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A NOTE ON REMOTAL SETS IN BANACH SPACES

Marco Baronti

Abstract. We give some conditions for a remotal set to be a singleton. Moreover, we give necessary and sufficient conditions providing that in a Hilbert space every bounded and closed set is remotal.

0. Introduction. Let X be a real Banach space and let $T \subset X$ be a bounded nonempty set. The set$-$valued map defined by:

$$Q_T(x) = \big\{q_T(x) \in T : \ \|x - q_T(x)\| = \sup\{\|x-t\| : \ t \in T\}\big\}$$

is called the farthest point map of T. If for any $x \in X$ the set $Q_T(x)$ is nonempty, then T is said to be remotal. If for $x \in X$ the set $Q_T(x)$ is a singleton, then T is said to be uniquely remotal. Of course, if T is compact, then T is remotal. We point out that remotal sets need not to be closed.

A Chebyshev centre of T is an $x_0 \in X$ for which:

$$\sup\{\|x_0 - t\| : \ t \in T\} = \inf\{\sup\{\|x-t\| : \ t \in T\} : \ x \in X\} = r(T)$$

The number $r(T)$ is said to be Chebyshev radius of T. So $r(T)$ is the radius of the smallest ball in X which contains the set T. The collection of Chebyshev centres of T is denoted by $E(T)$. We say that X admits centres, if for any nonempty bounded subset T we have that $E(T)$ is nonempty. Necessary and sufficient conditions for the existence and uniqueness of Chebyshev centres are well known [**4**].

The following well known questions have not been answered yet:

1) What conditions, on the space X, can ensure that each closed and bounded set is remotal?

2) If T is a uniquely remotal set in a normed space, then can one conclude that T is a singleton?

The Question 2 is essential in the theory of farthest points, because it is connected with the following question:

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3) In a Hilbert space, is every Chebyshev set convex? (We recall that T is a Chebyshev set if for any $x \in X$ there is a unique $t \in T$ which is the best approximation of x in T).

Klee has shown that an affirmative answer to Question 2 implies an affirmative answer to Question 3. Also, there are some partially affirmative answers to Question 2. ([**1**], [**2**], [**3**] and [**4**]).

We quote the following result of Klee:

THEOREM. *Let X be a normed space and let T be a nonempty subset of X. Then, if T is a compact and uniquely remotal set, then T is a singleton.*

We shall give the answer to Question 1 when X is a Banach space admitting a monotone basis. We recall that $\{e_n : n \in N\}$ is a monotone basis if $\{e_n : n \in N\}$ is a basis such that:

$$\mathrm{span}[e_1, e_2, \dots, e_n] \perp \mathrm{span}[e_{n+1}, e_{n+2}, \dots, e_{n+m}] \quad \text{for any } n, m \in \mathbf{N}$$

where $x \perp y$ means $\|x\| \le \|x + ty\|$ for any t in $\mathbf{R}$. Then we shall improve some results of Panda and Kapoor [**8**].

1. Main results. We give an answer to Question 1 in a particular case which includes Hilbert spaces.

THEOREM 1. *Let X be a Banach space admitting a monotone basis. Every nonempty closed and bounded subset T is remotal if and only if X is finite-dimensional.*

Proof. "If part" is obvious. For the "Only if" part we suppose that X is an infinite-dimensional Banach space. Let $\{e_n : n \in N\}$ be a monotone normalized basis; we denote by T the following set $T = \{(1 - 1/n)e_n : n \in N\}$. Of course T is bounded and closed, so T is remotal. We have

$$\sup\{\|(1 - 1/n)e_n\| : n \in N\} = 1 > \|(1 - 1/n)e_n\| \text{ for any } n \in N$$

Hence T is not remotal and this contradiction proves the theorem.

LEMMA 2. *Let X be a Banach space and let x and t be in X, such that $\|x\| \ge \|x - t\|$. Then $k\|x\| \ge \|kx - t\|$ for any $k \ge 1$.*

Proof. We set $F : [0, +\infty) \to R, F(k) = \|kx - t\| - \|kx\|$. F is a convex function such that $F(0) = \|t\| \ge F(k)$ for any k, and $F(1) \le 0$. Thus we have $F(k) \le 0$ for any $k \ge 1$.

LEMMA 3. *Let X be a Banach space and let x and t be in X, such that $\|x\| \le \|x - t\|$. Then $k\|x\| \le \|kx - t\|$ for any k in* [0,1].

Proof. We can use an argument similar to the one in the preceding proof.

