

GENERALIZED MIRON'S d -CONNECTION IN THE RECURRENT K -HAMILTON SPACES

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Abstract. For the generalized Miron's d -connection in recurrent K -Hamilton spaces, the torsion tensor and the connection coefficients are determined. For special cases the already known results are obtained.

1. Introduction. This paper is a generalization of [8], based on works of Miron, Anastasiei, Janus, Kirhovits and others, in the Hamilton geometry. Here the generalization is going in different directions:

- a) The transformation of the coordinate system is given by (2.1), where (2.1b) is more general than in former investigations in this field.
- b) In the metric tensor the blocks over and under the diagonal are not necessarily equal to zero.
- c) The connection coefficients are introduced in such a way that $\nabla_X: T(E^*) \rightarrow T(E^*)$ by which we can obtain eight kinds of connection coefficients.
- d) A consequence of **c**) is that the torsion tensor has also eight of components.
- e) The field $\lambda(x, p)$, the vector field of recurrency is introduced.

The main result is that the connection coefficients for such a general case are obtained explicitly. For some special cases the already known results are obtained. Theorems 7.6 and 7.7 give the relations between the metric tensor and the nonlinear connection of such K -Hamilton spaces, which allow torsion-free Miron's d -connection.

2. Adapted bases in $T(E^*)$ and $T^*(E^*)$. Let E^* be an $(n + mK)$ -dimensional differentiable manifold. If u is one point of E^* , then, in some local chart, u has the coordinates

$$u = ((x^i), (p_a^1), (p_a^2), \dots, (p_a^K)) = ((x^i), (p_a^\alpha)) = (x, p).$$

where $(x^i) = (x^1, x^2, \dots, x^n) = (x)$, $(p_a^\alpha) = (p_1^\alpha, \dots, p_m^\alpha) = (p^\alpha)$, and $a, b, c, d, e, f = \overline{1, m}$; $i, j, h, k, l = \overline{1, n}$; $\alpha, \beta, \gamma, \delta = \overline{1, K}$.

We shall consider the following coordinate transformation. If $((x^{i'}), (p_{a'}^\alpha)) = (x', p')$ are the coordinates of the same point u in the new coordinate system, then,

$$(2.1) \quad (a) \quad x^{i'} = x^{i'}(x^1, \dots, x^n), \quad \text{rank} \left[\partial x^{i'} / \partial x^i \right] = n$$

$$(b) \quad p_{a'}^\alpha = M_{a'}^{(\alpha)}(x^1, \dots, x^n) p_a^\alpha, \quad \text{rank} [\partial p_{a'}^\alpha / \partial p_a^\alpha] = m.$$

The Einstein summation convention will be used for all three kinds of indices, except in the case when the index appears in brackets. If (2.1) is valid, then an inverse transformation exists, i.e.

$$(2.2) \quad (a) \quad x^i = (x^{1'}, \dots, x^{n'}) \quad (b) \quad p_a^\alpha = M_a^{(\alpha)}(x^{1'}, \dots, x^{n'}) p_{a'}^\alpha.$$

The natural basis $\overline{B} = \{(\partial_i), (\partial_1^a), \dots, (\partial_K^a)\}$ of $T(E^*)$ is formed by n vectors of the type $\partial_i = \partial / \partial x^i$ and $m \cdot K$ vectors of the type $\partial_\alpha^a = \partial / \partial p_a^\alpha$. Any vector field $X \in T(E^*)$ may be represented in the form

$$(2.3) \quad X = \overline{X}^i \partial_i + \overline{X}_a^\alpha \partial_\alpha^a.$$

With respect to the coordinate transformations (2.1) and (2.2), the basic vectors of \overline{B} obey the following law of transformation:

$$(2.4) \quad \begin{bmatrix} \partial_i \\ \partial_1^a \\ \vdots \\ \partial_K^a \end{bmatrix} = \begin{bmatrix} \partial x^{i'} / \partial x^i & \begin{pmatrix} (1) \\ \partial_i M_{a'}^a \end{pmatrix} p_a^1 & \dots & \begin{pmatrix} (K) \\ \partial_i M_{a'}^a \end{pmatrix} p_a^K \\ 0 & M_{a'}^{(1)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & M_{a'}^{(K)} \end{bmatrix} \begin{bmatrix} \partial_{i'} \\ \partial_1^{a'} \\ \vdots \\ \partial_K^{a'} \end{bmatrix}$$

$$(2.5) \quad \begin{bmatrix} \partial_{i'} \\ \partial_1^{a'} \\ \vdots \\ \partial_K^{a'} \end{bmatrix} = \begin{bmatrix} \partial x^j / \partial x^{i'} & \begin{pmatrix} (1) \\ \partial_{i'} M_b^{b'} \end{pmatrix} p_b^1 & \dots & \begin{pmatrix} (K) \\ \partial_{i'} M_b^{b'} \end{pmatrix} p_b^K \\ 0 & M_b^{(1)} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & M_b^{(K)} \end{bmatrix} \begin{bmatrix} \partial_j \\ \partial_1^b \\ \vdots \\ \partial_K^b \end{bmatrix}$$

Substituting (2.5) into (2.4), we obtain,

$$(2.6) \quad (a) \quad \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{i'}} = \delta_i^j \quad (b) \quad M_{a'}^{(\alpha)} M_b^{(\alpha)} = \delta_b^a$$

$$(c) \quad \begin{pmatrix} (\alpha) \\ \partial_{i'} M_b^{b'} \end{pmatrix} p_b^\alpha \frac{\partial x^{i'}}{\partial x^i} + \begin{pmatrix} (\alpha) \\ \partial_i M_{a'}^a \end{pmatrix} p_a^\alpha M_b^{(\alpha)} = 0,$$

where (2.6c) is the consequence of (2.4) and (2.6b).

From (2.4) and (2.5) it is obvious that ∂_i and $\partial_{i'}$ do not transform as tensors, so we introduce a new, the so-called adapted basis $B = \{(\delta_i), (\partial_1^\alpha), \dots, (\partial_K^\alpha)\}$ of $T(E^*)$, where by definition

$$(2.7) \quad \delta_i = \partial_i - N_{a_i}^\alpha(x, p)\partial_\alpha^a,$$

and $N_{a_i}^\alpha(x, p)$ are the coefficients of the nonlinear connection. These are the arbitrary functions, which under the coordinate transformations (2.1) and (2.2) transform in the following manner:

$$(2.8) \quad \begin{aligned} (a) \quad N_{a' i'}^\alpha(x', p') &= M_{a'}^\alpha \frac{\partial x^i}{\partial x^{i'}} N_{a_i}^\alpha(x, p) - p_a^\alpha \frac{\partial M_{a'}^\alpha}{\partial x^i} \frac{\partial x^i}{\partial x^{i'}} \\ (b) \quad N_{b' j}^\alpha(x, p) &= M_{b'}^\alpha \frac{\partial x^{j'}}{\partial x^j} N_{b' j'}^\alpha(x', p') + p_a^\alpha M_b^\alpha \frac{\partial M_{a'}^\alpha}{\partial x^j}. \end{aligned}$$

