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VARIATIONAL INEQUALITIES OF STRONGLY NONLINEAR ELLIPTIC OPERATORS OF INFINITE ORDER

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Abstract. We study the solvability of variational inequalities for strongly nonlinear elliptic operators of infinite order with liberal growth on their coefficients.

1. Introduction. In a number of papers Dubinskiĭ [cf. [**3**], [**4**]) has considered the nontriviality of Sobolev spaces of infinite order corresponding to the boundary value problems for linear differential equations of infinite order and obtained the solvability of those problems in the case when the coefficients of the equation grow polynomially with respect to the derivatives.

Chan Dyk Van [**2**], extended the results of Dubinskiĭ to include the case of operators with rapidly (slowly) increasing coefficients.

We generalize the results above to cover the solvability of variational inequalities for strongly nonlinear operators of the form

$$Au(x) + Bu(x), \qquad x \in \Omega \tag{1.1}$$

where

$$Au(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u(x)), \qquad |\gamma| \le |\alpha|, \tag{1.2}$$

$$Bu(x) = \sum_{|\alpha| \le M} (-1)^{|\alpha|} D^\alpha B_\alpha(x, D^\alpha u(x)), \qquad M \text{ fixed}, \tag{1.3}$$

with more liberal growth on the coefficients. Here Ω is a bounded domain in $\mathbf{R}^n$.

2. Preliminaries. Let Ω be a bounded domain in $\mathbf{R}^n$ $(n \ge 2)$ for which the cone and the strong local lipschitz properties hold [**1**].

An N-function is any continuous map $\Phi: \mathbf{R} \longrightarrow \mathbf{R}$ which is even, convex and satisfies $\Phi(t)/t \to 0$ (resp. $+\infty$) as $t \to 0$ (resp. $+\infty$). The conjugate or

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AMS Subject Classification (1990): Primary 35 G 30, 46 E 35 complementary N-function of Φ and its nonnegative reciprocal will be denoted by $\bar{\Phi}$ and Φ^{-1}, respectively [**1**].

When Φ and Ψ are two N-functions we shall write $\Psi \ll \Phi$ if for any $\varepsilon > 0$

$$\text{Lim}_{t\to\infty} \Psi(t)/\Phi(\varepsilon t) = 0$$

The Orlicz space $L_{\Phi_\alpha}(\Omega)$ corresponding to N-functions Φ_α is defined as the set of all measurable functions $u: \ \Omega \to \mathbf{R}$ such that

$$\|u\|_{\Phi_\alpha} = \inf\Big\{\lambda > 0; \ \int_\Omega \Phi_\alpha(\frac{u}{\lambda}) \leq 1\Big\} < \infty$$

Let $E_{\Phi_\alpha}(\Omega)$ be the (norm) closure in $L_{\Phi_\alpha}(\Omega)$ of $L^\infty(\Omega)$-functions with compact support in $\bar{\Omega}$.

The Sobolev-Orlicz spaces of functions u such that u and its distributional derivatives $D^\alpha u, \ |\alpha| \leq m$, lie in $L_{\Phi_\alpha}(\Omega)$ (resp. $E_{\Phi_\alpha}(\Omega)$). These are Banach spaces for the norm

$$\|u\|_{m,\Phi_\alpha} = \left(\sum_{|\alpha|\leq m} \|D^\alpha u\|^2_{\Phi_\alpha}\right)^{1/2}$$

and they are identified to subspaces of the product

$$\prod_{|\alpha|\leq m} L_{\Phi_\alpha}(\Omega) = \prod L_{\Phi_\alpha}$$

Denote by $C^\infty(\Omega)$ the space of infinitely differentiable functions on Ω, $D(\Omega)$ the space $C^\infty(\Omega)$ with compact support in Ω and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

