A NOTE ON REMOTAL SETS IN BANACH SPACES

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Abstract. We give some conditions for a remotal set to be a singleton. Moreover, we give necessary and sufficient conditions providing that in a Hilbert space every bounded and closed set is remotal.

0. Introduction. Let $X$ be a real Banach space and let $T \subset X$ be a bounded nonempty set. The set-valued map defined by:

$$Q_T(x) = \{q_T(x) \in T : \|x - q_T(x)\| = \sup\{\|x - t\| : t \in T\}\}$$

is called the farthest point map of $T$. If for any $x \in X$ the set $Q_T(x)$ is nonempty, then $T$ is said to be remotal. If for $x \in X$ the set $Q_T(x)$ is a singleton, then $T$ is said to be uniquely remotal. Of course, if $T$ is compact, then $T$ is remotal. We point out that remotal sets need not to be closed.

A Chebyshev centre of $T$ is an $x_0 \in X$ for which:

$$\sup\{\|x_0 - t\| : t \in T\} = \inf\{\sup\{\|x - t\| : t \in T\} : x \in X\} = r(T)$$

The number $r(T)$ is said to be Chebyshev radius of $T$. So $r(T)$ is the radius of the smallest ball in $X$ which contains the set $T$. The collection of Chebyshev centres of $T$ is denoted by $E(T)$. We say that $X$ admits centres, if for any nonempty bounded subset $T$ we have that $E(T)$ is nonempty. Necessary and sufficient conditions for the existence and uniqueness of Chebyshev centres are well known [4].

The following well known questions have not been answered yet:

1) What conditions, on the space $X$, can ensure that each closed and bounded set is remotal?

2) If $T$ is a uniquely remotal set in a normed space, then can one conclude that $T$ is a singleton?

The Question 2 is essential in the theory of farthest points, because it is connected with the following question:

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3) In a Hilbert space, is every Chebyshev set convex? (We recall that $T$ is a Chebyshev set if for any $x \in X$ there is a unique $t \in T$ which is the best approximation of $x$ in $T$).

Klee has shown that an affirmative answer to Question 2 implies an affirmative answer to Question 3. Also, there are some partially affirmative answers to Question 2. ([1], [2], [3] and [4]).

We quote the following result of Klee:

**Theorem.** Let $X$ be a normed space and let $T$ be a nonempty subset of $X$. Then, if $T$ is a compact and uniquely remotal set, then $T$ is a singleton.

We shall give the answer to Question 1 when $X$ is a Banach space admitting a monotone basis. We recall that $\{e_n : n \in N\}$ is a monotone basis if $\{e_n : n \in N\}$ is a basis such that:

$$\text{span}[e_1, e_2, \ldots, e_n] \perp \text{span}[e_{n+1}, e_{n+2}, \ldots, e_{n+m}]$$

for any $n, m \in \mathbb{N}$

where $x \perp y$ means $\|x\| \leq \|x + ty\|$ for any $t \in \mathbb{R}$. Then we shall improve some results of Panda and Kapoor [8].

**1. Main results.** We give an answer to Question 1 in a particular case which includes Hilbert spaces.

**Theorem 1.** Let $X$ be a Banach space admitting a monotone basis. Every nonempty closed and bounded subset $T$ is remotal if and only if $X$ is finite-dimensional.

**Proof.** "If part" is obvious. For the "Only if" part we suppose that $X$ is an infinite-dimensional Banach space. Let $\{e_n : n \in N\}$ be a monotone normalized basis; we denote by $T$ the following set $T = \{(1 - 1/n)e_n : n \in N\}$. Of course $T$ is bounded and closed, so $T$ is remotal. We have

$$\sup\{|(1 - 1/n)e_n| : n \in N\} = 1 > |(1 - 1/n)e_n|$$

for any $n \in N$

Hence $T$ is not remotal and this contradiction proves the theorem.

**Lemma 2.** Let $X$ be a Banach space and let $x$ and $t$ be in $X$, such that $\|x\| \geq \|x - t\|$. Then $k\|x\| \geq \|kx - t\|$ for any $k \geq 1$.

**Proof.** We set $F : [0, +\infty) \to \mathbb{R}, F(k) = \|kx - t\| - \|kx\|$. $F$ is a convex function such that $F(0) = \|t\| \geq F(k)$ for any $k$, and $F(1) \leq 0$. Thus we have $F(k) \leq 0$ for any $k \geq 1$.

**Lemma 3.** Let $X$ be a Banach space and let $x$ and $t$ be in $X$, such that $\|x\| \leq \|x - t\|$. Then $k\|x\| \leq \|kx - t\|$ for any $k$ in $[0,1]$.

**Proof.** We can use an argument similar to the one in the preceding proof.

Now we introduce a new statement. Let $X$ be a Banach space and let $T$ be a bounded subset of $X$. Moreover, let $x$ be in $X$ and $k$ in $R$, we say that the condition $P(x,d)$ is true if and only if:

$$q_T(x) \in Q_T(x) \Rightarrow q_T(x) \in Q_T(x + k(-x + q_T(x))), \text{ for any } k \leq d.$$
**Theorem 4.** Let $X$ and $T$ be the same as above. Then, for any $x$ in $X$, $P(x,0)$ holds true.