Any vector field $X \in T(E^*)$ in the adapted basis B is given by

$$(2.9) \quad X = X^i \delta_i + X_a^\alpha \partial_\alpha^a.$$

The coordinates of the vector X given by (2.9) and the elements of basis B transform as tensors in the following manner:

$$(2.10) \quad \begin{aligned} \delta_i &= \frac{\partial x^{i'}}{\partial x^i} \delta_{i'} & \partial_\alpha^a &= M_{a'}^\alpha(x) \partial_\alpha^{a'} \\ X^i &= \frac{\partial x^i}{\partial x^{i'}} X^{i'} & X_a^\alpha &= M_a^{\alpha'}(x') X_{a'}^{\alpha'}. \end{aligned}$$

From (2.3) and (2.9) we obtain the relation between coordinates of the field X , in the bases \overline{B} and B . They are connected by the relations $X^i = \overline{X}^i$, $X_a^\alpha = \overline{X}_a^\alpha + N_{a_i}^\alpha \overline{X}^i$. The subspace of $T(E^*)$ spanned by $\{\delta_i\}$, shall be denoted by $T_H(E^*)$ (the horizontal part) and the subspace spanned by $\{\partial_\alpha^a\}$, by ${}_{(\alpha)}T_V(E^*)$ (the vertical α -part). So, we have $T(E^*) = T_H(E^*) \oplus T_V(E^*)$, where,

$$T_V(E^*) = \sum_{\alpha=1}^K {}_{(\alpha)}T_V(E^*), \quad \dim T_H(E^*) = n, \quad \dim {}_{(\alpha)}T_V(E^*) = m.$$

Here, $X^i \delta_i$ is the horizontal and $X_a^\alpha \partial_\alpha^a$ the vertical part of the field X . Now (2.9) may be written in the form

$$X = X_H + X_V, \quad X_H = X^i \delta_i \quad X_V = X_a^\alpha \partial_\alpha^a.$$

Let us consider the dual tangent space of E^* , the space $T^*(E^*)$. The natural basis in $T^*(E^*)$ is

$$\overline{B}^* = \{dx^1, \dots, dx^n, dp_1^1, \dots, dp_m^1, \dots, dp_1^K, \dots, dp_m^K\} = \{dx^i, dp_a^1, \dots, dp_a^K\}.$$

From (2.1) we obtain

$$(2.11) \quad (a) \quad dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i \quad (b) \quad dp_{a'}^\alpha = \frac{\partial M_{a'}^\alpha(x)}{\partial x^i} p_a^\alpha dx^i + M_{a'}^\alpha(x) dp_a^\alpha.$$

From (2.11b) it is obvious that dp_a^α do not transform as tensors, so we introduce a new basis $B^* = \{(dx^1), (\delta p_a^1), \dots, (\delta p_a^K)\}$, where,

$$(2.12) \quad \delta p_a^\alpha = dp_a^\alpha + N_{a_i}^\alpha(x, p) dx^i.$$

Through the coordinate transformation (2.1) the bases \overline{B}^* and B^* are related by (2.11a) and

$$(2.13) \quad (a) \quad \delta p_a^\alpha = M_{a'}^\alpha(x') \delta p_{a'}^\alpha \quad (b) \quad \delta p_{a'}^\alpha = M_{a'}^\alpha(x) \delta p_a^\alpha.$$

The proof of (2.13) is obtained by using (2.12) and (2.8). Any field $w \in T^*(E^*)$ can be written in the bases \overline{B}^* and B^* in the following manner:

$$(2.14) \quad w = \overline{w}_i dx^i + \overline{w}_\alpha^\alpha dp_a^\alpha = w_i dx^i + w_\alpha^\alpha \delta p_a^\alpha$$

where $w_i = \overline{w}_i - N_{a_i}^\alpha \overline{w}_\alpha^\alpha$, $\overline{w}_\alpha^\alpha = w_\alpha^\alpha$.

The subspace of $T^*(E^*)$ spanned by $\{(dx^i)\}$ shall be denoted by $T_H^*(E^*)$ and the subspace spanned by $\{(\delta p_a^\alpha)\}$ by ${}_{(\alpha)}T_V^*(E^*)$. So, we have

$$T^*(E^*) = T_H^*(E^*) \oplus T_V^*(E^*), \quad \text{where} \quad T_V^*(E^*) = \sum_{\alpha=1}^K \oplus_{(\alpha)} T_V^*(E^*).$$

Now, (2.14) may be written in the form

$$w = w_H + w_V, \quad w_H = w_i dx^i, \quad w_V = w_\alpha^\alpha \delta p_a^\alpha.$$

If $\{(dx^i), (\delta p_a^1), \dots, (\delta p_a^K)\}$ and $\{(dx^{i'}), (\delta p_{a'}^1), \dots, (\delta p_{a'}^K)\}$ are two bases in $T^*(E^*)$, related by (2.11a) and (2.13), then any $w \in T^*(E^*)$ satisfies the relation

$$(2.15) \quad w = w_i dx^i + w_\alpha^\alpha \delta p_a^\alpha = w_{i'} dx^{i'} + w_{\alpha'}^{\alpha'} \delta p_{a'}^{\alpha'}.$$

Substituting $dx^{i'}$ from (2.11a) and $\delta p_{a'}^\alpha$ from (2.13b) into (2.15) and comparing the coefficients of the basis vectors, we obtain

$$(2.16) \quad w_i = w_{i'} \partial x^{i'} / \partial x^i, \quad w_\alpha^\alpha = M_{a'}^\alpha w_{\alpha'}^{\alpha'}.$$

By a straightforward calculation, we can prove the following

PROPOSITION 2.1. *The adapted bases $\{(\delta_i), (\partial_1^a), \dots, (\partial_K^a)\}$ and $\{(dx^i), (\delta p_a^1), \dots, (\delta p_a^K)\}$ are dual to each other, i.e.*

$$(2.17) \quad \begin{aligned} \langle \delta_i, dx^j \rangle &= \delta_i^j, & \langle \delta_i, \delta p_a^\alpha \rangle &= 0, \\ \langle \partial_\alpha^a, dx^j \rangle &= 0, & \langle \partial_\alpha^a, \delta p_b^\beta \rangle &= \delta_b^a \delta_\alpha^\beta. \end{aligned}$$

3. Tensor Fields on E^* . a) A horizontal tensor field t_H has the local representation:

$$t_H = t^{i_1 \dots i_p j_1 \dots j_q}(x, p) \delta_{i_1} \otimes \dots \otimes \delta_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q};$$

it is defined on

$$\underbrace{T_H(E^*) \otimes \dots \otimes T_H(E^*)}_{p \text{ times}} \otimes \underbrace{T_H^*(E) \otimes \dots \otimes T_H^*(E^*)}_{q \text{ times}}$$

By the coordinate transformations given by (2.1) and (2.2) the coordinates of the field t_H have the following transformation law

$$t^{i'_1 \dots i'_p j'_1 \dots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}} t^{i_1 \dots i_p j_1 \dots j_q}.$$

b) The α -vertical tensor field $(\alpha)t_V$ has the local representation

$$(\alpha)t_V = (\alpha)t_{b_1 \dots b_s}^{\alpha \dots \alpha a_1 \dots a_r} \partial_{\alpha}^{b_1} \otimes \dots \otimes \partial_{\alpha}^{b_s} \otimes \delta p_{a_1}^{\alpha} \otimes \dots \otimes \delta p_{a_r}^{\alpha}$$

