PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 53 (67), 1993, 4–16

NON-COMPLETE EXTENDED *P*-SUM OF GRAPHS, GRAPH ANGLES AND STAR PARTITIONS

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Abstract. The NEPS (Non-complete Extended P-Sum) of graphs is a graph operation in which the vertex set of the resulting graph is the Cartesian product of the vertex sets of starting graphs. The paper contains a survey on NEPS and some new results concerning graph angles and star partitions of NEPS.

1. Introduction. This is a paper on a very general graph operation called NEPS (non-complete extended p-sum) [**18**].

Definition 1.1. Let $\mathcal{B} \subseteq \{0,1\}^n \setminus \{(0,\dots,0)\}$ be a set of binary n-tuples. NEPS with basis $\mathcal{B}$ of graphs $G_1,\dots,G_n$ is the graph whose vertex set is the Cartesian product of the vertex sets of graphs $G_1,\dots,G_n$ in which two vertices, say $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$, are adjacent if and only if there exists an n-tuple $(\beta_1,\dots,\beta_n) \in \mathcal{B}$ such that $x_i = y_i$ holds whenever $\beta_i = 0$, and x_i is adjacent to y_i (in G_i) whenever $\beta_i = 1$.

Let G be a graph with vertices $1,\dots,n$ and let $A = A(G)$ be the (0,1)-adjacency matrix of G. Let $\mu_1,\dots,\mu_m$ $(\mu_1 > \dots > \mu_m)$ be the distinct eigenvalues of G with corresponding eigenspaces $\mathcal{E}(\mu_1),\dots,\mathcal{E}(\mu_m)$. Let $\{e_1,\dots,e_n\}$ be the standard orthonormal basis of $\mathbf{R}^n$.

Definition 1.2. The numbers $\alpha_{ij} = \cos\beta_{ij}$ $(i = 1,\dots,m;\ j = 1,\dots,n)$, where β_{ij} is the angle between $\mathcal{E}(\mu_i)$ and e_j, are called angles of G. The $m \times n$ matrix $\mathcal{A} = (\alpha_{ij})$ is called the angle matrix of G.

We may order the columns of $\mathcal{A}$ lexicographically so that $\mathcal{A}$ becomes a graph invariant. Rows of $\mathcal{A}$ are associated with eigenvalues and are called eigenvalue angle sequences, while columns of $\mathcal{A}$ are associated with vertices and are called vertex angle sequences.

Definition 1.3. The main angles of G are the cosines of the angles between the eigenspaces of G and the all-1 vector.

AMS Subject Classification (1990): Primary 05 C 50

Let

$$A = \mu_1 P_1 + \ldots + \mu_m P_m \tag{1.1}$$

be the usual spectral decomposition of the adjacency matrix A of a graph G; the matrix $P_i = (p_{jk}^{[i]})$ represents the orthogonal projection onto the eigenspace $\mathcal{E}(\mu_i)$ $(i = 1, \ldots, m)$. We have $\alpha_{ij} = \|P_i e_j\|$ and (when $P_i e_j$, $P_i e_k$, are non-zero)

$$p_{jk}^{[i]} = \alpha_{ij}\alpha_{ik}\cos\gamma_{jk}^{[i]},$$

where $\gamma_{jk}^{[i]}$ is the angle between $P_i e_j$ and $P_i e_k$. In particular, $p_{jj}^{[i]} = \alpha_{ij}^2$. Hence, the main diagonal of P_i contains squares of members of the angle sequence of μ_i.

Let V be a Euclidean space. The span of the subset $\{v_1, \ldots, v_k\}$ is denoted by $\langle v_i \big| \ i = 1, \ldots, k\rangle$. Let (x, y) be the inner product of vectors $x, y \in V$ and let $\|x\|$ be the norm of x. A star in V is a finite set of vectors which span V. For a star $\mathcal{S} = \{v_1, \ldots, v_k\}$

1^o $\mathcal{S}$ is orthogonal, if $(v_i, v_j) = 0$ $(i \neq j)$;

2^o $\mathcal{S}$ is spherical, if $\|v_i\| = \|v_j\|$ for all i, j.

Let U be a non-trivial subspace of V. A eutactic star in U is the orthogonal projection onto U of an orthogonal spherical basis of V.

