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COMPLETELY REGULAR AND ORTHODOX CONGRUENCES ON REGULAR SEMIGROUPS

Branka P. Alimpić and **Dragica N. Krgović**

Abstract. Let S be a regular semigroup and $E(S)$ the set of all idempotents of S. Let $\mathrm{Con}\, S$ be the congruence lattice of S, and let T, K, U and V be equivalences on $\mathrm{Con}\, S$ defined by $\rho T\xi \Leftrightarrow \mathrm{tr}\,\rho = \mathrm{tr}\,\xi$, $\rho K\xi \Leftrightarrow \ker\rho = \ker\xi$, $\rho U\xi \Leftrightarrow \rho\cap \leq = \xi \cap \leq$ and $V = U \cap K$, where $\mathrm{tr}\,\rho = \rho\,|_{E(S)}$, $\ker\rho = E(S)\rho$, and $\leq$ is the natural partial order on $E(S)$. It is known that T, U and V are complete congruences on $\mathrm{Con}\, S$ and T-, K-, U- and V-classes are intervals $[\rho_T, \rho^T]$, $[\rho_K, \rho^K]$, $[\rho_U, \rho^U]$, and $[\rho_V, \rho^V]$, respectively ([**13**], [**10**], [**9**]). In this paper U-classes for which ρ^U is a semilattice congruence, and V-classes for which ρ^V is an inverse congruence are considered. It turns out that the union of all such U-classes is the lattice $\mathrm{CRCon}\, S$ of all completely regular congruences on S, and the union of all such V-classes is the lattice $\mathrm{OCon}\, S$ of all orthodox congruences on S. Also, some complete epimorphisms of the form $\rho \to \rho^U$ and $\rho \to \rho^V$ are obtained.

1. Preliminaries. In the following we shall use the terminology and notation of [**4**] and [**11**]. Throughout the paper, S stands for a regular semigroup. If $\rho \in \mathrm{Con}\, S$, and α is an equivalence on S/ρ, then the equivalence $\bar\alpha$ on S is defined by $a\bar\alpha b \Leftrightarrow (a\rho)\alpha(b\rho)$, $(a, b \in S)$. If α is a relation on S then α^* denotes the congruence on S generated by α. If α is an equivalence on S, then α° denotes the greatest congruence on S contained in α, and if $T \subseteq S$, then $T\alpha$ denotes the union of α-classes of all elements of T. If $\rho, \xi \in \mathrm{Con}\, S$ and $\rho \subseteq \xi$, then the relation ξ/ρ on S/ρ defined by $(a\rho)\xi/\rho(b\rho) \Leftrightarrow a\xi b$ $(a, b \in S)$ is a congruence. If $\rho, \xi, \zeta \in \mathrm{Con}\, S$, $\rho \subseteq \xi$ and $\rho \subseteq \zeta$, then $(\xi \vee \zeta)/\rho = (\xi/\rho) \vee (\zeta/\rho)$ and $(\xi \circ \zeta)/\rho = (\xi/\rho) \circ (\zeta/\rho)$.

Let $\mathcal{C}$ be a class of semigroups, and let $\rho \in \mathrm{Con}\, S$. Then ρ is a $\mathcal{C}$- congruence if $S/\rho \in \mathcal{C}$, and ρ is over $\mathcal{C}$ if $(\forall e \in E(S))e\rho \in \mathcal{C}$.

In the paper, σ, η, α, ν, Y, o denote the least group, semilattice, rectangular band, Clifford, inverse and orthodox congruences on S, respectively. Also, μ and τ denote the greatest idempotent separating and idempotent pure congruences on S, respectively.