(not summing over α); it is defined on

$$(\alpha) \underbrace{T_V(E^*) \otimes \dots \otimes T_V(E^*)}_{s \text{ times}} \otimes (\alpha) \underbrace{T_V^*(E^*) \otimes \dots \otimes T_V^*(E^*)}_{r \text{ times}}$$

By changing of coordinates of kind (2.1) and (2.2) the coordinates of the field $(\alpha)t_V$, given above, have the following transformation law

$$(\alpha)t_{\alpha \dots \alpha b'_1 \dots b'_s}^{a'_1 \dots a'_r \alpha \dots \alpha} = (\alpha)t_{\alpha \dots \alpha b_1 \dots b_s}^{a_1 \dots a_r \alpha \dots \alpha} M_{a_1}^{(\alpha) a'_1} \dots M_{a_r}^{(\alpha) a'_r} M_{b_1}^{(\alpha) b'_1} \dots M_{b_s}^{(\alpha) b'_s}.$$

c) A vertical tensor field t_V on $T_V(E^*) \otimes T_V^*(E^*)$ has the form

$$t_V = t_{a_1 \beta}^{\alpha b_1} \partial_{\alpha}^{a_1} \otimes \delta p_{b_1}^{\beta} \quad (\text{summing over } \alpha \text{ and } \beta).$$

The coordinate transformation of the tensor t_V is given by

$$t_{a'_1 \beta}^{\alpha b'_1} = t_{a_1 \beta}^{\alpha b_1} M_{a_1}^{(\alpha) a'_1} M_{b_1}^{(\beta) b'_1}.$$

d) A tensor field t on

$$\underbrace{T_H(E^*) \otimes \dots \otimes T_H(E^*)}_{p \text{ times}} \otimes \underbrace{T_H^*(E^*) \otimes \dots \otimes T_H^*(E^*)}_{q \text{ times}} \otimes \underbrace{T_V(E^*) \otimes \dots \otimes T_V(E^*)}_{s \text{ times}} \times \underbrace{T_V^*(E^*) \otimes \dots \otimes T_V^*(E^*)}_{r \text{ times}},$$

is given by

$$t = t^{i_1 \dots i_p j_1 \dots j_q \beta_1 \dots \beta_s a_1 \dots a_r}(x, p) \delta_{i_1} \otimes \dots \otimes \delta_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q} \otimes \partial_{\beta_1}^{b_1} \otimes \dots \otimes \partial_{\beta_s}^{b_s} \otimes \delta p_{a_1}^{\alpha_1} \otimes \dots \otimes \delta p_{a_r}^{\alpha_r}.$$

The summation is performed over all indices. The coordinate transformation of the above tensor is given by

$$t^{i_1 \dots i_p}_{j_1 \dots j_q}{}^{\beta_1 \dots \beta_s a_1 \dots a_r}{}_{b_1 \dots b_s \alpha_1 \dots \alpha_r} = t^{i_1 \dots i_p}_{j_1 \dots j_q}{}^{\beta_1 \dots \beta_s a_1 \dots a_r}{}_{b_1 \dots b_s \alpha_1 \dots \alpha_r} \frac{\partial x^{i_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j_1}} \dots \frac{\partial x^{j_q}}{\partial x^{j_q}} M_{b_1}^{(\beta_1)}{}_{b_1} \dots M_{b_s}^{(\beta_s)}{}_{b_s} M_{a_1}^{(\alpha_1)}{}_{a_1} \dots M_{a_r}^{(\alpha_r)}{}_{a_r}.$$

The order of spaces $T_H(E^*)$, $T_H^*(E^*)$, ${}_{(\alpha)}T_V^*(E^*)$ and $T_V^*(E^*)$ can be taken arbitrarily. It has the influence on the order of the indices of the tensor t , which is defined on their tensor product.

4. The Metric Tensor. In the space $T^*(E^*) \otimes T^*(E^*)$, the metric tensor G , with respect to the basis $= \{(dx^i), (\delta p_a^1), \dots, (\delta p_a^K)\}$, has the form

$$(4.1) \quad G = [(dx^i), (\delta p_a^1), \dots, (\delta p_a^K)] \begin{bmatrix} [g_{ij}] & [g_{i1}^b] & \dots & [g_{iK}^b] \\ [g_{1j}^a] & [g_{11}^{ab}] & \dots & [g_{1K}^{ab}] \\ \vdots & \vdots & \ddots & \vdots \\ [g_{Kj}^a] & [g_{K1}^{ab}] & \dots & [g_{KK}^{ab}] \end{bmatrix} \otimes \begin{bmatrix} dx^j \\ \delta p_b^1 \\ \vdots \\ \delta p_b^K \end{bmatrix} = g_{ij} dx^i \otimes dx^j + g_{\alpha j}^a \delta p_a^\alpha \otimes dx^j + g_{i\beta}^b dx^i \otimes \delta p_b^\beta + g_{\alpha\beta}^{ab} \delta p_a^\alpha \otimes \delta p_b^\beta.$$

The matrices $[g_{ij}]$, $[g_{i\beta}^b]$, $[g_{\alpha j}^a]$ and $[g_{\alpha\beta}^{ab}]$ have the formats $n \times n$, $n \times m$, $m \times n$ and $m \times m$. As G is a tensor, its coordinates in the new coordinate system (x', p') transform in the following manner:

$$(4.2) \quad g_{i'j'} = g_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}, \quad g_{\alpha'j'}^a = g_{\alpha j}^a M_a^{(\alpha)}{}_{a'} \frac{\partial x^j}{\partial x^{j'}}, \\ g_{i'\beta}^b = g_{i\beta}^b \frac{\partial x^i}{\partial x^{i'}} M_b^{(\beta)}{}_{b'}, \quad g_{\alpha\beta}^{a'b'} = g_{\alpha\beta}^{ab} M_a^{(\alpha)}{}_{a'} M_b^{(\beta)}{}_{b'}$$

We shall suppose that G is a symmetric, positive definite tensor field of rank $n + mK$. From the symmetry, it follows that $g_{ij} = g_{ji}$, $g_{i\beta}^b = g_{\beta i}^b$, $g_{\alpha\beta}^{ab} = g_{\beta\alpha}^{ba}$. The "covariant" coordinates of the field $X = X^i \delta_i + X_a^\alpha \partial_\alpha^a$ are given by

$$X_i = g_{ij} X^j + g_{i\alpha}^a X_a^\alpha, \quad X_\alpha^a = g_{\alpha i}^a X^i + g_{\alpha\beta}^{ab} X_b^\beta.$$

The inverse matrix of G (appearing in (4.1)) is given by

$$\begin{bmatrix} [g^{jk}] & [g_c^{j1}] & \dots & [g_c^{jK}] \\ [g_b^{1k}] & [g_{bc}^{11}] & \dots & [g_{bc}^{1K}] \\ \vdots & \vdots & \ddots & \vdots \\ [g_b^{Kk}] & [g_{bc}^{K1}] & \dots & [g_{bc}^{KK}] \end{bmatrix}$$