We define $W_0^m L_{\Phi_\alpha}(\Omega)$ as the $\sigma(\Pi L_{\Phi_\alpha}, \Pi E_{\bar{\Phi}_\alpha})$ closure of $D(\Omega)$ in $W^m L_{\Phi_\alpha}(\Omega)$ and $W_0^m E_{\Phi_\alpha}(\Omega)$ as the norm closure of $D(\Omega)$ in $W^m L_{\Phi_\alpha}(\Omega)$. The Sobolev-Orlicz spaces of infinite order is defined by:

$$W^\infty L_{\Phi_\alpha}(\Omega) = \Big\{u \in C^\infty(\Omega): \ \sum_{|\alpha|=0}^{\infty} \int_\Omega \Phi_\alpha\big(D^\alpha u(n)\big) \ dx < \infty\Big\},$$

and

$$W_0^\infty L_{\Phi_\alpha}(\Omega) = \Big\{u \in D(\Omega): \ \|u\|_{\infty,\Phi_\alpha} = \sum_{|\alpha|=0}^{\infty} \|D^\alpha u\|_{\Phi_\alpha} < \infty\Big\}$$

They are Banach spaces with the norm $\|\cdot\|_{\infty,\Phi_\alpha}$. Similar definition of $W_0^\infty E_{\Phi_\alpha}(\Omega)$ is obvious. The dual of $W_0^\infty L_{\Phi_\alpha}(\Omega)$ (resp. $W_0^\infty E_{\Phi_\alpha}(\Omega)$) will be denoted by $W^{-\infty} E_{\bar{\Phi}_\alpha}(\Omega)$ (resp. $W^{-\infty} L_{\bar{\Phi}_\alpha}(\Omega)$), where

$$W^{-\infty} E_{\bar{\Phi}_\alpha}(\Omega) \ \text{(resp. } W^{-\infty} L_{\bar{\Phi}_\alpha}(\Omega)) =$$
$$\Big\{h \in \mathcal{D}'(\Omega): \ h(x) = \sum_{|\infty|=0}^{\infty} (-1)^{|\alpha|} D^\alpha h_\alpha, h_\alpha \in E_{\bar{\Phi}_\alpha}(\Omega) \ \text{(resp. } L_{\bar{\Phi}_\alpha}(\Omega))\Big\}$$

These spaces are Banach spaces with the norm

$$\|h\|_{-\infty,\bar{\Phi}_\alpha} = \sum_{|\alpha|=0}^{\infty} \|h_\alpha\|_{\bar{\Phi}_\alpha} < \infty$$

The duality of $W_0^\infty L_{\Phi_\alpha}(\Omega)$ and $W^{-\infty} E_{\bar{\Phi}_\alpha}(\Omega)$ is defined by

$$\langle h, u\rangle = \sum_{|\alpha|=0}^{\infty} \int_\Omega h_\alpha(x) D^\alpha u(x)\ dx.$$

Let $1 \le p < \infty$. The Sobolev spaces of infinite order are defined by

$$W_0^\infty(a_\alpha, p)(\Omega) = \left\{ u \in \mathcal{D}(\Omega) :\ \|u\|_{\infty,p}^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega |D^\alpha u(x)|^p dx < \infty \right\},$$

where $a_\alpha \ge 0$ is a sequence of numbers. We formally define the spaces dual to $W_0^\infty(a_\alpha, p)(\Omega)$ via:

$$W^{-\infty}(a_\alpha, p')(\Omega) = \left\{ h :\ h = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha h_\alpha;\ h_\alpha \in L^{p'}(\Omega) : \right.$$
$$\left. \|h\|_{-\infty,p'}^{p'} = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{L^{p'}(\Omega)}^{p'} < \infty \right\}, \quad p' = p/p-1.$$

For more details we may refer to [**1**-**3**].

Let $l, M \in \mathbf{N}, M$ be fixed. By λ_1 and λ_2 we denote the number of multi-indices α with $|\alpha| \le l$, $|\alpha| \le M$, respectively.

3. Conditions on the coefficients. To define the operator (1.2) more precisely we introduce either the following set of hypotheses:

$A_1)$ For all $l \in \mathbf{N}$ and $|\gamma| \le |\alpha|$, each $A_\alpha(x, \xi_\gamma)$ is a Caratheodory function, i.e., $A_\alpha(x, \xi_\gamma)$ is measurable in $x \in \Omega$ for all fixed $\xi_\gamma \in \mathbf{R}^{\lambda_1}$, and continuous in ξ_γ for almost all (a.a.) fixed $x \in \Omega$. Moreover there exists a function $h_1 \in L^1(\Omega)$, independent of l, and a sequence of positive numbers $(S_l)_{l\in\mathbf{N}}$ with $\sum_l \lambda_1 S_l < \infty$ such that

$$\sup_{|\xi_\gamma| \le S_l^{-1}} |A_\alpha(x, \xi_\gamma)| \le h_1(x) S_l.$$

$A_2)$ There exists a constant $C_0 > 0$ and a function $h_2 \in L^1(\Omega)$, both independent of l, such that

$$\sum_{|\alpha|=0}^{l} A_\alpha(x, \xi_\gamma)\xi_\alpha \ge C_0 \sum_{|\alpha|=0}^{l} a_\alpha |\xi_\alpha|^P - h_2(x)$$

for all $x \in \Omega$, $\xi_\gamma \in \mathbf{R}^{\lambda_1}$.

A_3) For all $l \in \mathbf{N}$, a.a. $x \in \Omega$ and all distinct ξ_γ, $\xi^*_\gamma \in \mathbf{R}^{\lambda_1}$

$$\sum_{|\alpha|=0}^{l} (A_\alpha(x,\xi_\gamma) - A_\alpha(x,\xi^*_\gamma))(\xi_\alpha - \xi^*_\alpha) > 0;$$

or the following one:

$A_1)^*$ For all $l \in \mathbf{N}$, each $A_\alpha(x,\xi_\gamma)$ is a real-valued Caratheodory function defined on $\Omega \times \mathbf{R}^{\lambda_1}$. There exist two N-functions Φ_α, Ψ_α with $\Psi_\alpha \ll \Phi_\alpha$; functions $a_\alpha(x)$ in $E_{\bar{\Phi}_\alpha}(\Omega)$ for $|\alpha| = l$, in $L_{\bar{\Phi}_\alpha}(\Omega)$ for $|\alpha| < l$; and positive constants c_1, c_2, both independent of l, such that

if $|\alpha| = l$, then

$$|A_\alpha(x,\xi_\gamma)| \le a_\alpha(x) + c_1 \sum_{|\beta|=l} \bar{\Phi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta) + c_1 \sum_{|\beta|<l} \bar{\Psi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta),$$

if $|\alpha| < l$, then

$$|A_\alpha(x,\xi_\gamma)| \le a_\alpha(x) + c_1 \sum_{|\beta|<l} \bar{\Phi}_\alpha^{-1}\Psi_\alpha(c_2\xi_\beta) + c_1 \sum_{|\beta|<l} \bar{\Phi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta)$$

for a.a. $x \in \Omega$ and all $\xi_\gamma \in \mathbf{R}^{\lambda_1}$.

$A_2)^*$ There exist functions b_α in $E\bar{\Phi}_\alpha(\Omega)$ for $|\alpha| = l$, in $L_{\bar{\Phi}_\alpha}(\Omega)$ for $|\alpha| < l$; function $h_3 \in L^1(\Omega)$ and positive constants d_1, d_2, independent of l, such that

$$\sum_{|\alpha|=0}^{l} A_\alpha(x,\xi_\gamma)\xi_\alpha \ge d_1 \sum_{|\alpha|=0}^{l} \Phi_\alpha(d_2\xi_\alpha) - \sum_{|\alpha|=0}^{l} b_\alpha(x)\xi_\alpha - h_3(x)$$

for a.a. $x \in \Omega$ and all $\xi_\gamma \in \mathbf{R}^{\lambda_1}$.

$A_3)^*$ As in A_3).

For the operator (1.3) we impose the following assumption:

B_1) $B_\alpha(x,\xi_\alpha)$ is a Caratheodory function defined on $\Omega \times \mathbf{R}^{\lambda_2}$. There exists a function h_4 in $L^1(\Omega)$ such that: $|B_\alpha(x,\xi_\alpha)| \le h_4(x)P_\alpha(\xi_\alpha)$ for some continuous function $P_\alpha : \mathbf{R}^{\lambda_2} \longrightarrow \mathbf{R}$ and $B_\alpha(x,\xi_\alpha)\xi_\alpha \ge 0$, $x \in \Omega$, $|\alpha| \le M$.