More generally, suppose that A is a real symmetric matrix with distinct eigenvalues $\mu_1 > \ldots > \mu_m$, and corresponding eigenspaces $\mathcal{E}(\mu_1), \ldots, \mathcal{E}(\mu_m)$. Let k_i $(= \dim(\mathcal{E}(\mu_i))$ be the multiplicity of μ_i and let $V = \mathbf{R}^n$ with $(x, y) = x^T y$. Denote by P_i the orthogonal projection of V onto $\mathcal{E}(\mu_i)$. Let $\mathcal{E}$ be the standard basis $\{e_1, \ldots, e_n\}$ of V, and let $\mathcal{F}_i$ be the eutactic star $\{P_i e_1, \ldots, P_i e_n\}$ obtained by projecting the star $\mathcal{E}$ onto $\mathcal{E}(\mu_i)$. ($P_i e_j$ is one of the arms of the star $\mathcal{F}_i$ and the angle α_{ij} is the norm $\|P_i e_j\|$ of this arm.) In [**21**] the following question was posed.

Question. Given A, is it possible to find a basis $\mathcal{B}$ of $\mathbf{R}^n$ consisting of vectors from $\cup_{i=1}^m \mathcal{F}_i$ such that (for $P_i e_s$, $P_j e_t \in \mathcal{B}$) the condition

$$P_i e_s \neq P_j e_t \Rightarrow s \neq t$$

holds?

If such a basis $\mathcal{B}$ exists, then there is a 1-1 correspondence between $\mathcal{E}$ and $\mathcal{B}$. If a set X is partitioned into sets $X, \ldots, X_n$ we shall write $X = X_1 \dot\cup \cdots \dot\cup X_n$ and call $X_1, \ldots, X_n$ the cells of the partition. A cell X_i is called non-trivial if $|X_i| > 1$.

Definition 1.4. A star basis of $\mathbf{R}^n$, corresponding to a symmetric matrix A, is a basis $\mathcal{B} = \{P_i e_s \big| s \in X_i, \ i = 1, \ldots, m\}$, where P_i is defined as above, and $X = X_1 \dot\cup \cdots \dot\cup X_m$ is a partition of the set $\{1, \ldots, n\}$. The corresponding partition is called a star partition. Cells in a star partition are called star cells.

The existence of star bases and star partitions has been proved in [**21**]. The following theorem from [**21**] will be useful in further considerations.

PROPOSITION 1.5. *If $X = X_1 \dot\cup \cdots \dot\cup X_m$ is a partition of $\{1, \ldots, n\}$, the vertex set of a graph G, such that μ_i is not an eigenvalue of $G - X_i$ $(i = 1, \ldots, m)$, then this partition is a star partition of G.*

Section 2 represents a survey of results on NEPS. In Section 3 (Section 4) we study the angles (the star partitions) in a NEPS.

2. A survey of results on NEPS. NEPS was introduced in [**18**] and has been rediscovered in [**38**]. It generates a lot of binary graph operations in which the vertex set of the resulting graph is the Cartesian product of vertex sets of graphs on which the operation is performed (see [**16**, p.p. 65–66], and the references cited in [**16**]).

We now recall some special cases of NEPS. Let G be the NEPS with basis $\mathcal{B}$ of graphs $G_1, \dots, G_n$.

In particular, if $n = 2$ and

if $\mathcal{B} = \{(0,1),(1,0)\}$, G $(= G_1 + G_2)$ becomes the sum of G_1 and G_2;

if $\mathcal{B} = \{(1,1)\}$, G $(= G_1 \times G_2)$ becomes the product of G_1 and G_2;

and if $\mathcal{B} = \{(0,1),(1,0),(1,1)\}$, G $(= G_1 * G_2)$ becomes the strong product of G_1 and G_2.

The p-sum of graphs is a NEPS with the basis containing all n-tuples with exactly p 1's.

The odd (even) sum of graphs is a NEPS with the basis containing all n-tuples with an odd (even) number of 1's.

The mixed sum of graphs is a NEPS with the basis containing all n-tuples in which the number of 1's is congruent to 1 or 2 modulo 4.

The J-sum (or extended p-sum) of graphs, where J is a subset of $\{1, \dots, n\}$, is a NEPS with the basis containing all n-tuples in which the number of 1's belongs to J.

The notion of NEPS has arisen in a natural way when studying spectral properties of graphs obtained by binary operations of the mentioned type. Main ideas are essentially described in [**2**]. Early references [**3**–**5**] have been summarized and generalized in [**6**].

In [**19**], the definition of NEPS of graphs has been extended to digraphs (digraphs may have multiple arcs and/or loops) and in [**27**] and [**35**] to infinite graphs in two different ways.

There are some other graph operations in which the resulting graph has as its vertex set the Cartesian product of vertex sets of the starting graphs. In [**8**], the so called Boolean operations of graphs are defined, while [**40**] introduces a more general operation called the generalized direct product of graphs containing NEPS and Boolean operations as special cases. The generalized direct product of graphs has been extended to digraphs in [**31**–**34**]. The generalized direct product of graphs does not posess some useful properties of NEPS and therefore we consider here only NEPS.

The following two theorems are taken from [**18**].