The matrices $[g^{jk}]$, $[g^{j\gamma}_c]$, $[g^{\gamma k}_c]$ and $[g^{\beta\gamma}_{bc}]$ have formats $n \times n$, $n \times m$, $m \times n$ and $m \times m$. Now we have:

$$\begin{aligned} g_{ij}g^{jk} + g_{i\beta}^b g^{\beta k} &= \delta_i^k, & g_{\alpha j}^a g^{j\gamma}_c + g_{\alpha\beta}^{ab} g^{\beta\gamma}_{bc} &= \delta_c^a \delta_\alpha^\gamma, \\ g_{ij}g^{j\gamma}_c + g_{i\beta}^b g^{\beta\gamma}_{bc} &= 0, & g_{\alpha j}^a g^{jk} + g_{\alpha\beta}^{ab} g^{\beta k} &= 0. \end{aligned}$$

The contravariant coordinates $w = w_i dx^i + w_\alpha^\alpha \delta p_\alpha^\alpha$ are given by

$$w^i = g^{ij} w_j + g^{i\alpha} w_\alpha^a, \quad w_\alpha^\alpha = g^{j\alpha} w_j + g^{\alpha\beta} w_\beta^b.$$

Using (2.10), (2.16) and (4.2) the validity of the following transformation laws can be shown:

$$X_{i'} = X_i \frac{\partial x^i}{\partial x^{i'}}, \quad X_\alpha^{a'} = X_\alpha^a M_a^{a'}, \quad w^{i'} = w^i \frac{\partial x^i}{\partial x^{i'}}, \quad w_{\alpha'}^\alpha = w_\alpha^\alpha M_a^{a'}.$$

Definition 3.1. The differentiable manifold E^* (in which the coordinate transformations of type (2.1) are allowed) supplied with arbitrary nonlinear connection N (which satisfies (2.8)) and the metric tensor G (given by (4.1)) is called a K -Hamilton space and is denoted by (E^*, N, G) .

It is a generalization of the K -Hamilton space defined in [9], [10] etc., because here the metric tensor G is not necessarily obtained from the K -Hamiltonian $H(x, p)$. If the K -Hamilton function $H(x, p)$ is given in the space (E^*) , then the metric tensor G can be defined in the following way:

$$\begin{aligned} g_{ij}(x, p) &= g_{ij}(x), & g_{i\beta}^b &= 0, & g_{\alpha j}^a &= 0, \\ g_{\alpha\beta}^{ab} &= \frac{1}{2} \partial_\alpha^a \partial_\beta^b H^2(x, p), & \text{for every } \alpha, \beta &= \overline{1, K}, \end{aligned}$$

where $g_{ij}(x)$ is some metric tensor defined on M and M is the π^* projection of E^* :

$$\pi^*(E^*) = M, \quad \pi^*((x^i), (p_a^1), \dots, (p_a^K)) = (x^i).$$

We can not define $g_{i\beta}^b(x, p) = \frac{1}{2} \delta_i \partial_\beta^b H^2(x, p)$, $g_{ij}(x, p) = \frac{1}{2} \delta_i \delta_j H^2(x, p)$, because the above quantities do not transform as tensors. Using the metric tensor G determined by (4.1), we define the scalar product (X, Y) of fields $X, Y \in T(E^*)$ by

$$(4.3) \quad (X, Y) = g_{ij} X^i Y^j + g_{i\beta}^b X^i Y_b^\beta + g_{\alpha j}^a X_\alpha^\alpha Y^j + g_{\alpha\beta}^{ab} X_\alpha^\alpha Y_b^\beta.$$

The length of X , $|X|$ is defined by $|X|^2 = (X, X)$ and $\cos \theta$, where θ is the angle between X and Y by

$$(4.4) \quad \cos \theta = (X, Y) / (|X| \cdot |Y|).$$

When $\cos \theta = 0$, we say that X and Y are mutually orthogonal. For the horizontal field X_H we have $X_H = X^i \partial_i$, $|X_H|^2 = g_{ij} X^i X^j$ and for the vertical

vector X_V we have $X_V = X_a^\alpha \partial_\alpha^a$, $|X_V|^2 = g_{\alpha\beta}^{ab} X_a^\alpha X_b^\beta$. For the field ${}_{(\alpha)}X_V \in {}_{(\alpha)}T_V(E^*)$ we have

$${}_{(\alpha)}X_V = X_a^\alpha \partial_\alpha^a \quad |{}_{(\alpha)}X_V|^2 = g_{\alpha\alpha}^{ab} X_a^\alpha X_b^\alpha \quad (\text{not summing over } \alpha).$$

THEOREM 4.1. *The subspaces $T_H(E^*)$, ${}_{(1)}T_V(E^*)$, \dots , ${}_{(K)}T_V(E^*)$ are mutually orthogonal with respect to the metric tensor G , if and only if $[g_{ij}^b] = 0$, $[g_{\alpha j}^a] = 0$, $[g_{\alpha\beta}^{ab}] = 0$ for every $\alpha, \beta = \overline{1, K}$, $\alpha \neq \beta$.*

The proof follows from (4.3) and (4.4).

5. Generalized Miron's d -connection in $T(E^*)$. The distinguished connection ∇ , or the d -connection, in the K -Hamilton space in [15], [9], [16] and others is defined as a function $\nabla : (X, Y) \rightarrow \nabla_X Y$ $X, Y, \nabla_X Y \in T(E^*)$ for which, beside the usual conditions for the linear connection, the following restrictions are valid:

$$\begin{aligned} \nabla_X Y_H \in T_H(E^*), \quad \nabla_X Y_V \in T_V(E^*) \text{ for all } X \in T(E^*), \\ \text{all } Y_H \in T_H(E^*), \text{ and all } Y_V \in T_V(E^*). \end{aligned}$$

For the generalized linear d -connection in $T(E^*)$ in the K -Hamilton space the above restrictions need not be satisfied.

Definition 5.1. The generalized linear Miron's d -connection in $T(E^*)$ is defined by

$$\begin{aligned} (5.1) \quad \nabla_{\delta_i} \delta_j &= F_j^k \delta_k + F_{jc}^\gamma \partial_\gamma^c, \\ \nabla_{\delta_i} \partial_\alpha^a &= F_\alpha^k \delta_k + F_{\alpha c}^\gamma \partial_\gamma^c, \\ \nabla_{\partial_\alpha^a} \delta_j &= C_j^k \delta_k + C_{jc\alpha}^{\gamma a} \partial_\gamma^c, \\ \nabla_{\partial_\alpha^a} \partial_\beta^b &= C_{\beta\alpha}^{bka} \delta_k + C_{\beta c\alpha}^{b\gamma a} \partial_\gamma^c. \end{aligned}$$