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LEMMA 1. *Let $\mathcal{C}$ be a class of semigroups closed for homomorphisms and let $\gamma = \gamma_S$ be the least $\mathcal{C}$-congruence on S. If $\rho \in \operatorname{Con} S$, then*

(1) *ρ is a $\mathcal{C}$-congruence on $S \Leftrightarrow \rho \supseteq \gamma$,*

(2) *$\gamma_{S/\rho} = (\rho \vee \gamma)/\rho$.*

Proof. (1) Clearly, if $S/\rho \in \mathcal{C}$, then $\rho \supseteq \gamma$. Conversely, if $\rho \supseteq \gamma$, then $S/\gamma/(\rho/\gamma) \in \mathcal{C}$, i.e. $S/\rho \in \mathcal{C}$.

(2) From (1) it follows that $(\rho \vee \gamma)/\rho \supseteq \gamma_{S/\rho}$. Since $\rho \subseteq \bar{\gamma}_{S/\rho}$ and $\bar{\gamma}_{S/\rho}$ is a $\mathcal{C}$-congruence on S, we get $\rho \vee \gamma \subseteq \bar{\gamma}_{S/\rho}$, which yields $(\rho \vee \gamma)/\rho \subseteq \gamma_{S/\rho}$.

For $a \in S$, let $V(a)$ denote the set of all inverses of a in S. Let $\mathcal{U}$ and $\mathcal{V}$ be equivalences on S defined by $a\mathcal{U}b \Leftrightarrow V(a)\mathcal{H} = V(b)\mathcal{H}$, and $a\mathcal{V}b \Leftrightarrow V(a) = V(b)$.

RESULT 1. [**9**] *If $\rho, \xi \in \operatorname{Con} S$, then*

(1) $\rho T \xi \Leftrightarrow \bar{\mathcal{H}}_{S/\rho} = \bar{\mathcal{H}}_{S/\xi}$ *and* $\rho^T = \bar{\mathcal{H}}^\circ_{S/\rho}$, $\rho_T = (\operatorname{tr}\rho)^*$,

(2) $\rho K \xi \Leftrightarrow \bar{\tau}_{S/\rho} = \bar{\tau}_{S/\xi}$ *and* $\rho^K = \bar{\tau}_{S/\rho}$,

(3) $\rho U \xi \Leftrightarrow \bar{\mathcal{U}}_{S/\rho} = \bar{\mathcal{U}}_{S/\xi}$ *and* $\rho^U = \bar{\mathcal{U}}^\circ_{S/\rho}$, $\rho_U = (\rho \cap \leq)^*$,

(4) $\rho V \xi \Leftrightarrow \bar{\mathcal{V}}_{S/\rho} = \bar{\mathcal{V}}_{S/\xi}$ *and* $\rho^V = \bar{\mathcal{V}}^\circ_{S/\rho}$.

In particular we have $\varepsilon^T = \mathcal{H}^\circ = \mu$, $\varepsilon^K = \tau$, $\varepsilon^U = \mathcal{U}^\circ$, $\omega_U = (\leq)^*$ and $\varepsilon^V = \mathcal{V}^\circ$, where ε is the equality, and ω is the universal relation on S.

COROLLARY 1. *For $\rho \in \operatorname{Con} S$ we have $\rho^T/\rho = \mathcal{H}^\circ_{S/\rho} = \mu_{S/\rho}$, $\rho^K/\rho = \tau_{S/\rho}$, $\rho^U/\rho = \mathcal{U}^\circ_{S/\rho}$, $\rho^V/\rho = \mathcal{V}^\circ_{S/\rho}$.*

RESULT 2. [**9**] *A congruence ρ on S is over completely simple semigroups if and only if $\rho \subseteq \mathcal{U}$.*

RESULT 3. [**8**] *Let L be a complete lattice and C be a complete congruence on L. Then for any $x \in L$ the C-class xC is the interval $[x_C, x^C]$ of L, and for any $A \subseteq L$,*

$$(\bigvee_{x \in A} x)_C = \bigvee_{x \in A} x_C, \quad (\bigwedge_{x \in A} x)^C = \bigwedge_{x \in A} x^C.$$

RESULT 4. [**7**] *Let S be a regular semigroup and $\rho, \xi \in \operatorname{Con} S$. Then*

(i) *if ξ is idempotent separating, then $\rho \vee \xi = \rho \circ \xi \circ \rho$. In particular $\rho \vee \mu = \rho \circ \mu \circ \rho$;*

(ii) *if S is completely regular, then $\rho \vee \eta = \rho \circ \eta \circ \rho$;*

(iii) *if S is orthodox, then $\rho \vee Y = \rho \circ Y \circ \rho$.*

2. Completely regular congruences. In this section we establish certain characterizations of completely regular congruences.