THEOREM 2.1. *Let $A_1, \dots, A_n$ be adjacency matrices of graphs $G_1, \dots, G_n$, respectively. The NEPS G with basis $\mathcal{B}$ of graphs $G_1, \dots, G_n$ has the adjacency* *matrix given by*

$$A = \sum_{\beta \in \mathcal{B}} A_1^{\beta_1} \otimes \cdots \otimes A_n^{\beta_n}.$$

Here $A_i^0 = I$ (of the same order), $A_i^1 = A_i$, and $\otimes$ denotes the Kronecker product of matrices.

THEOREM 2.2. *If, for $i = 1, \dots, n$, $\lambda_{i1}, \dots, \lambda_{in_i}$ is the spectrum of G_i (n_i being its number of vertices), then the spectrum of G which is the NEPS of $G_i, \dots, G_n$ with basis $\mathcal{B}$ consists of all possible values $\Lambda_{i_1,\dots,i_n}$ where*

$$\Lambda_{i_1,\dots,i_n} = \sum_{\beta \in \mathcal{B}} \lambda_{1i_1}^{\beta_1} \dots \lambda_{ni_n}^{\beta_n} \qquad (i_k = 1, \dots, n_k;\ k = 1, \dots, n).$$

In particular, if $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_m$ are the eigenvalues of G and H, respectively, then

$\lambda_i + \mu_j$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ are eigenvalues of $G + H$;

$\lambda_i \mu_j$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ are eigenvalues of $G \times H$;

$\lambda_i + \mu_j + \lambda_i \mu_j$ $(i = 1, \dots, n;\ j = 1, \dots, m)$ are eigenvalues of $G * H$.

Theorem 2.2 has been extended in [**26**,**27**,**35**,**36**] to infinite digraphs.

It is a well-known fact that the product of two connected graphs G_1 and G_2 is disconnected if both G_1 and G_2 are bipartite. Similar situations appear in a NEPS. The conditions under which a NEPS is connected or bipartite can be efficiently studied through graph eigenvalues, as it was done in papers [**3**], [**6**], [**7**] (see also Section 7.4 in [**16**]). For the same type of problems with Boolean functions and generalized direct product of graphs and digraphs see [**8**], [**19**], [**2**], [**3**].

Two graphs are said to be almost cospectral if their non-zero eigenvalues (and their multiplicities) coincide. In [**11**], it was conjectured that, if the NEPS of bipartite graphs is disconnected, its components are almost cospectral. This conjecture is true for the product of graphs (see [**11**]).

The notion of the NEPS has been used in [**10**] (see also [**15**, pp. 54–59]) to construct some strongly regular graphs.

THEOREM 2.3. *The odd sum F_n of n ($n \geq 2$) copies of the graph K_4 is a strongly regular graph with eigenvalues $2^{2n-1} - (-1)^{n-1} 2^{n-1}$, 2^{n-1}, -2^{n-1}.*

THEOREM 2.4. *The mixed sum H_s of $4s$ ($s \geq 1$) copies of the graph K_2 is a strongly regular graph with eigenvalues $2^{4s-1} - (-1)^s 2^{2s-1}$, 2^{2s-1}, -2^{2s-1}.*

It was noted in [**10**] that F_{4p} and H_{2p} are cospectral, and the problem of solving graph equation $O_{4p}(G) = M_{2p}(H)$, where $O_n(G)$ and $M_n(G)$ denote the odd and the mixed sum of n copies of the graph G, was posed.

Spectra of regular graphs can be used for determining the number of spanning trees. The next theorem can be found, for example, in [**16**, p. 39].

THEOREM 2.5. *If $\lambda_1 = r$, $\lambda_2, \dots, \lambda_n$ is the spectrum of a regular graph G of degree r, then the number of spanning trees of G is equal to*

$$(1/n)(r - \lambda_2) \cdots (r - \lambda_n).$$

It is easy to see that the NEPS of regular graphs is again a regular graph and Theorem 2.5 can be applied. Several interesting graphs can be expressed in the form of a NEPS of regular graphs and their numbers of spanning trees have been determined in [**7**]. See also [**16**, Section 7.6].

It is known (see, for example, [**16**, p. 44]) that the number N_k of walks of length k in a graph with distinct eigenvalues $\mu_1, \dots, \mu_m$ is given by

$$C_1\mu_1^k + \dots + C_m\mu_m^k, \tag{2.1}$$

where C_i $(i = 1, \dots, m)$ are constants.