PROPOSITION 5.1. *If $X, Y \in T(E^*)$, where X is given by (2.9), and $Y = Y^j \delta_j + Y_b^\beta \partial_\beta^b$, then*

$$(5.2) \quad \nabla_X Y = \left(Y_{|i}^k X^i + Y^k |_\alpha X^\alpha \right) \delta_k + \left(Y_{c|i}^\gamma X^i + Y_c^\gamma |_\alpha X^\alpha \right) \partial_\gamma^c,$$

where,

$$\begin{aligned} (5.3) \quad Y_{|i}^k &= \delta_i Y^k + F_{j|i}^k Y^j + F_{\beta i}^{bk} Y_b^\beta, \\ Y^k |_\alpha &= \partial_\alpha^a Y^k + C_{j\alpha}^{ka} Y^j + C_{\beta\alpha}^{bka} Y_b^\beta, \\ Y_{c|i}^\gamma &= \delta_i Y_c^\gamma + F_{jc|i}^\gamma Y^j + F_{\beta c i}^{b\gamma} Y_b^\beta, \\ Y_c^\gamma |_\alpha &= \partial_\alpha^a Y_c^\gamma + C_{jc\alpha}^{\gamma a} Y^j + C_{\beta c\alpha}^{b\gamma a} Y_b^\beta. \end{aligned}$$

Proof. (5.2) and (5.3) are proved by using the linearity of the connection ∇ and (5.1).

PROPOSITION 5.2. *If $((x^i), (p_a^\alpha))$ and $((x'^i), (p_{a'}^\alpha))$ are two coordinate systems connected by (2.1) and (2.2), then*

$$(5.4) \quad \nabla_{X'} Y' = \nabla_X Y$$

iff $Y_{|i}^k, Y^k|_\alpha, Y_c^\gamma|_i$ and $Y_c^\gamma|_\alpha$ transform as tensors, i.e. if

$$\begin{aligned} Y_{|i}^{k'} &= Y^k|_i (\partial_{i'} x^i) (\partial_k x^{k'}), & Y^{k'}|_{a'} &= Y^k|_a (\partial_k x^{k'}) M_{a'}^{(\alpha)}, \\ Y_{c'}^\gamma|_{i'} &= Y_c^\gamma|_i M_{c'}^{(\gamma)} (\partial_{i'} x^i), & Y_{c'}^\gamma|_{a'} &= Y_c^\gamma|_a M_{c'}^{(\gamma)} M_a^{(\alpha)}, \end{aligned}$$

or, equivalently, iff the connection coefficients have the following transformation law

$$\begin{aligned} (5.5) \quad (a) \quad F_{j' i}^{k'} &= F_{j' c' i'}^{k'} (\partial_j x^{j'}) (\partial_{k'} x^k) (\partial_i x^{i'}) + (\partial_i \partial_j x^{k'}) (\partial_{k'} x^k), \\ (b) \quad F_{j' c' i}^\gamma &= F_{j' c' i'}^\gamma M_{c'}^{(\gamma)} (\partial_i x^{i'}), \\ (c) \quad F_{\beta' i}^{b' k'} &= F_{\beta' c' i'}^{b' k'} M_{b'}^{(\beta)} (\partial_{k'} x^k) (\partial_i x^{i'}), \\ (d) \quad F_{\beta' c' i}^{b' \gamma} Y_b^\beta &= F_{\beta' c' i'}^{b' \gamma} M_{b'}^{(\beta)} M_{c'}^{(\gamma)} (\partial_i x^{i'}) Y_b^\beta + (\partial_i M_{c'}^{(\gamma)}) M_{c'}^{(\gamma)} Y_d^\gamma, \\ (e) \quad C_{j' \alpha}^{k' a} &= C_{j' c' \alpha'}^{k' a'} (\partial_j x^{j'}) (\partial_{k'} x^k) M_{a'}^{(\alpha)}, \\ (f) \quad C_{j' c' \alpha}^{\gamma a} &= C_{j' c' \alpha'}^{\gamma a'} M_{c'}^{(\gamma)} M_{a'}^{(\alpha)}, \\ (g) \quad C_{\beta' \alpha}^{b' k' a} &= C_{\beta' c' \alpha'}^{b' k' a'} M_{b'}^{(\beta)} (\partial_{k'} x^k) M_{a'}^{(\alpha)}, \\ (h) \quad C_{\beta' c' \alpha}^{b' \gamma a} &= C_{\beta' c' \alpha'}^{b' \gamma a'} M_{b'}^{(\beta)} M_{c'}^{(\gamma)} M_{a'}^{(\alpha)}. \end{aligned}$$

As Y_b^β and Y_d^γ are the coordinates of the arbitrary vector Y , from (5.5d) it follows that:

$$(5.5.d') \quad F_{(\alpha) d' i'}^a = F_{(\alpha) d i}^a M_a^{(\alpha)} M_{d'}^{(\alpha)} (\partial_{i'} x^i) + \left(\partial_{i'} M_{d'}^{(\alpha)} \right) M_{d'}^{(\alpha)}$$

$$(5.5.d'') \quad F_{\alpha d' i'}^{a' \gamma} = F_{\alpha d i}^{a' \gamma} M_a^{(\alpha)} M_{d'}^{(\gamma)} (\partial_i x^i), \quad \alpha \neq \gamma.$$

Proof. The proof is obtained by a direct calculation, using (5.2)–(5.4), (2.7) and (2.10).

The formulae (5.5) may be obtained by using the transformation law of the basic vectors and the linearity of the connection ∇ .

If one nonlinear connection N_{α}^{α} is given, we have an adapted basis and using this basis, we define the linear connection ∇ . Another nonlinear connection $\bar{N}_{\alpha}^{\alpha}$ may be obtained from (5.5d). If we take $Y^{\beta} = p_p^{\beta}$, and introduce the notation $F_{\beta c}^{\gamma} p_b^{\beta} = \bar{N}_{\alpha}^{\alpha}$, then (5.5d) becomes

$$\bar{N}_{\alpha}^{\alpha} = \bar{N}_{\alpha' i}^{\alpha} M_{\alpha}^{\alpha'} (\partial_i x^{i'}) + \left(\partial_i M_{\alpha'}^{\alpha} \right) M_{\alpha}^{\alpha'} p_d^{\alpha},$$

which, compared to (2.8), shows, that $\bar{N}_{\alpha}^{\alpha}$ transform as a nonlinear connection.