LEMMA 2. *For a regular semigroup S, the following are equivalent:*

(i) *S is completely regular,*

(ii) *$\eta \subseteq \mathcal{U}$,*

(iii) *$\eta = \mathcal{U}$.*

Proof. Since any completely regular semigroup is a semilattice of completely simple semigroups, the equivalence (i)$\Leftrightarrow$(ii) follows immediately from Result 2.

By definition of $\mathcal{U}$, it follows that $\mathcal{U} \subseteq \mathcal{D}$, and if S is completely regular, then $\mathcal{D} = \eta$. Hence, (ii)$\Leftrightarrow$(iii).

THEOREM 1. *For $\rho \in \operatorname{Con} S$, the following are equivalent:*

(i) *ρ is a completely regular congruence,*

(ii) *ρ^U is a semilattice congruence,*

(iii) *$\rho^U = \rho \vee \eta$,*

(iv) *$\rho U(\rho \vee \eta)$.*

Proof. (i)$\Rightarrow$(iii)
$$\begin{aligned} \rho \in \operatorname{CRCon} S &\Leftrightarrow S/\rho \text{ is completely regular} \\ &\Leftrightarrow \mathcal{U}_{S/\rho} = \eta_{S/\rho} && \text{(by Lemma 2)} \\ &\Leftrightarrow \bar{\mathcal{U}}_{S/\rho} = \rho \vee \eta && \text{(by Lemma 1)} \\ &\Rightarrow \rho^U = \rho \vee \eta && \text{(by Result 1).} \end{aligned}$$

(iii)$\Rightarrow$(iv) This is evident.

(iv)$\Rightarrow$(ii) $\rho U(\rho \vee \eta) \Rightarrow \rho \vee \eta \subseteq \rho^U \Rightarrow \rho^U$ is a semilattice congruence.

(ii)$\Rightarrow$(i)
$$\begin{aligned} \eta \subseteq \rho^U &\Rightarrow \rho \vee \eta \subseteq \rho^U \\ &\Rightarrow (\rho \vee \eta)/\rho \subseteq \rho^U/\rho \\ &\Rightarrow \eta_{S/\rho} \subseteq \mathcal{U}_{S/\rho} \quad \text{(by Lemma 1 and Corollary 1)} \\ &\Rightarrow S/\rho \quad \text{is completely regular} \qquad \text{(by Lemma 2).} \end{aligned}$$

Let $\operatorname{SCon} S$ denote the lattice of all semilattice congruences on S.

COROLLARY 2.
(1) $(\forall \rho, \xi \in \operatorname{CRCon} S)(\rho U \xi \Leftrightarrow \rho \vee \eta = \xi \vee \eta)$,

(2) $\operatorname{CRCon} S = [\eta_U, \omega] = [\eta, \omega]U$ and $\eta U = [\eta_U, \eta]$,

(3) $\operatorname{SCon} S = [\eta, \omega] = \{\rho \in \operatorname{CRCon} S \mid \rho^U = \rho\}$,

(4) $\mu \subseteq \mathcal{U}^\circ \subseteq \eta$.