4. Main results. THEOREM 1. *Let K be a closed convex subset of $W_0^\infty(a_\alpha, p)(\Omega)$ containing the origin. Suppose that $A_1) - A_3)$ and $B_1)$ hold. Let $f \in W^{-\infty}(a_\alpha, p')(\Omega)$ be given. Then there exists at least one solution $u \in K$ of*

$$\langle A(u), v-u\rangle + \langle B(u), v-u\rangle \ge \langle f, v-u\rangle \qquad \forall v \in K \tag{4.1}$$

Proof. Consider a partial sum of order $2l$ of the series (4.1):

$$\langle A_{2l}(u_l), v-u_l\rangle + \langle B(u_l), v-u_l\rangle \ge \langle f^l, v-u_l\rangle \qquad \forall v \in K \tag{4.2}$$

where

$$A_{2l}(u_l)(x) = \sum_{|\alpha|=0}^{l} (-1)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u_l), \qquad |\gamma| \leq |\alpha|,$$

$$B(u_l)(x) = \sum_{|\alpha|\leq M<l} (-1)^{|\alpha|} D^\alpha B_\alpha(x, D^\alpha u_l),$$

and

$$f^l = \sum_{|\alpha|=0}^{l} (-1)^{|\alpha|} a_\alpha D^\alpha f_\alpha \in W^{-l}(a_\alpha, p')(\Omega).$$

For solvability of (4.2), in view of $A_1) - A_3)$ and $B_1)$, we refer to [**5**] and [**6**]. Set $v = 0$ in (4.2), and use $A_3)$ and $B_1)$; we obtain an a priori bound

$$\|u_l\|_{W_0^l(a_\alpha,p)(\Omega)} \leq \text{const.}$$

Since $u_l \in W^l(a_\alpha, p)(\Omega)$ implies $u_l \in W^1(a_\alpha, p)(\Omega)$ we get from compactness of $W^1(a_\alpha, p)(\Omega) \longrightarrow C(\bar{\Omega})$, the uniform convergence of $u_l(x) \longrightarrow u(x)$ on $\bar{\Omega}$ as $l \to \infty$. Similarly, by compactness of $W^l(a_\alpha, p)(\Omega) \longrightarrow C^{l-m}(\bar{\Omega})$, for large enough l and $m \in \mathbf{N}$, we obtain,

$$D^\alpha u_l(x) \longrightarrow D^\alpha u(x) \qquad \text{uniformly on } \bar{\Omega} \text{ as } l \to \infty \tag{4.3}$$

Using the definition of $W_0^\infty(a_\alpha, p)(\Omega)$ we get $u \in W_0^\infty(a_\alpha, p)(\Omega)$ and by closedness of $K, u \in K$.

It remains to show that u is a solution of (4.1). For this purpose it suffices to prove the assertions:

$$\text{Lim}_l \ \langle A_{2l}(u_l), z\rangle = \langle A(u), z\rangle \tag{4.4}$$

$$\text{Lim}_l \ \langle B(u_l), z\rangle = \langle B(u), z\rangle \tag{4.5}$$

$$\text{Lim}_l \ \inf\langle A_{2l}(u_l), u_l\rangle \geq \langle A(u), u\rangle \tag{4.6}$$

and

$$\text{Lim}_l \ \inf\langle B(u_l), u_l\rangle \geq \langle B(u), u\rangle \tag{4.7}$$

for all $z \in K$. To prove (4.4) we use the inequality:

$$|A_\alpha(x, D^\gamma u_l)| \leq \sup_{|\xi_\gamma|\leq S_l^{-1}} |A_\alpha(x, \xi_\gamma)| + S_l A_\alpha(x, D^\gamma u_l) D^\alpha u_l$$

as well as the uniform boundedness of $\{\langle A_{2l}u_l, u_l\rangle\}$ in $L^1(\Omega)$, to obtain the uniform equi-integrability of $\{A_\alpha(x, D^\gamma u_l)\}$ in $L^1(\Omega)$ provided that $\sum S_l \lambda_1(.,l) < \infty$. Now, in view of Vitali's convergence theorem, (4.7) follows.

To prove (4.5) we have

$$\sum_{|\alpha|\leq M} \int_\Omega |B_\alpha(x, D^\alpha u_l)| \leq \int_\Omega |h_4(x) P_\alpha(D^\alpha u_l)| \leq$$

$$\|h_4\|_{L^1(\Omega)} \sum_{|\alpha|\leq M} \|P_\alpha(D^\alpha u_l)\|_{L^\infty(\Omega)} \leq \text{ const.},$$

and (4.5) follows from the dominated convergence theorem.