THEOREM 2.6 (c.f. [**4**],[**6**]). *Let* $\sum_{i_j} C_{ji_j}\lambda_{ji_j}^k$ $(j = 1, \dots, n)$ *denote the number of walks of length* k *for the graph* G_j. *Then the NEPS with the basis* $\mathcal{B}$ *of graphs* $G_1, \dots, G_n$ *contains*

$$\sum_{i_1,\dots,i_n} C_{1i_1}\dots C_{ni_n}\left(\sum_{\beta\in\mathcal{B}} \lambda_{1i_1}^{\beta_1}\dots\lambda_{ni_n}^{\beta_n}\right)^k$$

walks of length k.

A formula for the number of walks of length k, between two specified vertices in a NEPS, has been derived in [**9**]. It was applied to the NEPS of complete graphs thus solving a problem of enumeration of ways in which a rook (chess piece) can make a series of k moves between two specified cells of a chess-board. A similar problem for a knight was solved in [**22**, pp. 67–68]. Here we reproduce (Problem 2.7) our solution to Problem E 2392 [**37**]. See also [**1**] for some comments to different solutions of this problem.

Problem 2.7. Let the distance between the two cells of the (infinite) chess-board be defined as the minimum number of steps for a knight to move from one cell to another. Denote by $D(0, P)$ the distance between cells $0 = (0, 0)$ and $P = (a, b)$

Solution: To determine $D(0, P)$, we first determine $N^k_{(0,0),(a,b)}$, the number of ways for a knight to move from the cell $(0, 0)$ to the cell (a, b) (or, equivalently, from the cell (1,1) to the cell $(a + 1, b + 1)$) in exactly k steps. For this purpose, we shall observe, for sufficiently large n, a chess-board of dimension $n \times n$ on a torus, and a graph which represents the knight movement on this chess-board. The adjacency matrix $\mathcal{A}$ of this graph can be represented in the form

$$\mathcal{A} = A \otimes (A^2 - 2I) + (A^2 - 2I) \otimes A, \tag{2.2}$$

where A is the adjacency matrix of a cycle of length n. It is well known that the eigenvalues λ_i of the matrix A are given by $\lambda_i = 2\cos(2\pi i/n)$ $(i = 1, \dots, n)$, while $x_{il} = \frac{1}{n}\exp\left(\frac{2\pi i}{n} lj\right)$, $l = 1, \dots, n$, $j^2 = -1$, are the coordinates of the eigenvector corresponding to λ_i. Hence, the eigenvalues and the eigenvectors of the matrix (2.2) are determined by

$$\Lambda_{i,l} = (\lambda_i + \lambda_l)(\lambda_i\lambda_l - 2), \quad X_{il} = x_i \otimes x_l, \quad i, l = 1, \dots, n.$$

It can easily be checked that, for $0 \le a, b \le n$ and $k < n$, we have

$$N^k_{(0,0),(a,b)} = (\mathcal{A}^k)_{(1,1),(a+1,b+1)} = \sum_{p,q=1}^{n} x_{p1} x_{q1} \overline{x}_{p,a+1} \overline{x}_{q,b+1} \Lambda^k_{p,q}$$

$$= \sum_{p,q=1}^{n} \frac{4^k}{n^2} \exp\left(\frac{2\pi}{n}(-pa - qb)j\right) \left(\cos\frac{2\pi}{n}p + \cos\frac{2\pi}{n}q\right)^k \cdot$$

$$\cdot \left(2\cos\frac{2\pi}{n}p \cos\frac{2\pi}{n}q - 1\right)^k .$$

By letting $n \to +\infty$, we obtain

$$\begin{aligned} &N^k_{(0,0),(a,b)} \\ &= \frac{4^{k-1}}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \exp(-j(ax+by))(\cos x + \cos y)^k (2\cos x \cos y - 1)^k dx\, dy \\ &= \sum_{t=0}^{k} (-2)^{k-t} \binom{k}{t} \sum_{s_1,s_2=0}^{k} \frac{k!}{s_1! s_2!} \binom{s_1 + t}{\frac{1}{2}(s_1 + t - a)} \binom{s_2 + t}{\frac{1}{2}(s_2 + t - b)}, \end{aligned} \tag{2.3}$$

where $\binom{m}{\alpha} = 0 \quad$ if $\alpha \notin \{0, 1, \dots, m\}$.

Thus

$$D(0, P) = \min\{k \mid N^k_{(0,0),(a,b)} \ne 0\}. \tag{2.4}$$

Without a loss of generality, we may assume that $a \ge b$ and $a > 2$. By a simple, but tedious analysis of formula (2.3), making use of (2.4), we obtain

$$D(0,P) = \begin{cases} a + b - 2[(a+b)/3], & a \le 2b \\ a + b - 2\left[\frac{a}{2}\right] - 2\left[\frac{1}{2}\left(b - \left[\frac{a}{2}\right]\right)\right], & a \ge 2b. \end{cases}$$

An alternative form of this formula was given in [**1**], together with an equivalent formula due to M. Goldberg.