PROPOSITION 1. *For $\rho \in \operatorname{CRCon} S$, $\rho \vee \eta = \rho \circ \eta \circ \rho$.*

Proof. Let δ denote the least completely regular congruence on S. By Corollary 2 we have $\delta = \eta_U \subseteq \rho$. According to Lemma 1 we get $\eta_{S/\delta} = (\delta \vee \eta)/\delta = \eta/\delta$. So we have

$$\begin{aligned} & (\rho/\delta) \vee (\eta/\delta) = (\rho/\delta) \circ (\eta/\delta) \circ (\rho/\delta) \quad \text{(by Result 4(ii))} \\ \Leftrightarrow \quad & (\rho \vee \eta)/\delta = (\rho \circ \eta \circ \rho)/\delta \\ \Leftrightarrow \quad & \rho \vee \eta = \rho \circ \eta \circ \rho. \end{aligned}$$

From (1) of Corollary 2 and the implication (i)$\Rightarrow$(iii) of Theorem 1 we get Proposition 8.1 of [**9**].

COROLLARY 3. *For S the following are equivalent:*

(i) *S is completely regular,*

(ii) *$(\forall \rho \in \operatorname{Con} S)\rho^U = \rho \vee \eta$,*

(iii) $\eta = \mathcal{U}^\circ$,

(iv) $\varepsilon U \eta$.

The next result is an analogue of Theorem 1 of [**1**].

THEOREM 2. *For $\rho \in \operatorname{Con} S$, the following are equivalent:*

(i) *ρ is a completely simple congruence,*

(ii) *ρ^T is a rectangular band congruence,*

(iii) $\rho^T = \rho \vee \alpha$,

(iv) $\rho T(\rho \vee \alpha)$,

(v) $\rho U \omega$.

Proof. (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv) follows immediately from [1].

(i)$\Leftrightarrow$(v) ρ is a completely simple congruence

$\Leftrightarrow \omega/\rho$ is over completely simple semigroups

$\Leftrightarrow \rho U \omega$ (by Theorem 4.10 of [**9**]).

Let $\operatorname{CSCon} S$ ($\operatorname{RBCon} S$) denote the lattice of all completely simple (rectangular band) congruences on S. Then we have

COROLLARY 4. (1) $(\forall \rho, \xi \in \operatorname{CSCon} S) \rho T \xi \Leftrightarrow \rho \vee \alpha = \xi \vee \alpha$,

(2) $\operatorname{CSCon} S = [\alpha_T, \omega] = [\alpha, \omega]T$ *and* $\alpha T = [\alpha_T, \alpha]$,

(3) $\operatorname{RBCon} S = [\alpha, \omega] = \{\rho \in \operatorname{CSCon} S \mid \rho^T = \rho\}$,

(4) $\alpha_T = \omega_U = (\operatorname{tr} \alpha)^* = (\leq)^*$.

COROLLARY 5. *For S the following are equivalent:*

(i) *S is completely simple,*

(ii) $(\forall \rho \in \operatorname{Con} S) \rho^T = \rho \vee \alpha$,

(iii) $\mu = \alpha$,

(iv) $\mathcal{U}^\circ = \omega$,

(v) $\varepsilon U \omega$.

PROPOSITION 2. *For $\rho \in CS\operatorname{Con} S$, $\rho \vee \alpha = \rho \circ \alpha \circ \rho$.*

Proof. Let ζ denote the least completely simple congruence on S. By Corollary 4 we have $\zeta = \alpha_T \subseteq \rho$. According to Lemma 1 we get $\alpha_{S/\zeta} = (\zeta \vee \alpha)/\zeta = \alpha/\zeta$. Hence, by Corollary 5, $\mu_{S/\zeta} = \alpha/\zeta$. Thus

$$
\begin{aligned}
& (\rho/\zeta) \vee (\alpha/\zeta) = (\rho/\zeta) \circ (\alpha/\zeta) \circ (\rho/\zeta) \qquad \text{(by Result 4(i))} \\
\Leftrightarrow \quad & (\rho \vee \alpha)/\zeta = (\rho \circ \alpha \circ \rho)/\zeta \\
\Leftrightarrow \quad & \rho \vee \alpha = \rho \circ \alpha \circ \rho.
\end{aligned}
$$

3. Orthodox congruences. Now we describe orthodox congruences on S in terms of K and V. Let $\rho \in \operatorname{Con} S$. It is easy to see that ρ is an orthodox congruence if and only if $\ker \rho$ is a subsemigroup of S.