The assertions (4.6) and (4.7) are direct consequences of Fatou's lemma in view of the uniform convergence (4.3), and the proof is completed.

The result above enables us to state the following theorem.

THEOREM 2. *Let K be a convex $\sigma(W^{\infty}L_{\Phi_\alpha}(\Omega),\ W^{-\infty}E_{\bar{\Phi}_\alpha}(\Omega))$ sequentially closed subset of $W^{\infty}L_{\Phi_\alpha}(\Omega)$ such that $K \cap W_0^{\infty}E_{\Phi_\alpha}(\Omega)$ is $\sigma(W^{\infty}L_{\Phi_\alpha}(\Omega),$ $W^{-\infty}L_{\bar{\Phi}_\alpha}(\Omega))$ dense in K and $0 \in K$. Let $f \in W^{-\infty}E_{\bar{\Phi}_\alpha}(\Omega)$ be given, and let the hypotheses $A_1)^* - A_3)^*$ hold. Then there exists at least one solution $u \in K$ such that:*

$$\langle Au, v-u\rangle - \langle f, v-u\rangle \geq 0 \qquad \forall v \in K \tag{4.8}$$

Outline of proof. As in Theoerem 1, we may consider the auxiliary variational inequality

$$\langle A_{2m}(u_m), v-u_m\rangle - \langle f^m, v-u_m\rangle \geq 0 \qquad \forall v \in K \tag{4.9}$$

The solvability of (4.9) is a consequence of [**7**]. Thus, there exists $u_m \in K$ solving (4.9). Put $v = 0$ in (4.9) and make use of $A_2)^*$; we have

$$\int_\Omega \Phi_\alpha(c_2 D^\alpha u_m) \leq c_3, \quad \text{where} \quad c_3 = c_3(\|f\|_{W^{-\infty}L_{\bar{\Phi}_\alpha}(\Omega)})$$

Hence, there exists a subsequence of u_m such that $u_m \longrightarrow u$ in $C^{\infty}(\Omega)$. By the definition of $W^{\infty}L_{\Phi_\alpha}(\Omega)$ and the $\sigma(W^{\infty}L_{\Phi_\alpha}(\Omega),\ W^{-\infty}E_{\bar{\Phi}}(\Omega))$ sequential closedness of K, we get $u \in K$. To show that u solves (4.8) it remains to prove the assertions (4.4) and (4.6) of Theorem 1. A similar argument as in the proof of Theorem 1, may be used to finish the proof.

Example. As a particular example which can be treated by Theorem 1, outside the scope of [**3**], one may consider the nonlinear Dirichlet boundary-value problem

$$\sum_{l=0}^{\infty}\sum_{|\alpha|=l}(-1)^{|\alpha|}D^\alpha(a_\alpha S_l^P|D^\alpha u|^{P-2}D^\alpha u) + |u|e^{|u|} = f(x)$$

where $(S_l)_{l\in\mathbf{N}}$ is a sequence described in $A_1)$. In fact

$$A_\alpha(x, D^\gamma u) := a_\alpha S_l^p|D^\alpha u|^{P-2}D^\alpha u, \qquad |\gamma| = |\alpha|$$
$$B_\alpha(x, D^\alpha u) := |u|e^{|u|}$$

By the Sobolev's embedding theorem, for $u \in W^l(a_\alpha, p)(\Omega)$ $(lp > n)$, the functions $D^\alpha u$ are bounded for all $|\alpha| \leq l$. Therefore $A_\alpha(x, \xi_\gamma)$ and $B_\alpha(x, \xi_\gamma)$ are $L^\infty(\Omega)$ - functions and hence $A_1)$ and $B_1)$ follow. Condition $A_2)$ is obvious, while $A_3)$ follows in view of the inequality

$$|x|^P + |y|^P - xy(|x|^{P-2} + |y|^{P-2}) > 0 \qquad \text{for } x \neq y$$

Thus the hypotheses of Theorem 1 are satisfied.

Our example falls outside the scope of [**3**] beacause the term $|u|\ e^{|u|}$ does not verify the polynomial growth condition of [**3**].

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