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ON THE RATE OF CONVERGENCE FOR MODIFIED SZÁSZ-MIRAKYAN OPERATORS ON FUNCTIONS OF BOUNDED VARIATION

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Abstract. Recently Gupta and Agrawal [**4**] gave an estimate of the rate of convergence for modified Szász-Mirakyan operators for functions of bounded variation, using probabilistic approach. Because of some mistakes in their paper, we were motivated to correct and improve their results.

1. Introduction. Let f be a function defined on $[0,\infty)$. The Szász-Mirakyan operator S_n applied to f is

$$S_n(f;x)=\sum_{k=0}^{\infty}p_k(nx)f(k/n),$$

where

$$p_k(t)=e^{-t}t^k/k!$$

Kasana et al. [**5**] proposed modified Szász-Mirakyan operators to approximate functions integrable on $[0,\infty)$:

$$M_n(f;x)=n\sum_{k=0}^{\infty}p_k(nx)\int_0^{\infty}p_k(nt)f(t)dt.$$

We shall study $M_n(f;x)$ for functions of bounded variation on every finite subinterval of $[0,\infty)$, giving quantitative estimates of the rate of convergence. Results of this type for Fourier series were obtained by Bojanić [**2**].

2. Auxiliary results. We shall need the following lemmas in the proof of our theorem.

Lemma 1. [**1**, pp. 104, 110] and [**3**, p. 304]: Berry–Essen Theorem. *Let $X_1, X_2, \dots, X_n$ be n indenpendently and identically distributed (i.i.d.) random*

AMS Subject Classification (1985): Primary 41 A 30, 41 A 36 *variables with zero mean and a finite absolute third moment. If* $\rho_2 \equiv E(X_1^2) > 0$, *then* $\mathrm{Sup}_{x\in R}|F_n(x)-\phi(x)| \leq (.41)l_{3,n}$, *where* $F_n(x)$ *is the distribution function of* $(X_1 + X_2 + \ldots + X_n)(n\rho_2)^{-1/2}$; $\phi(x)$ *is the standard normal distribution function, and* $l_{3,n}$ *is the Liapounov ratio* $l_{3,n} = \rho_3\rho_2^{-3/2}n^{-1/2}$, $\rho_3 = E|X_1|^3$.

We may note here that Gupta and Agrawal (1991) mistook $\beta_3 = E(X_1^3)$ for ρ_3 and since $\rho_3 > \beta_3$ for all nondegenerate distribution functions of i.i.d. X_i's, this led to an error in the estimate, e.g., in their Lemma 1, it should have read $p_k(nx) \leq 5/2(\sqrt{nx})$.

Lemma 2. *For every* $x \in (0,\infty)$, *we have* $p_k(nx) \leq \phi(x)/\sqrt{n}$, *where* $\phi(x) = 2x^{-1/2}(4x^2 + 3x + 1)$.

Proof. Let $\{\xi_k\}$ be a sequence of i.i.d. random variables all having the same Poisson distribution with parameter x. Let $\eta_n = \sum_{k=1}^n \xi_k$; then

$$p(\eta_n = k) = e^{-nx}(nx)^k/k! = p_k(nx),$$

wherefrom

$$\begin{aligned}\rho_2 = x;\ \rho_3 = E|\xi_1 - x|^3 &\leq E(\xi_1^3) + 3xE(\xi_1^2) + 3x^2E(\xi_1) + x^3\\ &= 8x^3 + 6x^2 + x, \quad \text{and}\end{aligned}$$

$$l_{3,n} = (8x^3 + 6x^2 + x)/\sqrt{n}(x^{3/2}).$$

So, we have

$$p_k(nx) = P(k-1 < \eta_n \leq k) = P\Big(\frac{k-1-nx}{\sqrt{nx}} < \frac{\eta_n - nx}{\sqrt{nx}} \leq \frac{k-nx}{\sqrt{nx}}\Big).$$

By using Lemma 1 we have

$$\Big|p_k(nx) - \frac{1}{\sqrt{2\pi}}\int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}dt\Big| < 2(0.41)\frac{8x^2+6x+1}{\sqrt{nx}} < \frac{8x^2+6x+1}{\sqrt{nx}}.$$

Also

$$\frac{1}{\sqrt{2\pi}}\int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}dt \leq \frac{1}{\sqrt{2knx}} \leq \frac{1}{\sqrt{nx}}.$$

Therefore,

$$p_k(nx) \leq \frac{8x^2+6x+1}{\sqrt{nx}} + \frac{1}{\sqrt{nx}} < \frac{\phi(x)}{\sqrt{n}}.$$

Lemma 3. *For* $n \geq 2$, *we have* $2x/n \leq M_n((t-x)^2;x) \leq (2x+1)/n$.

