geq 1, \tag{3.12}$$

then M^3 is either totally geodesic, or $R_{11} = R_{22} = 1,\ R_{33} = 2$ and $\|\sigma\|^2 = 2$ on M^3 and the second fundamental form is given by (2.8).

COROLLARY 3.1 [**12**]. *Let M^3 be a 3-dimensional compact minimal submanifold in S^{3+p}. If the Ricci curvature of M^3 satisfies*

$$Ric(M^3) > 1, \tag{3.13}$$

then M^3 is totally geodesic.

Proof of Theorem 3.3. By $b_{kk} \geq -f(v)$ and the 3-dimensional minimality, we can see that

$$b_{22} \leq 0,\ b_{33} \leq 0,\ \sum_{k\neq 1}(b_{kk})^2 \leq \left(\sum_{k\neq 1} b_{kk}\right)^2 = (b_{11})^2. \tag{3.14}$$

By the definition of b_{ij} (see Lemma 1.2), we have from (2.4)

$$-\sum_{\alpha,k\neq 1}(h^{\alpha}_{1k})^2 = R_{11} - 2 + b_{11}.$$

From (3.14) and (1.11), we get

$$\sum_{\alpha,k\neq 1} b^{\alpha}_{kk}(h^{\alpha}_{1k})^2 \geq \frac{1}{2}\sum_{k\neq 1} b_{kk}(b_{11} - b_{kk}) = -\frac{1}{2}\sum_{k}(b_{kk})^2. \tag{3.16}$$

Substituting (3.15) into (2.13) in case of $n = 3$ and using (3.14), we come to

$$\frac{1}{2}(\Delta L)_{1111} \geq -f(v) + 2\sum_{\alpha,k\neq 1} b_{kk}(h^{\alpha}_{1k})^2 + 2f(v)R_{11}. \tag{3.17}$$

Applying (2.15) and (3.16) on (3.17), by (3.14)

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq 2f(v)R_{11} - f(v) + f(v)(R_{11} - 2 + b_{11}) - \frac{1}{2}\sum_{k}(b_{kk})^2\\ &\geq 3f(v)(R_{11} - 1).\end{aligned} \tag{3.18}$$

By Lemma 1.2, (3.12) and (3.18) imply that either $f(v) = 0$, i.e. M^3 is totally geodesic, or $R_{11} = 1$. In the latter case, (3.14) - (3.18) all are identities. By a similar argument as in the proof of Theorem 3.2, we have

$$R_{11} = R_{22} = 1,\ \ R_{33} = 2. \tag{3.19}$$

Thus $\|\sigma\|^2 = 6 - R = 2$ on M^3. So, we complete the proof of Theorem 3.3 from Theorem 2.2.

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