Now we introduce a new statement. Let X be a Banach space and let T be a bounded subset of X. Moreover, let x be in X and k in R, we say that the condition $P(x, d)$ is true if and only if:

$$q_T(x) \in Q_T(x) \Rightarrow q_T(x) \in Q_T(x + k(-x + q_T(x))), \quad \text{for any } k \le d.$$

THEOREM 4. *Let X and T be the same as above. Then, for any x in $X, P(x,0)$ holds true.*

Proof. Let $q_T(x)$ be in $Q_T(x)$, $h \leq 0$, and t in T. Setting $k = 1-h$, $x' = x - q_T(x)$, $t' = t - q_T(x)$ we get: $k \geq 1$, $\|x'\| \geq \|x' - t'\|$ and, hence, by using Lemma 2, we obtain $k\,\|x'\| \geq \|kx' - t'\|$.

THEOREM 5. *Let X be a Banach space admitting centres. Let T be a remotal set. If there is a centre c in $E(T)$, such that $P(c,d)$ is true with $d > 0$, then T is a singleton.*

Proof. Let c be an element of $E(T)$ such that $P(c,d)$ is true with $d > 0$. Let $q_T(c)$ be in $Q_T(c)$. Of course $\|c - q_T(c)\|$ is the Chebyshev radius of T, so $T \subseteq B(c, \|c - q_T(c)\|)$, where $B(x,r) = \{y \in X : \ \|x-y\| \leq r\}$. Let k be such that $0 < k < \min\{1,d\}$. Now we suppose that $\|c - q_T(c)\| > 0$; $(1-k)\|c - q_T(c)\| < \|c - q_T(c)\|$. Then, by definition of the Chebyshev radius, we have: $T \not\subseteq B\ (c - k(c - q_T(c)), (1-k)\|c - q_T(c)\|)$, so there is a t_0 in T such that $\|c - k(c - q_T(c)) - t_0\| > (1-k)\|c - q_T(c)\|$. From the fact that $P(c,d)$ is true, one obtains

$$\begin{aligned}\|c - k(c - q_T(c)) - t_0\| &\leq \|c - k(c - q_T(c)) - q_T(c)\| \\ &= (1-k)\|c - q_T(c)\|.\end{aligned}$$

This contradiction implies that $\|c - q_T(c)\| = 0$, that is T is a siglenton.

Remark 6. In Theorem 5 it is enough to suppose that $Q_T(c)$ is nonempty.

Remark 7. We point out that any convex remotal set T has a unique centre if X is a strictly convex dual space [**7**].

In [**8**] Panda and Kapoor proved the following result

THEOREM 8. *Let X be a normed space admitting centres, and let T be a nonempty unique remotal subset. If the farthest point map q_T is inner radial upper semi-continuous (I.R.U. continuous) on $T + r(T)B(0,1)$, then T is a singleton.*

We recall that q_T is I.R.U. continuous at x_0 if for any v_0 in $q_T(x_0)$ and an open set W containing $q_T(x_0)$ there is a neighbourhood N of x_0 such that $q_T(x) \subseteq W$ for any x in $N \cap \{v_0 + k(-v_0 + x_0) : \ 0 \leq k \leq 1\}$. We observe that if $P(c,d)$ is true, then q_T is I.R.U. continuous at c, but in spite of that, Theorem 5 improves Theorem 8 because we suppose that only T is remotal (or that $Q_T(c)$ is nonempty) and that the map q_T is continuous only at one element of X.

THEOREM 9. *Let X be a strictly convex Banach space. Let T be a nonempty remotal subset. If for any x in X there is a $d > 0$ such that $P(x,d)$ is true, then T is a unique remotal set.*

Proof. If T is a singleton, then we have the assertion so we can suppose that T is not a singleton. Let $x \in X, q_T(x) \in Q_T(x)$, and $t \in T$, such that $t \neq q_T(x)$. Set $v = x - t$, $u = -x + q_T(x)$. We have $\|v\| \leq \|u\|$. If we suppose that $\|v\| = \|u\|$ and that $P(x,d)$ is true, then we have $\|ku + v\| \leq (1-k)\|u\|$ for any $k < d' = \min\{d, 1\}$. So, the function $F : (-\infty, d'] \to R$, defined by $F(h) = \|hu + v\| - (1-h)\|u\|$ is convex and $F(h) \leq 0$ for any $h < d'$. If we use Lemma 3 with $k = 1-h$, $x' = u$

and $t' = u + v$, we obtain $k\,\|x'\| \leq \|kx' - t'\|$, that is $F(h) \geq 0$ and so $F(h) = 0$ for any $h < d'$. Set $k = 1$. Then,

$$\|u - v\| = 2\|u\| = 2\|v\| \Rightarrow \left\| \frac{u}{\|u\|} + \frac{-v}{\|-v\|} \right\| = 2.$$

But X is strictly convex and, hence, $u = -v \Rightarrow -x + q_T(x) = -x + t \Rightarrow q_T(x) = t$. This contradiction implies that $\|v\| < \|u\|$, that is $\|x - q_T(x)\| < \|x - t\|$ for any $t \neq q_T(x)$, so we have the assertion.

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Istituto di Matematica
Facolta Ingegneria
Piazzale Kennedy Pad. D
16100 Genova
Italia

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