The torsion tensor $T(X, Y)$ is, as usual, given by

$$(5.6) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

THEOREM 5.1. *The torsion tensor $T(X, Y)$ for the connection ∇ has the form*

$$T(X, Y) = T_{HHH} + T_{HHV} + T_{HVV} + T_{HVV} + T_{VHH} + T_{VHV} + T_{VVH} + T_{VVV},$$

where, for instance, $T_{VVH} = [T(X_V, Y_V)]_H$, $T_{VHH} = [T(X_V, Y_H)]_H$. The components of the torsion tensor are

$$(5.7) \quad \begin{aligned} (a) \quad T_{HHH} &= (F_j^k{}_i - F_i^k{}_j) X^i Y^j \delta_k = T_j^k{}_i X^i Y^j \delta_k \\ (b) \quad T_{HHV} &= (F_{j c}^{\gamma}{}_i - F_{i c}^{\gamma}{}_j + \delta_i N_c^{\gamma}{}_j - \delta_j N_c^{\gamma}{}_i) X^i Y^j \partial_{\gamma}^c = T_{j c}^{\gamma}{}_i X^i Y^j \partial_{\gamma}^c \\ (c) \quad T_{HVV} &= (F_{\beta}^b{}_i{}^k - C_i^k{}^{\beta}) X^i Y_b^{\beta} \delta_k = T_{\beta}^b{}_i X^i Y_b^{\beta} \delta_k \\ (d) \quad T_{HVV} &= (F_{\beta c}^b{}_i{}^{\gamma} - C_{i c}^{\gamma}{}^b - \partial_{\beta}^b N_c^{\gamma}{}_i) X^i Y_b^{\beta} \partial_{\gamma}^c = T_{\beta c}^b{}_i X^i Y_b^{\beta} \partial_{\gamma}^c \\ (e) \quad T_{VHH} &= (C_j^k{}^{\alpha}{}_i - F_{\alpha}^k{}_j) X_{\alpha}^{\alpha} Y^j \delta_k = T_j^k{}^{\alpha}{}_i X_{\alpha}^{\alpha} Y^j \delta_k \\ (f) \quad T_{VHV} &= (C_{j c}^{\gamma}{}^{\alpha} - F_{\alpha c}^{\gamma}{}_j + \partial_{\alpha}^{\alpha} N_c^{\gamma}{}_j) X_{\alpha}^{\alpha} Y^j \partial_{\gamma}^c = T_{j c}^{\gamma}{}^{\alpha} X_{\alpha}^{\alpha} Y^j \partial_{\gamma}^c \\ (g) \quad T_{VVH} &= (C_{\beta}^b{}_i{}^k{}^{\alpha} - C_{\alpha}^k{}^{\beta}) X_{\alpha}^{\alpha} Y_b^{\beta} \delta_k = T_{\beta}^b{}_i{}^k{}^{\alpha} X_{\alpha}^{\alpha} Y_b^{\beta} \delta_k \\ (h) \quad T_{VVV} &= (C_{\beta c}^b{}_i{}^{\alpha}{}^{\gamma} - C_{\alpha c}^{\gamma}{}^b) X_{\alpha}^{\alpha} Y_b^{\beta} \partial_{\gamma}^c = T_{\beta c}^b{}_i{}^{\alpha}{}^{\gamma} X_{\alpha}^{\alpha} Y_b^{\beta} \partial_{\gamma}^c. \end{aligned}$$

Proof. The proof is obtained by a direct calculation by using (5.2), (5.3) and (5.6).

Definition 5.2. The K -Hamiltonian space (E^*, N, G) , supplied with the linear connection ∇ defined by (5.1) and an arbitrary torsion tensor T , given by (5.7) is denoted by (E^*, N, G, ∇, T) .

6. Generalized Miron's d -connection in $T^*(E^*)$ Using the duality of the bases B and B^* of $T(E^*)$, and $T^*(E^*)$, respectively expressed by (2.17) one can obtain

THEOREM 6.1. *The connection on $T^*(E^*)$ acts in the following way:*

$$(6.1) \quad \begin{aligned} \nabla_{\delta_i} dx^j &= -F_k^j|_i dx^k - F_\gamma^c|_i \delta p_c^\gamma, & \nabla_{\delta_i} \delta p_b^\beta &= -F_{k_b}^\beta|_i dx^k - F_{\gamma_b}^c|_i \delta p_c^\gamma, \\ \nabla_{\partial_a^c} dx^j &= -C_k^j|_a dx^k - C_\gamma^c|_a \delta p_c^\gamma, & \nabla_{\partial_a^c} \delta p_b^\beta &= -C_{k_b}^{\beta a} dx^k - C_{\gamma_b}^c|_a \delta p_c^\gamma. \end{aligned}$$

Using the properties of the linear connection ∇ and relations (4.1) and (6.1), we see that the following relations are satisfied:

$$\begin{aligned} \nabla_X G &= (g_{ij|k} X^k + g_{ij}|_\gamma X_c^\gamma) dx^i \otimes dx^j + (g_{j\beta|k} X^k + g_{i\beta}|_\gamma X_c^\gamma) dx^i \otimes \delta p_b^\beta + \\ &\quad (g_{\alpha j|k} X^k + g_{\alpha j}|_\gamma X_c^\gamma) \delta p_a^\alpha \otimes dx^j + (g_{\alpha\beta|k} X^k + g_{\alpha\beta}|_\gamma X_c^\gamma) \delta p_a^\alpha \otimes \delta p_b^\beta, \end{aligned}$$

where,

$$(6.2) \quad \begin{aligned} g_{ij|k} &= \delta_k g_{ij} - g_{hj} F_i^h|_k - g_{\delta j}^d F_i^\delta|_k - g_{ih} F_j^h|_k - g_{i\delta}^d F_j^\delta|_k, \\ g_{ij}|_\gamma &= \partial_\gamma^c g_{ij} - g_{hj} C_i^h|_\gamma - g_{\delta j}^d C_i^\delta|_\gamma - g_{ih} C_j^h|_\gamma - g_{i\delta}^d C_j^\delta|_\gamma, \\ g_{i\beta|k} &= \delta_k g_{i\beta} - g_{h\beta} F_i^h|_k - g_{\delta\beta}^d F_i^\delta|_k - g_{ih} F_\beta^h|_k - g_{i\delta}^d F_\beta^\delta|_k, \\ g_{i\beta}|_\gamma &= \partial_\gamma^c g_{i\beta} - g_{h\beta} C_i^h|_\gamma - g_{\delta\beta}^d C_i^\delta|_\gamma - g_{ih} C_\beta^h|_\gamma - g_{i\delta}^d C_\beta^\delta|_\gamma, \\ g_{\alpha\beta|k} &= \delta_k g_{\alpha\beta} - g_{h\beta} F_\alpha^h|_k - g_{\delta\beta}^d F_\alpha^\delta|_k - g_{\alpha h} F_\beta^h|_k - g_{\alpha\delta}^d F_\beta^\delta|_k, \\ g_{\alpha\beta}|_\gamma &= \partial_\gamma^c g_{\alpha\beta} - g_{h\beta} C_\alpha^h|_\gamma - g_{\delta\beta}^d C_\alpha^\delta|_\gamma - g_{\alpha h} C_\beta^h|_\gamma - g_{\alpha\delta}^d C_\beta^\delta|_\gamma. \end{aligned}$$

Definition 6.1. The space (E^*, N, G, ∇, T) is called the recurrent K -Hamilton space and denoted by $(E^*, N, G, \nabla, T, \lambda)$, if tensor fields $\lambda_k(x, p)$ and $\lambda_\gamma^c(x, p)$ exist, so that

$$(6.3) \quad \begin{aligned} g_{ij|k} &= \lambda_k g_{ij}, & g_{ij}|_\gamma &= \lambda_\gamma^c g_{ij}, & g_{i\beta|k} &= \lambda_k g_{i\beta}, & g_{i\beta}|_\gamma &= \lambda_\gamma^c g_{i\beta}, \\ g_{\alpha j|k} &= \lambda_k g_{\alpha j}, & g_{\alpha j}|_\gamma &= \lambda_\gamma^c g_{\alpha j}, & g_{\alpha\beta|k} &= \lambda_k g_{\alpha\beta}, & g_{\alpha\beta}|_\gamma &= \lambda_\gamma^c g_{\alpha\beta}. \end{aligned}$$