Lemma 3. *For a regular semigroup S, the following are equivalent:*

(i) *S is orthodox*,

(ii) $Y = \mathcal{V}$,

(iii) $Y \subseteq \mathcal{V}$,

(iv) $Y \subseteq \tau$.

Proof. (i)$\Rightarrow$(ii) It is proved in [**3**] and [**14**].

(ii)$\Rightarrow$(iii) This is evident.

(iii)$\Rightarrow$(iv) $Y \subseteq \mathcal{V} \Rightarrow Y \subseteq \mathcal{V}^{\circ} = \varepsilon^{V} \subseteq \varepsilon^{K} = \tau$.

(iv)$\Rightarrow$(i) Since Y is orthodox, $\ker Y$ is a subsemigroup of S. Thus $Y \subseteq \tau \Leftrightarrow \ker Y = E(S) \Rightarrow E(S)$ is a subsemigroup of $S \Leftrightarrow S$ is orthodox.

Theorem 3. *For $\rho \in \operatorname{Con} S$ the following are equivalent:*

(i) *ρ is an orthodox congruence,*

(ii) *ρ^{V} is an inverse congruence,*

(iii) $\rho^{V} = \rho \vee Y$,

(iv) $\rho V(\rho \vee Y)$,

(v) $\rho K(\rho \vee Y)$,

(vi) *ρ^{K} is an inverse congruence.*

Proof. (i)$\Rightarrow$(iii)

$$\begin{aligned} \rho \text{ is orthodox} &\Leftrightarrow \mathcal{V}_{S/\rho} = Y_{S/\rho} && \text{(by Lemma 3)} \\ &\Leftrightarrow \bar{\mathcal{V}}_{S/\rho} = \rho \vee Y && \text{(by Lemma 1)} \\ &\Rightarrow \rho^{V} = \rho \vee Y. && \text{(by Result 1).} \end{aligned}$$

(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(vi) This is evident.

(vi)$\Rightarrow$(i)

$$\begin{aligned} Y \subseteq \rho^{K} &\Rightarrow \rho \vee Y \subseteq \rho^{K} \\ &\Rightarrow (\rho \vee Y)/\rho \subseteq \rho^{K}/\rho \\ &\Leftrightarrow Y_{S/\rho} \subseteq \tau_{S/\rho} && \text{(by Lemma 1 and Corollary 1)} \\ &\Leftrightarrow \rho \ \text{ is orthodox} && \text{(by Lemma 3).} \end{aligned}$$

(iii)$\Rightarrow$(ii)$\Rightarrow$(vi) This is evident.

Let $\operatorname{ICon} S$ denote the set of all inverse congruences on S.

Corollary 6. (1) $(\forall \rho, \xi \in \operatorname{OCon} S)\rho V \xi \Leftrightarrow \rho \vee Y = \xi \vee Y$,

(2) $\operatorname{OCon} S = [o, \omega] = [Y, \omega]V$ *and* $YV = [o, Y]$,

(3) $\operatorname{ICon} S = [Y, \omega] = \{\rho \in \operatorname{OCon} S \mid \rho^{V} = \rho\}$,

(4) $\mathcal{V}^{\circ} \subseteq Y$.

From Corollary 6(1) and the implication (i)$\Rightarrow$(iii) of Theorem 3 we get Proposition 8.5 of [**9**], and from (i)$\Rightarrow$(v) of Theorem 3 we get oKY [**6**] and Lemma 2.1 of [**2**].

Corollary 7. *For S the following are equivalent:*

(i) *S is orthodox,*

(ii) $(\forall \rho \in \operatorname{Con} S)\rho^{V} = \rho \vee Y$,

(iii) $Y = \mathcal{V}^\circ$,

(iv) $\varepsilon V Y$.