Definition 2.8. An eigenvalue (of a graph) is called main if its eigenspace contains a vector not being orthogonal to $(1, \dots, 1)$.

The next proposition was proved in [**29**, **30**].

PROPOSITION 2.9. *An eigenvalue of a NEPS of graphs is main if and only if, when expressed in terms of eigenvalues of graphs on which the operation is performed, it depends only on main eigenvalues of these graphs.*

Let nearly equal mean differing by at most one. A graph is cordial if there exists a partition of its vertex set into two nearly equal in size subsets V_1 and V_2, such that the set edges with both end-points in V_1 or both end-points in V_2 is nearly equal in size to the set of edges which have one end point in V_1 and the other in V_2.

It was proved in [**24**] that the class of balanced symmetric cordial graphs is closed under NEPS. Some similar results are proved too. In proving these results, spectral techniques were used.

Angles of a NEPS have been considered in [**20**]. The result from [**20**] will be extended and generalized in Section 3.

An elementary proof of Lloyd's theorem from coding theory is given in [**17**]. It uses the spectrum of a NEPS. A version of this proof is given in [**16**, pp. 131-132] and reproduced in [**22**, pp. 129-130].

The definition and some properties of NEPS appear in textbooks [**12**], [**13**], [**22**]. Expository texts on NEPS are contained in monographs [**16**], [**15**], in bibliographic survey [**23**], in the paper [**28**] and in the report [**25**].

We conclude this review section by the following theorem which summarizes some known results and adds some new ones.

THEOREM 2.10. *The following classes of graphs are closed under NEPS operation:*

1^o *regular graphs,*
2^o *eulerian graphs,*
3^o *balanced symmetric cordial graphs,*
4^o *singular graphs,*
5^o *integral graphs,*
6^o *transitive graphs,*
7^o *walk-regular graphs,*
8^o *quasi-group graphs.*

1^o and 2^o follow from the fact that vertex degrees in a NEPS can be expressed in terms of vertex degrees of graphs on which the operation is performed. 3^o was proved in [**24**]. 4^o and 5^o follow from Theorem 2.2. 6^o and 8^o were proved in [**38**]. 7^o follows from the result on angles (see Section 3, Remark 3.5) and the fact that a graph is walk-regular if and only if all vertices have the same vertex angle sequences [**20**].

Remark 2.11. In Theorem 2.10 we call a graph eulerian if all its vertices have even degrees. A graph is singular if zero belongs to its spectrum. A graph is called integral if all its eigenvalues are integers. If the number of closed walks of length k $(k = 1, 2, \dots)$ starting and terminating at a vertex is independent of the vertex, a graph is called walk-regular.

3. Angles. The following preparatory lemma is a direct consequence of the distributivity of the Kronecker product w.r.t. the matrix addition.

LEMMA 3.1. *Given matrices $A_1, \dots, A_k$ (in particular, the adjacency matrices of graphs $G_1, \dots, G_k$, respectively) with spectral decompositions*

$$\sum_{s_1=1}^{m_1} \mu_{s_1}^{(1)} P_{s_1}^{(1)}, \dots, \sum_{s_k=1}^{m_k} \mu_{s_k}^{(k)} P_{s_k}^{(k)}.$$

Then

$$A_1 \otimes \cdots \otimes A_k = \sum_{(s_1,\dots,s_k)} \mu_{s_1}^{(1)} \dots \mu_{s_k}^{(k)} \ (P_{s_1}^{(1)} \otimes \cdots \otimes P_{s_k}^{(k)}). \tag{3.1}$$

Remark 3.2. If some of the products $\mu_{s_1}^{(1)} \dots \mu_{s_k}^{(k)}$ $(1 \le s_i \le m_i;\ i = 1, \dots, k)$ coincide, then (3.1) can be written in the form

$$A_1 \otimes \cdots \otimes A_k = \sum_r \mu_r P_r, \tag{3.2}$$

where $\mu_1 > \ldots > \mu_t$ are different products of the form $\mu_{s_1}^{(1)} \ldots \mu_{s_k}^{(k)}$ $(= \mu_r$ for some $r)$; $P_r = \sum P_{s_1}^{(1)} \otimes \cdots \otimes P_{s_k}^{(k)}$ where the sum is taken over all k-tuples $(s_1, \ldots, s_k)$ such that $\mu_{s_1}^{(1)} \ldots \mu_{s_k}^{(k)} = \mu_r$. Actually, (3.2) is a spectral decomposition of $A_1 \otimes \cdots \otimes A_k$.