Proof. It is easily verified that: $M_n(1;x) = 1$, $M_n(t;x) = (1+nx)/n$ and $M_2(t^2;x) = (x^2n^2 + 4nx + 2)/n^2$. Hence, $M_n((t-x)^2;x) = 2x/n + 2/n^2$, which leads to the assertion of the lemma.

Lemma 4. *Let* $k_n(x;t) = n\sum_{k=0}^{\infty} p_k(nx)p_k(nt)$. *Then*

(i) *For* $0 \leq y < x$, *we have*

$$\int_0^y k_n(x,t)dt \leq \frac{2x+1}{n(x-y)^2} \tag{2.1}$$

(ii) *For* $x < z < \infty$, *we have*

$$\int_z^\infty k_n(x,t)dt \leq \frac{2x+1}{n(z-x)^2} \tag{2.2}$$

Proof. The results follow from straightforward application of Lemma 3 and from the fact that $(t-x)^2 \geq (x-y)^2$ in (i) and $(t-x)^2 \geq (z-x)^2$ in (ii).

LEMMA 5. *For* $x \in (0,\infty)$, *we have*

$$n\int_x^\infty p_k(nt)dt = \sum_{j=0}^k p_j(nx).$$

Proof. The above equality can be obtained by repeated integration of its left-hand side by parts. Also,

$$n\int_0^x p_k(nt)dt = 1 - n\int_x^\infty p_k(nt)dt = \sum_{j=k+1}^\infty p_j(nx).$$

3. Main Result. Our main result may be stated as follows.

THEOREM. *Let* f *be a function of bounded variation on every finite subinterval of* $[0,\infty)$ *and let* $f(t) = O(e^{\alpha t})$ *for some* $\alpha > 0$ *as* $t \to \infty$. *If* $x \in (0,\infty)$, *then for* n *sufficiently large we have*

$$\begin{aligned}\left|M_n(f;x) - \frac{1}{2}\{f(x+) + f(x-)\}\right| &\leq \frac{(x^2+6x+3)x^{-2}}{n}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} g_x \\ &+ \frac{(4x^2|3x+1|)x^{-1/2}}{\sqrt{n}}|f(x+)-f(x-)| + O(1)e^{-\alpha x}\frac{(2x+1)x^{-2}}{n};\end{aligned} \tag{3.1}$$

Where $V_a^b(g_x)$ *is total variation of* g_x *on* $[a,b]$ *and*

$$g_x \equiv g_x(t) = \begin{cases} f(t) - f(x+), & x < t < \infty; \\ 0, & t = x; \\ f(t) - f(x-), & 0 \leq t < x. \end{cases}$$

Proof. First,

$$\begin{aligned}&|M_n(f;x) - \frac{1}{2}(f(x+) + f(x-))| \\ &\quad\leq |M_n(g_x;x)| + \frac{1}{2}|f(x+) - f(x-)||M_n(\mathrm{Sign}(t-x);x)|\end{aligned} \tag{3.2}$$

Thus to estimate the left-hand-side, we need estimates for $M_n(g_x;x)$ and $M_n(\mathrm{Sign}(t-x);x)$. Now,

$$\begin{aligned} M_n(\mathrm{Sign}(t-x);x) &= \int_0^\infty \mathrm{Sign}(t-x)k_n(x;t)dt \\ &= \int_x^\infty k_n(x;t)dt - \int_0^x k_n(x;t)dt \\ &= A_n(x) - B_n(x), \text{ say.} \end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned} A_n(x) &= \int_x^\infty k_n(x,t)dt \\ &= n\sum_{k=0}^\infty p_k(nx)\int_x^\infty p_k(nt)dt = \sum_{k=0}^\infty p_k(nx)\sum_{j=0}^k p_j(nx) \\ &= p_0^2 + p_1(p_0+p_1) + p_2(p_0+p_1+p_2) + \dots \\ &= p_0^2 + p_1^2 + p_2^2 + \dots + p_0(p_1+p_2+\dots) + p_1(p_2+p_3+\dots) + \dots \end{aligned}$$

Let $1 = (p_0+p_1+p_2+\dots)(p_0+p_1+p_2+\dots)$. Then $1-A_n(x) = p_0(p_1+p_2+\dots)+p_1(p_2+p_3+\dots)+\dots$. Further, $A_n(x)-(1-A_n(x)) = 2A_n(x)-1 = p_0^2+p_1^2+p_2^2+\dots$, and $A_n(x)+B_n(x) = 1$; so we get

$$|A_n(x)-B_n(x)| = |2A_n(x)-1| = \sum_{k=0}^\infty p_k^2(nx) \le \frac{\phi(x)}{\sqrt{n}}\sum_{k=0}^\infty p_k(nx) = \frac{\phi(x)}{\sqrt{n}}.$$