Using Result 4(iii), the proof of the following proposition is similar to the proof of the Proposition 1.

PROPOSITION 3. *For* $\rho \in \mathrm{OCon}\, S$, $\rho \vee Y = \rho \circ Y \circ \rho$.

In the following we describe orthodox completely regular (i.e. orthogroup) congruences and orthodox completely simple congruences on S. By [**5**], any orthodox completely simple semigroup is a rectangular group and conversely.

THEOREM 4. *For* $\rho \in \mathrm{Con}\, S$, *the following are equivalent:*

(i) ρ *is an orthogroup congruence,*

(ii) ρ^V *is a Clifford congruence,*

(iii) $\rho^V = \rho \vee \nu$,

(iv) $\rho V(\rho \vee \nu)$.

Proof. (i)⇒(ii) By Theorem 3, ρ^V is an inverse completely regular congruence, i.e. a Clifford congruence.

(ii)⇒(iii)

$$\begin{aligned} \rho^V \text{ is a Clifford congruence} &\Rightarrow \rho^V = \rho \vee Y && \text{(by Theorem 3)} \\ &\Rightarrow \rho^V \subseteq \rho \vee \nu \subseteq \rho^V && \text{(since } Y \subseteq \nu) \\ &\Rightarrow \rho^V = \rho \vee \nu. \end{aligned}$$

(iii)⇒(iv) This is evident.

(iv)⇒(i) By Theorem 3, $\rho V(\rho \vee \nu)$ implies that ρ is orthodox. On the other hand,

$$\begin{aligned} \rho V(\rho \vee \nu) &\Rightarrow \rho U(\rho \vee \nu) && \text{(since } V \subseteq U) \\ &\Rightarrow \rho \quad \text{is completely regular} && \text{(by Theorem 1).} \end{aligned}$$

Hence ρ is an orthogroup congruence.

Let $\mathrm{OGCon}\, S$ ($\mathrm{SGCon}\, S$) denote the lattice of all orthogroup (Clifford) congruences on S. Then we have

COROLLARY 8. (1) $(\forall \rho, \xi \in \mathrm{OGCon}\, S)\rho V \xi \Leftrightarrow \rho \vee \nu = \xi \vee \nu$.
(2) $\mathrm{OGCon}\, S = [\nu_V, \omega] = [\nu, \omega]V$ *and* $\nu V = [\nu_V, \nu]4$.
(3) $\mathrm{SGCon}\, S = [\nu, \omega] = \{\rho \in \mathrm{OGCon}\, S \mid \rho^V = \rho\}$.

COROLLARY 9. *For S the following are equivalent:*

(i) S *is an orthogroup,*

(ii) $(\forall \rho \in \mathrm{Con}\, S)\rho^V = \rho \vee \nu$,

(iii) $\mathcal{V}^\circ = \nu$,

(iv) $\varepsilon V \nu$.

THEOREM 5. *For* $\rho \in \mathrm{Con}\, S$, *the following are equivalent:*

(i) ρ *is a rectangular group congruence,*

(ii) ρ^V *is a group congruence,*

(iii) $\rho^V = \rho \vee \sigma$,

(iv) $\rho V(\rho \vee \sigma)$.

Proof. (i)⇒(ii) By Theorem 3, ρ^V is an inverse completely simple congruence, i.e. ρ^V is a group congruence.

(ii)⇒(iii) If ρ^V is a group congruence, then $\rho^V = \rho^K = \rho \vee \sigma$, by Theorem 3 of [**1**].

(iii)⇒(iv) This is evident.

(iv)⇒(i) $\rho V(\rho \vee \sigma) \Rightarrow \rho$ is orthodox (by Theorem 3). On the other hand,

$$\begin{aligned} \rho V(\rho \vee \sigma) &\Rightarrow \rho U(\rho \vee \sigma) U \omega && \text{(by Theorem 2)} \\ &\Rightarrow \rho \text{ is completely simple.} && \text{(by Theorem 2).} \end{aligned}$$

Hence, ρ is a rectangular group congruence.