To prove (3.2) it is sufficient to prove that $P_i P_j = \delta_{ij} I$. For this purpose we make use of the fact that

$$\Big(\sum X_{i_1} \otimes \cdots \otimes X_{i_k}\Big) \Big(\sum Y_{j_1} \otimes \cdots \otimes Y_{j_k}\Big) = \sum (X_{i_1} \cdot Y_{j_1}) \otimes \cdots \otimes (X_{i_k} \cdot Y_{j_k}).$$

Remark 3.2 justfies the following definition.

Definition 3.3. A NEPS of some graphs is called coincidence-free if no two eigenvalues obtained by Theorem 2.2 coincide.

In [**20**] (see Proposition 5) the same result as in Remark 3.2 was deduced for the Kronecker product of two matrices, but expressed in the terms of graph angles. Namely, the following proposition has been proved in [**20**].

PROPOSITION 3.4. *Let A, B be real symmetric square matrices of orders m, n respectively. Let λ be an eigenvalue of A and let μ be an eigenvalue of B. Let $\lambda_{i_1}, \ldots, \lambda_{i_t}$ and $\mu_{i_1}, \ldots, \mu_{i_t}$ be all the distinct eigenvalues of A, B respectively such that $\lambda\mu = \lambda_{i_1}\mu_{i_1} = \cdots = \lambda_{i_t}\mu_{i_t}$. Let $\alpha_{i_k g}$ be the angle corresponding to the eigenvalue λ_{i_k} and the coordinate g $(g = 1, 2, \ldots, m;\ k = 1, 2, \ldots, t)$ and let $\beta_{i_k h}$ $(h = 1, 2, \ldots, n;\ k = 1, 2, \ldots, t)$ be the angle corresponding to the eigenvalue μ_{i_k} and the coordinate h. Then the angle γ_{gh} corresponding to the eigenvalue $\lambda\mu$ of $A \otimes B$ and the coordinate pair (g, h) satisfies*

$$\gamma_{gh}^2 = \alpha_{i_1 g}^2 \beta_{i_1 h}^2 + \alpha_{i_2 g}^2 \beta_{i_2 h}^2 + \cdots + \alpha_{i_t g}^2 \beta_{i_t h}^2.$$

Having in mind that diagonal entries of projection matrices P_i are squares of angles, Remark 3.2, and especially formula (3.2), yield Proposition 3.4 for $k = 2$. Both Remark 3.2 and Proposition 3.4 can be generalized to hold for any NEPS, but we avoid such more general statements in order to avoid technical difficulties which would not contribute much to the essence of the matter.

Remark 3.5. We see that in a coincidence-free NEPS the angles are the products of the corresponding angles on which the operation is performed. Moreover, we see that the angles of a NEPS are independent of the basis if the change of the basis does not affect the coincidences between the eigenvalues.

Now we turn to the main angles. From (1.1) we get

$$A^k = \mu_1^k P_1 + \ldots + \mu_m^k P_m. \tag{3.3}$$

Comparing (2.1) with (3.3) we get that C_i from (2.1) is equal to the sum of all entries of P_i from (1.1). On the other hand, it is known [**14**] that

$$N_k = n \sum_{i=1}^{m} \beta_i^2 \mu_i^k, \tag{3.4}$$

where β_i is the main angle of the eigenvalue μ_i. Hence we can formulate the following proposition.

PROPOSITION 3.6. *We have*

$$\beta_i = \sqrt{\frac{1}{n}\mathrm{sum}\, P_i} \quad (i = 1, \dots, m),$$

where sum X *denotes the sum of all entries of the matrix* X.

It is easy to see that $\mathrm{sum}(X \otimes Y) = \mathrm{sum}\ X\ \mathrm{sum}\ Y$. In a coincidence-free NEPS we get, from Lemma 3.1 and Proposition 3.6, that the main angle of the eigenvalue $\mu_{s_1}^{(1)} \cdots \mu_{s_k}^{(k)}$ is equal to

$$\sqrt{\frac{1}{n}\mathrm{sum}\,(P_{s_1}^{(1)} \otimes \cdots \otimes P_{s_k}^{(k)})} = \sqrt{\frac{1}{n_1}\mathrm{sum}\, P_{s_1}^{(1)} \cdots \frac{1}{n_k}\mathrm{sum}\, P_{s_k}^{(k)})} = \beta_{s_1}^{(1)} \dots \beta_{s_k}^{(k)},$$

where $n_1, \dots, n_k$ are numbers of vertices of graphs $G_1, \dots, G_k$. Hence, we have proved the following proposition.

PROPOSITION 3.7. *In a coincidence-free NEPS, main angles are products of the corresponding main angles of graphs on which the operation is performed.*

Remark 3.8. In the case of coinciding eigenvalues we have for the main angles the same effect as with the angles as described in Proposition 3.4. Note that the main angles of a NEPS are also independent of the basis, if the change of the basis does not affect the coincidences between eigenvalues.