To estimate $M_n(g_x;x)$, we first decompose $[0,\infty)$ into three parts, as follows

$$\begin{aligned} M_n(g_x;x) &= n\sum_{k=0}^\infty p_k(nx)\int_0^\infty p_k(nt)g_x(t)dt = \int_0^\infty g_x(t)k_n(x;t)dt \\ &= \int_0^{x-x/\sqrt{n}} g_x(t)k_n(x;t)dt + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)k_n(x;t)dt \\ &\quad + \int_{x+x/\sqrt{n}}^\infty g_x(t)k_n(x;t)dt \\ &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) g_x(t)k_n(x;t)dt \\ &= E_1 + E_2 + E_3, \text{ say.} \end{aligned}$$

First, we estimate E_2. For $t \in I_2$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x),$$

and so

$$\begin{aligned} E_2 &= \Big|\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)k_n(x;t)dt\Big| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} k_n(x;t)dt \\ &= V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t\lambda_n(x;t); \text{ where, } \lambda_n(x;t) = \int_0^t k_n(x;u)du. \end{aligned}$$

Since $\int_a^b d_t\,\lambda_n(x,t) \le 1$ for all $[a,b] \subseteq [0,\infty)$, we have

$$|E_2| = V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{3.3}$$

Let us estimate E_1. By using Lebesgue-Stieltjes integration by parts, with $y = x - x/\sqrt{n}$, we get

$$E_1 = g_x(y+)\lambda_n(x;y) - \int_0^y \lambda_n(x;t)d_t g_x(t).$$

Since, $|g_x(y+)| = |g_x(y+) - g_x(x)| \le V_{y+}^x(g_x)$, it follows that

$$|E_1| \le V_{y+}^x(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)d(-V_t^x g_x).$$

By (2.1) of Lemma 4, we have

$$|E_1| \le V_{y+}^x(g_x)\frac{2x+1}{n(x-y)^2} + \frac{2x+1}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)).$$

Integration by parts leads to the following

$$\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)) = \frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{\hat{V}_t^x(g_x)}{(x-t)^3}dt,$$

where $\hat{V}_t^x(g_x)$ is the normalized form of $V_t^x(g_x)$ and $\hat{V}_t^x(g_x) = V_t^x(g_x)$. Consequently, we have

$$\begin{aligned}
|E_1| &\le V_{y+}^x(g_x)\frac{2x+1}{n(x-y)^2} + \frac{2x+1}{n}\Big[\frac{-V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2} + 2\int_0^y \frac{V_t^x(g_x)}{(x-t)^3}dt\Big] \\
&= \frac{2x+1}{n}\Big[\frac{V_0^x(g_x)}{x^2} + 2\int_0^{x-x/\sqrt{n}} V_t^x(g_x)\frac{1}{(x-t)^3}dt\Big].
\end{aligned}$$

Substituting $x - x/\sqrt{t}$ for the variable t in the integral above, we get

$$\begin{aligned}
|E_1| &\le \frac{2x+1}{nx^2}(V_0^x(g_x) + \int_1^n V_{x-x/\sqrt{t}}^x(g_x)dt) \\
&\le \frac{2x+1}{nx^2}(V_0^x(g_x) + \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)) \\
&\le \frac{2(2x+1)}{nx^2}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x).
\end{aligned} \tag{3.4}$$

Finally, we evaluate E_3. Setting, $Z = x + x/\sqrt{n}$, we obtain

$$E_3 = \int_z^\infty g_x(t)k_n(x;t)dt = \int_z^\infty g_x(t)d_t\lambda_n(x;t).$$

We define $Q_n(x;t)$ on $[0,2x]$ as

$$\begin{aligned} Q_n(x,t) &= 1-\lambda_x(x;t-), \quad 0 \le t < 2x \\ &= 0 \qquad\qquad\qquad\;\; , \quad t = 2x, \end{aligned}$$

So

$$\begin{aligned} E_3 &= \int_z^{2x} g_x(t)d_tQ_n(x;t) - g_x(2x)\int_{2x}^{\infty} k_n(x;u)du + \int_{2x}^{\infty} g_x(t)d_t\lambda_n(x;t) \\ &= E_{31}+E_{32}+E_{33}; \ \text{ say.} \end{aligned} \tag{3.5}$$

Now, using partial integration for the first term, we get

$$E_{31} = g_x(z-)Q_n(x,z-) + \int_z^{2x} \hat{Q}_n(x;t)d_tg_x(t),$$

where $\hat{Q}_n(x;t)$ is the normalized form of $Q_n(x;t)$. Since $Q_n(x;z-)=Q_n(x;z)$ and $|g_x(z-)| \le V_x^{z-}(g_x)$, we have

$$|E_{31}| \le V_x^{z-}(g_x)Q_n(x;z) + \int_z^{2x} \hat{Q}_n(x;t)d_tV_x^t(g_x).$$