Definition 6.2. The recurrent K -Hamilton space will be called the metric K -Hamilton space and denoted by $(E^*, N, G, \nabla, T, 0)$, if

$$\begin{aligned} g_{ij|k} &= 0, & g_{ij}|_\gamma &= 0, & g_{i\beta|k} &= 0, & g_{i\beta}|_\gamma &= 0, \\ g_{\alpha j|k} &= 0, & g_{\alpha j}|_\gamma &= 0, & g_{\alpha\beta|k} &= 0, & g_{\alpha\beta}|_\gamma &= 0. \end{aligned}$$

THEOREM 6.2. *In the recurrent K -Hamilton space, the coordinates of the inverse metric tensor satisfy the following relations*

$$(6.4) \quad \begin{aligned} g^l{}_k|_h &= -\lambda_h g^l{}_k, & g^l{}_\beta|_h &= -\lambda_h g^l{}_\beta, & g^{\delta\beta}{}_{|h} &= -\lambda_h g^{\delta\beta}{}_{|h}, \\ g^l{}_k|_\gamma &= -\lambda_\gamma^c g^l{}_k, & g^l{}_\beta|_\gamma &= -\lambda_\gamma^c g^l{}_\beta, & g^{\delta\beta}{}_{|\gamma} &= -\lambda_\gamma^c g^{\delta\beta}{}_{|\gamma}. \end{aligned}$$

Proof. It follows, from (4.4), that

$$(6.5) \quad \begin{aligned} (a) \quad & \lambda_h \delta_i^k + g_{ij} g^{jk}|_h + g_{i\beta}^b g_b^{\beta k}|_h = 0 \\ (b) \quad & \lambda_h \delta_c^a \delta_a^\gamma + g_{\alpha j}^a g_c^{j\gamma}|_h + g_{\alpha\beta}^a g_b^{\beta\gamma}|_h = 0 \\ (c) \quad & g_{ij} g_c^{j\gamma}|_h + g_{i\beta}^b g_b^{\beta\gamma}|_h = 0 \\ (d) \quad & g_{\alpha j}^a g^{jk}|_h + g_{\alpha\beta}^a g_b^{\beta k}|_h = 0 \end{aligned}$$

The sum of (6.5a), multiplied by g^{il} , and (6.5d) multiplied by g_α^l , yields $g^l{}_{|h} = -\lambda_h g^{lk}$. The other relations in (6.4) may be obtained in a similar way.

The raising and lowering of the middle index of the connection coefficients are given by the following formulae

$$(6.6) \quad \begin{aligned} \begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^a \end{bmatrix} &= \begin{bmatrix} g_{jh} g_{j\delta}^d \\ g_{\alpha h}^a g_{\alpha\delta}^d \end{bmatrix} \begin{bmatrix} F_i^h{}^k \\ F_{i\delta k}^\delta \end{bmatrix} \Leftrightarrow \begin{bmatrix} F_{ik}^h \\ F_{i\delta k}^\delta \end{bmatrix} = \begin{bmatrix} g^{hj} g_a^{h\alpha} \\ g_d^{\delta j} g_{da}^{\delta\alpha} \end{bmatrix} \begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^a \end{bmatrix} \\ \begin{bmatrix} F_{\alpha ik}^a \\ F_{\alpha\beta k}^a \end{bmatrix} &= \begin{bmatrix} g_{ih} g_{i\delta}^d \\ g_{\beta h}^b g_{\beta\delta}^d \end{bmatrix} \begin{bmatrix} F_{\alpha}^h{}^k \\ F_{\alpha\delta k}^\delta \end{bmatrix} \Leftrightarrow \begin{bmatrix} F_{\alpha}^h{}^k \\ F_{\alpha\delta k}^\delta \end{bmatrix} = \begin{bmatrix} g^{hi} g_b^{h\beta} \\ g_d^{\delta i} g_{db}^{\delta\beta} \end{bmatrix} \begin{bmatrix} F_{\alpha ik}^a \\ F_{\alpha\beta k}^a \end{bmatrix}, \\ \begin{bmatrix} C_{ij\gamma}^c \\ C_{i\alpha\gamma}^a \end{bmatrix} &= \begin{bmatrix} g_{jh} g_{j\delta}^d \\ g_{\alpha h}^a g_{\alpha\delta}^d \end{bmatrix} \begin{bmatrix} C_i^{hc}{}_\gamma \\ C_{i\delta\gamma}^{\delta c} \end{bmatrix} \Leftrightarrow \begin{bmatrix} C_i^{hc}{}_\gamma \\ C_{i\delta\gamma}^{\delta c} \end{bmatrix} = \begin{bmatrix} g^{hj} g_a^{h\alpha} \\ g_d^{\delta j} g_{da}^{\delta\alpha} \end{bmatrix} \begin{bmatrix} C_{ij\gamma}^c \\ C_{i\alpha\gamma}^a \end{bmatrix}, \\ \begin{bmatrix} C_{\alpha i\gamma}^a \\ C_{\alpha\beta\gamma}^a \end{bmatrix} &= \begin{bmatrix} g_{ih} g_{i\delta}^d \\ g_{\beta h}^b g_{\beta\delta}^d \end{bmatrix} \begin{bmatrix} C_{\alpha}^{hc}{}_\gamma \\ C_{\alpha\delta\gamma}^{\delta c} \end{bmatrix} \Leftrightarrow \begin{bmatrix} C_{\alpha}^{hc}{}_\gamma \\ C_{\alpha\delta\gamma}^{\delta c} \end{bmatrix} = \begin{bmatrix} g^{hi} g_b^{h\beta} \\ g_d^{\delta i} g_{db}^{\delta\beta} \end{bmatrix} \begin{bmatrix} C_{\alpha i\gamma}^a \\ C_{\alpha\beta\gamma}^a \end{bmatrix} \end{aligned}$$

By using (6.1) and the above notations, (6.2) may be written in the following form:

$$(6.7) \quad \begin{aligned} g_{ij|k} &= \delta_k g_{ij} - F_{ijk} - F_{jki}, & g_{ij|_\gamma}^c &= \partial_\gamma^c g_{ij} - C_{ij\gamma}^c - C_{ji\gamma}^c, \\ g_{i\beta|k}^b &= \delta_k g_{i\beta}^b - F_{i\beta k}^b - F_{\beta ik}^b, & g_{i\beta|_\gamma}^b &= \partial_\gamma^c g_{i\beta}^b - C_{i\beta\gamma}^b - C_{\beta i\gamma}^b, \\ g_{\alpha\beta|k}^a &= \delta_k g_{\alpha\beta}^a - F_{\alpha\beta k}^a - F_{\beta\alpha k}^a, & g_{\alpha\beta|_\gamma}^a &= \partial_\gamma^c g_{\alpha\beta}^a - C_{\alpha\beta\gamma}^a - C_{\beta\alpha\gamma}^a. \end{aligned}$$