Let $\mathrm{RGCon}\, S$ ($\mathrm{GCon}\, S$) denote the lattice of all rectangular group (group) congruences on S. Then we have

Corollary 10. (1) $(\forall \rho, \xi \in \mathrm{RGCon}\, S)\rho V \xi \Leftrightarrow \rho \vee \sigma = \xi \vee \sigma$.
(2) $\mathrm{RGCon}\, S = [\sigma_V, \omega] = [\sigma, \omega]V$ *and* $\sigma V = [\sigma_V, \sigma]$,
(3) $\mathrm{GCon}\, S = [\sigma, \omega] = \{\rho \in \mathrm{RGCon}\, S \mid \rho^V = \rho\}$.

Corollary 11. *For S the following are equivalent:*

(i) *S is a rectangular group,*

(ii) $(\forall \rho \in \mathrm{Con}\, S)\rho^V = \rho \vee \sigma$,

(iii) $\mathcal{V}^\circ = \sigma$,

(iv) $\varepsilon V \sigma$.

4. Some complete epimorphisms. Using the results of Theorems 1–5 and the Result 2 we get the following

Theorem 6. *Let S be a regular semigroup. The mappings*

$$\begin{array}{llll} \varphi_1 : \mathrm{CRCon}\, S \longrightarrow \mathrm{SCon}\, S & \textit{defined by} & \varphi_1(\rho) = \rho \vee \eta, \\ \varphi_2 : \mathrm{CSCon}\, S \longrightarrow R\mathrm{RBon}\, S & \textit{defined by} & \varphi_2(\rho) = \rho \vee \alpha, \\ \varphi_3 : \mathrm{OCon}\, S \longrightarrow \mathrm{ICon}\, S & \textit{defined by} & \varphi_3(\rho) = \rho \vee Y, \\ \varphi_4 : \mathrm{OGCon}\, S \longrightarrow \mathrm{SGCon}\, S & \textit{defined by} & \varphi_4(\rho) = \rho \vee \nu, \\ \varphi_5 : RG\mathrm{Con}\, S \longrightarrow \mathrm{GCon}\, S & \textit{defined by} & \varphi_5(\rho) = \rho \vee \sigma \end{array}$$

are complete epimorphisms. The classes of the complete congruence $\bar{\varphi}$ induced by the epimorphism φ are U-classes, if $\varphi = \varphi_1$; T-classes, if $\varphi = \varphi_2$; and V-classes, if $\varphi = \varphi_i$, $i = 3, 4, 5$.

For a completely regular semigroup S, the statement concerning the mapping φ_1 is given in [**12**] (Theorem 4.3(i)).

Corollary 12.

$$\begin{array}{llll} \text{(i)} & (\cap_{\rho \in F}\rho) \vee \eta = \cap_{\rho \in F}(\rho \vee \eta) & (F \subseteq \mathrm{CRCon}\, S), \\ \text{(ii)} & (\cap_{\rho \in F}\rho) \vee \alpha = \cap_{\rho \in F}(\rho \vee \alpha) & (F \subseteq \mathrm{CSCon}\, S), \\ \text{(iii)} & (\cap_{\rho \in F}\rho) \vee Y = \cap_{\rho \in F}(\rho \vee Y) & (F \subseteq \mathrm{OCon}\, S), \\ \text{(iv)} & (\cap_{\rho \in F}\rho) \vee \nu = \cap_{\rho \in F}(\rho \vee \nu) & (F \subseteq \mathrm{OGCon}\, S), \end{array}$$

$$\text{(v)} \quad (\cap_{\rho \in F} \rho) \vee \sigma \;=\; \cap_{\rho \in F} (\rho \vee \sigma) \qquad (F \subseteq \mathrm{RGCon}\, S).$$

From (i) of this corollary we get Theorem 4.7 of [**7**], and from (iii) we get Theorem 2.4 of [**2**].

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367

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