Further, by using Lemma 4(ii) and the fact that $\hat{Q}_n(x;t) \le Q_n(x;t)$ on $[0,2x)$, we have

$$\begin{aligned} |E_{31}| \le& V_x^{z-}(g_x)\frac{(2x+1)}{n(x-z)^2} + \frac{(2x+1)}{n}\int_z^{2x-}\frac{1}{(t-x)^2}d_tV_x^t(g_x) \\ &+ \frac{1}{2}\Big(V_{2x-}^{2x}(g_x)\int_{2x}^{\infty}k_n(x;u)du\Big) \\ \le& V_x^{z-}(g_x)\frac{(2x+1)}{n(x-z)^2} + \frac{(2x+1)}{n}\int_z^{2x-}\frac{1}{(t-x)^2}d_tV_x^t(g_x) \\ &+ \frac{1}{2}\Big(V_{2x-}^{2x}(g_x)\frac{2x+1}{nx^2}\Big) \\ \le& V_x^{z-}(g_x)\frac{(2x+1)}{n(z-x)^2} + \frac{(2x+1)}{n}\int_z^{2x}\frac{1}{(x-t)^2}d_tV_x^t(g_x). \\ =& V_x^{z-}(g_x)\frac{(2x+1)}{n(z-x)^2} + \frac{(2x+1)}{n}\Big\{\frac{V_x^{2x}(g_x)}{x^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} \\ &\qquad\qquad + 2\int_z^{2x}\frac{1}{(t-x)^3}V_x^t(g_x)dt\Big\}. \end{aligned}$$

Thus

$$|E_{31}| \le \frac{(2x+1)}{n}\Big\{\frac{V_x^{2x}(g_x)}{x^2} + 2\int_z^{2x}\frac{1}{(t-x)^3}V_x^t(g_x)dt\Big\}.$$

Replacing the variable in the last integral by $x+x/\sqrt{u}$, we have

$$\begin{aligned} \int_{x+x/\sqrt{n}}^{2x} V_x^t(g_x)\frac{dt}{(t-x)^3} &= \frac{1}{2x^2}\int_1^n V_x^{x+x/\sqrt{u}}(g_x)du \\ &\le \frac{1}{2x^2}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \end{aligned}$$

Therefore

$$
\begin{aligned}
|E_{31}| &\leq \frac{(2x+1)}{nx^2}\Big\{V_x^{2x}(g_x) + \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x)\Big\} \\
&\leq \frac{2(2x+1)}{nx^2}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).
\end{aligned} \tag{3.6}
$$

Further, for evaluating E_{32}, using Lemma 4(ii) we obtain

$$
|E_{32}| \leq g_x(2x)\frac{(2x+1)}{nx^2} \leq \frac{(2x+1)}{nx^2}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{3.7}
$$

Finally, using Lemma 4(ii) and the assumption that $f(t) = O(e^{\alpha t})$, $(\alpha > 0)$ as $t \to \infty$, we find that for n sufficiently large,

$$
\begin{aligned}
|E_{33}| &\leq Mn\sum_{k=0}^{\infty} p_k(nx)\int_{2x}^{\infty} e^{\alpha t}\frac{e^{-nt}(nt)^k}{k!}dt \\
&= nM\sum_{k=0}^{\infty} p_k(nx)\int_{2x}^{\infty} e^{-(n-\alpha)t}\frac{(nt)^k}{k!}dt \\
&= mM\sum_{k=0}^{\infty} e^{-\alpha x}\cdot p_k(mx)\int_{2x}^{\infty} p_k(mt)\cdot(\frac{n}{m})^{2k+1}dt, \text{where } m = n-\alpha. \\
&\leq M\cdot e^{-\alpha x}\cdot m\sum_{k=0}^{\infty}(1+\frac{\alpha}{m})^{2k+1}p_k(mx)\int_{2x}^{\infty} p_k(mt)dt \\
&\leq M'\cdot\frac{(2x+1)e^{-\alpha x}}{nx^2}.
\end{aligned} \tag{3.8}
$$

Using (3.5) to (3.8), we have, for n sufficiently large,

$$
|E_3| \leq \frac{3x^{-2}(2x+1)}{n}\sum_{k=1}^{n} V_x^{x-x/\sqrt{k}}(g_x) + O(1)\frac{x^{-2}(2x+1)e^{-\alpha x}}{n}. \tag{3.9}
$$

Using (3.2) to (3.4) and (3.9) we are lead to the proposition (3.1) of the theorem which can easily be checked to be asymptotically the best.

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