We point out that we now have a new proof of Theorem 2.6. The proof is based on (3.3), Theorem 2.2 and Proposition 3.7.

Remark 3.8. The eigenvalue μ_i is main if and only if the corresponding main angle β_i is different from zero [**14**]. This fact, together with Proposition 3.7, provides another proof of Proposition 2.9.

4. Star partitions. We relate now the star partitions of some graphs to the star partitions of the graph obtained from them by NEPS. For this purpose let us recall (see [**21**]) that if $X_1 \dot\cup \dots \dot\cup X_m$ is a star partition of some graph, then its eigenvector matrix (its rows being the eigenvectors) can be represented in the form

$$\begin{array}{c c} & \begin{array}{cccc} X_1 & X_2 & \quad & X_m \end{array} \\ \begin{array}{c} \mathcal{E}_1 \\ \mathcal{E}_2 \\ \\ \mathcal{E}_m \end{array} & \begin{pmatrix} I_{k_1} & * & \dots & * \\ * & I_{k_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & I_{k_m} \end{pmatrix} \end{array}, \tag{4.1}$$

if the columns (i.e. vertices of a graph) are appropriately labeled. Here $*$ denotes a block of an appropriate size. In other words, if j is a vertex of G belonging to a star cell X_i, there exists an eigenvector of G (say $x^{(j)}$) corresponding to μ_i and assigned to j, such that all components of $x^{(j)}$, corresponding to vertices from X_i are zero, except one (which is equal to 1) which corresponds to j itself. In the following we shall say that such an eigenvector corresponds to the vertex j. The converse (as already remarked in [**21**] and proved in [**36**]) is also true.

THEOREM 4.1. *Suppose that the eigenvector matrix of a graph G has the form given by* (4.1). *Then $X_1 \dot\cup \cdots \dot\cup X_m$ is a star partition of G.*

Theorem 4.1 will be used to prove a theorem on star partitions of NEPS. Suppose that $G_1, \dots, G_k$ are graphs having

$$\mu_1^{(1)}, \dots, \mu_{m_1}^{(1)}; \; \dots \; ; \; \mu_1^{(k)}, \dots, \mu_{m_k}^{(k)}$$

as distinct eigenvalues and

$$X_1^{(1)} \dot\cup \dots \dot\cup X_{m_1}^{(1)}, \; \dots \; , \; X_1^{(k)} \dot\cup \dots \dot\cup X_{m_k}^{(k)},$$

as star partitions, respectively. If so, we have:

THEOREM 4.2. *If G is a graph obtained from $G_1, \dots, G_k$ by NEPS over any basis $\mathcal{B}$, then the sets $X_{s_1}^{(1)} \times \dots \times X_{s_k}^{(k)}$ $(1 \le s_1 \le m_1, \dots, 1 \le s_k \le m_k)$ comprise a star partition of G whenever the eigenvalues $\mu_{s_1,\dots,s_k}$ (obtained from $\mu_{s_1}^{(1)}, \dots, \mu_{s_k}^{(k)}$ by theorem* 2.2) *do not coincide (i.e., $\mu_{s_1,\dots,s_k} \ne \mu_{t_1,\dots,t_k}$ if $(s_1, \dots, s_k) \ne (t_1, \dots, t_k)$).*

Proof. Suppose $x_{j_1}^{(1)} \dots x_{j_k}^{(k)}$ are the eigenvectors of graphs $G_1, \dots, G_k$ corresponding to vertices $j_1, \dots, j_k$. Then $x_{j_1}^{(1)} \otimes \cdots \otimes x_{j_k}^{(k)}$ is an eigenvector corresponding (in G) to a vertex $(j_1, \dots, j_k)$. If there is no coincidence among the eigenvalues of G, then we get the required number of corresponding eigenvectors of G for any of its eigenvalues. In other words, the eigenvector matrix of G is then in the form (3.3). Thus, by Theorem 3.2, we complete the proof.

Definition 4.3. Star partition in a NEPS of graphs $G_1, \dots, G_k$, defined in the statement of Theorem 4.2, is called the Cartesian product of star partitions of graphs $G_1, \dots, G_k$.

Assuming this definition, Theorem 4.2 can be reformulated in the following way. If G is a graph obtained from graphs $G_1, \dots, G_k$ as a coincidence-free NEPS, then the Cartesian product of any star partitions of graphs $G_1, \dots, G_k$ is a star partition of G. We also see that such a star partition is independent of the basis of the NEPS, as long as the NEPS remains coincidence-free.

We shall now show that Theorem 4.2 is the best possible result in this context. Namely, in general, it is not true that a star partition of some graph (resulting from NEPS) can be obtained by amalgamating the aforementioned cells corresponding to the coinciding eigenvalues.