**Proof.** Let $q_T(x)$ be in $Q_T(x)$, $h \leq 0$, and $t$ in $T$. Setting $k = 1 - h$, $x' = x - q_T(x)$, $t' = t - q_T(x)$ we get: $k \geq 1$, $||x'|| \geq ||x' - t'||$ and, hence, by using Lemma 2, we obtain $k||x'|| \geq ||kx' - t'||$.

**Theorem 5.** Let $X$ be a Banach space admitting centres. Let $T$ be a remotal set. If there is a centre $c$ in $E(T)$, such that $P(c,d)$ is true with $d > 0$, then $T$ is a singleton.

**Proof.** Let $c$ be an element of $E(T)$ such that $P(c,d)$ is true with $d > 0$. Let $q_T(c)$ be in $Q_T(c)$. Of course $||c - q_T(c)||$ is the Chebyshev radius of $T$, so $T \subseteq B(c,||c - q_T(c)||)$, where $B(x,r) = \{ y \in X : ||x - y|| \leq r \}$. Let $k$ be such that $0 < k < \min\{1,d\}$. Now we suppose that $||c - q_T(c)|| > 0$; $(1 - k)||c - q_T(c)|| < ||c - q_T(c)||$. Then, by definition of the Chebyshev radius, we have: $T \subseteq B(c - k(c - q_T(c)),(1 - k)||c - q_T(c)||)$, so there is a $t_0$ in $T$ such that $||c - k(c - q_T(c)) - t_0|| > (1 - k)||c - q_T(c)||$. From the fact that $P(c,d)$ is true, one obtains

$$||c - k(c - q_T(c)) - t_0|| \leq ||c - k(c - q_T(c)) - q_T(c)||$$

$$= (1 - k)||c - q_T(c)||.$$

This contradiction implies that $||c - q_T(c)|| = 0$, that is $T$ is a singleton.

**Remark 6.** In Theorem 5 it is enough to suppose that $Q_T(c)$ is nonempty.

**Remark 7.** We point out that any convex remotal set $T$ has a unique centre if $X$ is a strictly convex dual space [7].

In [8] Panda and Kapoor proved the following result.

**Theorem 8.** Let $X$ be a normed space admitting centres, and let $T$ be a nonempty unique remotal subset. If the farthest point map $q_T$ is inner radial upper semi-continuous (I.R.U. continuous) on $T + r(T)B(0,1)$, then $T$ is a singleton.

We recall that $q_T$ is I.R.U. continuous at $x_0$ if for any $v_0$ in $q_T(x_0)$ and an open set $W$ containing $q_T(x_0)$ there is a neighbourhood $N$ of $x_0$ such that $q_T(x) \subseteq W$ for any $x$ in $N \cap \{v_0 + k(-v_0 + x_0) : 0 \leq k \leq 1\}$. We observe that if $P(c,d)$ is true, then $q_T$ is I.R.U. continuous at $c$, but in spite of that, Theorem 5 improves Theorem 8 because we suppose that only $T$ is remotal (or that $Q_T(c)$ is nonempty) and that the map $q_T$ is continuous only at one element of $X$.

**Theorem 9.** Let $X$ be a strictly convex Banach space. Let $T$ be a nonempty remotal subset. If for any $x$ in $X$ there is a $d > 0$ such that $P(x,d)$ is true, then $T$ is a unique remotal set.

**Proof.** If $T$ is a singleton, then we have the assertion so we can suppose that $T$ is not a singleton. Let $x \in X, q_T(x) \in Q_T(x)$, and $t \in T$, such that $t \neq q_T(x)$. Set $v = x - t, u = -x + q_T(x)$. We have $||v|| \leq ||u||$. If we suppose that $||v|| = ||u||$ and that $P(x,d)$ is true, then we have $||ku + v|| \leq (1 - k)||u||$ for any $k < d' = \min\{d,1\}$. So, the function $F : (-\infty,d] \rightarrow R$, defined by $F(h) = ||ku + v|| - (1 - h)||u||$ is convex and $F(h) \leq 0$ for any $h < d$. If we use Lemma 3 with $k = 1 - h, x' = u$
and \( t' = u + v \), we obtain \( k\|x'\| \leq \|kx' - t'\| \), that is \( F(h) \geq 0 \) and so \( F(h) = 0 \) for any \( h < d' \). Set \( k = 1 \). Then,

\[
\|u - v\| = 2\|u\| = 2\|v\| = \left\| \frac{u}{\|u\|} + \frac{-v}{\|v\|} \right\| = 2.
\]

But \( X \) is strictly convex and, hence, \( u = -v \Rightarrow -x + q_T(x) = -x + t \Rightarrow q_T(x) = t \).

This contradiction implies that \( \|v\| < \|u\| \), that is \( \|x - q_T(x)\| < \|x - t\| \) for any \( t \neq q_T(x) \), so we have the assertion.

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REFERENCES