Example 4.4. We have $K_2 \times K_2 = K_2 \cup K_2$. The only star partition of K_2 consists of two trivial cells, each corresponding to eigenvalues 1 or -1. Now, we indeed have the coincidence of eigenvalues in $K_2 \times K_2$ (since $1 = 1 \times 1 = (-1) \times (-1)$). By amalgamating the cells, as in Theorem 2.2, we get that each copy of K_2 in the resulting graph should be a cell corresponding to eigenvalues 1 or -1. The latter is impossible by Proposition 1.5.

This phenomenon is, at the first glance, not expected in view of the behaviour of angles and main angles under the coincidence of eigenvalues of NEPS (see Section 3). The explanation is as follows. When two eigenvalues coincide, the corresponding eigenspaces are joined in the sense of direct sum. The union of star bases of these eigenspaces is indeed a basis of the resulting eigenspace, but not necessarily a star basis. An actual star basis could correspond to vertices having no relation to the union of cells corresponding to original eigenspaces.

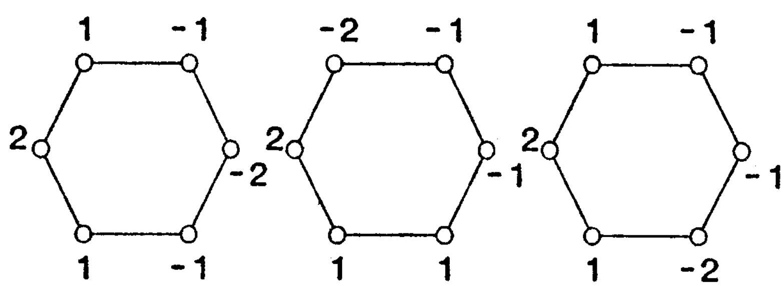
Remark 4.5. Note that the representation of graphs as a NEPS with a fixed basis of some graphs need not be unique. For example, we have $C_4 = C_4 + K_1 = K_2 + K_2$. In view of such non-uniqueness of representation the property of being coincidence-free makes sense only with respect to a fixed representation. For example, C_4 is coincidence-free with respect to the first representation, but not with respect to the second one.

In view of Theorem 3.2, it is reasonable to pose the following question.

Question. Is it true that any star partition in a coincidence-free NEPS is induced by the Cartesian product of some star partitions of the starting graphs?

The negative answer to this question is given by the following example.

Example 4.6. We have $K_3 \times K_2 = C_6$ and a few star partitions of C_6 are displayed in the figure below. Vertices in a star cell are labeled by the corresponding eigenvalue.



Only the first one is induced by (essentially unique) star partitions of K_3 and K_2, while the others are not. The fact that the given partitions are star partitions indeed, can easily be verified by Proposition 1.5.

THEOREM 4.7 *If $\mathcal{B}_i$ is a star basis of the graph G_i $(i = 1, \dots, k)$ then $\mathcal{B}_i \otimes \cdots \otimes \mathcal{B}_k$ is a star basis of NEPS of graphs $G_1, \dots, G_k$ provided there is no coincidence between eigenvalues of NEPS.*

Proof. Let $e_{i1}, \dots, e_{in_i}$ be the standard basis of $\mathbf{R}^{n_i}$ $(i = 1, \dots, k)$. Let $\sum_{s_i=1}^{m_i} \mu_{s_i}^{(i)} P_{s_i}^{(i)}$ be the spectral decomposition of the adjacency matrix of G_i $(i = 1, \dots, k)$. The set

$$\mathcal{B}_i = \{P_{s_i}^{(i)} e_{ij_i} \,|\, j_i \in X_{s_i}^{(i)},\ \ s_i = 1, \dots, m_i\}$$

is the star basis of G_i corresponding to the star partition $X_1^{(i)} \dot\cup \cdots \dot\cup X_{m_i}^{(i)}$.

In the case of non-coincidencing eigenvalues of NEPS (given by Theorem 2.2), the projection operator for the eigenvalue $\mu_{s_1,\dots,s_k}$ is equal to $P_{s_1}^{(1)} \otimes \cdots \otimes P_{s_k}^{(k)}$ (see (3.1)). Since

$$(P_{s_1}^{(1)} \otimes \cdots \otimes P_{s_k}^{(k)})(e_{1j_1} \otimes \cdots \otimes e_{kj_k}) = (P_{s_1}^{(1)} e_{1j_1}) \otimes \cdots \otimes (P_{s_k}^{(k)} e_{kj_k}),$$

where $j_i \in X_{s_i}^{(i)}$, $s_i = 1, \dots, m_i$, $i = 1, \dots, k$, we can easily complete the proof.

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(Received 14 09 1992)

Note added in proof. We have recently come accross the following two relevant references:

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