7. The Connection Coefficients in Recurrent k -Hamilton Spaces.

THEOREM 7.1. *In a recurrent K -Hamilton space $(E^*, N, G, \nabla, T, \lambda)$, the connection coefficients are determined by*

$$(7.1) \quad \begin{aligned} (a) \quad & 2F_{ijk} = (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}) - (\lambda_k g_{ij} + \lambda_i g_{jk} - \lambda_j g_{ki}) \\ & \quad + (F_{kij}^* + F_{ikj}^* + F_{kji}^*), \\ (b) \quad & F_{kij}^* = F_{kij} - F_{jik} = g_{ih} (F_k^h{}_j - F_j^h{}_k) + g_{i\delta}^d (F_{k\delta j}^\delta - F_{j\delta k}^\delta) \\ & \quad = g_{ih} T_k^h{}_j + g_{i\delta}^d T_{k\delta j}^\delta, \quad (\text{see 5.7}) \end{aligned}$$

$$(7.2) \quad (a) \quad 2F_{i\alpha k}^a = (\delta_k g_{i\alpha}^a + \delta_i g_{\alpha k}^a - \partial_\alpha^a g_{ik}) - (\lambda_k g_{i\alpha}^a + \lambda_i g_{\alpha k}^a - \lambda_\alpha^a g_{ik}) + A$$

$$(b) \quad A = (F_{i\alpha k}^a - F_{k\alpha i}^a) - (F_{\alpha k i}^a - C_{ik\alpha}^a) - (F_{\alpha i k}^a - C_{ki\alpha}^a) \\ = g_{\alpha h}^a T_{i\ k}^h + g_{\alpha\delta}^a T_{i\ dk}^\delta - g_{ih} T_{\alpha}^{ahk} - g_{kh} T_{\alpha}^{ah\ i} \\ - g_{i\delta}^d (T_{\alpha\ dk}^\delta + \partial_\alpha^a N_{dk}^\delta) - g_{k\delta}^d (T_{\alpha\ di}^\delta + \partial_\alpha^a N_{di}^\delta) \\ + g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta),$$

$$(7.3) \quad 2F_{\alpha ik}^a = (\delta_k g_{\alpha i}^a + \partial_\alpha^a g_{ik} - \delta_i g_{k\alpha}^a) - (\lambda_k g_{\alpha i}^a + \lambda_\alpha^a g_{ik} - \lambda_i g_{k\alpha}^a) - A,$$

$$(7.4) \quad (a) \quad 2F_{\alpha\beta k}^a = (\delta_k g_{\alpha\beta}^a + \partial_\alpha^a g_{\beta k}^a - \partial_\beta^b g_{k\alpha}^a)$$

$$- (\lambda_k g_{\alpha\beta}^a + \lambda_\alpha^a g_{\beta k}^a - \lambda_\beta^b g_{k\alpha}^a) + B,$$

$$(b) \quad B = (F_{\alpha\beta k}^a - C_{k\beta\alpha}^a) - (F_{\beta\alpha k}^a - C_{k\alpha\beta}^a) + (C_{\alpha k\beta}^a - C_{\beta k\alpha}^a) \\ = g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta - g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta + g_{\beta h}^b T_{\alpha}^{ah\ k} + g_{\beta\delta}^b T_{\alpha\ dk}^\delta \\ - g_{\alpha h}^a T_{\beta}^{bh\ k} - g_{\alpha\delta}^a T_{\beta\ dk}^\delta + g_{kh} T_{\alpha}^{ah\ \beta} + g_{k\delta}^d T_{\alpha\ d\beta}^\delta,$$

$$(7.5) \quad (a) \quad 2C_{ij\alpha}^a = (\partial_\alpha^a g_{ij} + \delta_i g_{j\alpha}^a - \delta_j g_{\alpha i}^a) - (\lambda_\alpha^a g_{ij} + \lambda_i g_{j\alpha}^a - \lambda_j g_{\alpha i}^a) + C,$$

$$(b) \quad C = (F_{i\alpha j}^a - F_{j\alpha i}^a) + (F_{\alpha ij}^a - C_{ji\alpha}^a) - (F_{\alpha ji}^a - C_{ij\alpha}^a) \\ = g_{\alpha h}^a T_{i\ j}^h + g_{\alpha\delta}^a (T_{i\ dj}^\delta + \delta_i N_{dj}^\delta - \delta_j N_{di}^\delta) + g_{ih} T_{\alpha}^{ah\ j} \\ - g_{jh} T_{\alpha}^{ah\ i} + g_{i\delta}^d (T_{\alpha\ dj}^\delta + \partial_\alpha^a N_{dj}^\delta) - g_{j\delta}^d (T_{\alpha\ di}^\delta + \partial_\alpha^a N_{di}^\delta)$$

$$(7.6) \quad (a) \quad 2C_{k\alpha\beta}^a = (\partial_\beta^b g_{k\alpha}^a + \delta_k g_{\alpha\beta}^a - \partial_\alpha^a g_{\beta k}^a) - (\lambda_\beta^b g_{k\alpha}^a + \lambda_k g_{\alpha\beta}^a - \lambda_\alpha^a g_{\beta k}^a) \\ - D - E,$$

$$(b) \quad D = (F_{\alpha\beta k}^a - C_{k\beta\alpha}^a) + (F_{\beta\alpha k}^a - C_{k\alpha\beta}^a) \\ = g_{\beta h}^b T_{\alpha}^{ah\ k} + g_{\alpha h}^a T_{\beta}^{bh\ k} + g_{\beta\delta}^b (T_{\alpha\ dk}^\delta + \partial_\alpha^a N_{dk}^\delta) \\ + g_{\alpha\delta}^a (T_{\beta\ dk}^\delta + \partial_\beta^b N_{dk}^\delta),$$

$$(c) \quad E = C_{\alpha k\beta}^a - C_{\beta k\alpha}^a = g_{kh} T_{\alpha}^{ah\ \beta} + g_{k\delta}^d T_{\alpha\ d\beta}^\delta,$$

$$(7.7) \quad 2C_{\alpha k\beta}^a = (\partial_\beta^b g_{\alpha k}^a + \partial_\alpha^a g_{k\beta}^a - \delta_k g_{\beta\alpha}^a) - (\lambda_\beta^b g_{\alpha k}^a + \lambda_\alpha^a g_{k\beta}^a - \lambda_k g_{\beta\alpha}^a) \\ + D + E,$$

$$(7.8) \quad (a) \quad 2C_{\alpha\beta\gamma}^a = (\partial_\gamma^c g_{\alpha\beta}^a + \partial_\alpha^a g_{\beta\gamma}^a - \partial_\beta^b g_{\gamma\alpha}^a) - (\lambda_\gamma^c g_{\alpha\beta}^a + \lambda_\alpha^a g_{\beta\gamma}^a - \lambda_\beta^b g_{\gamma\alpha}^a) \\ + (C_{\gamma\alpha\beta}^{*ca\ b} + C_{\alpha\gamma\beta}^{*ac\ b} - C_{\gamma\beta\alpha}^{*cb\ a}),$$

$$(c) \quad C_{\gamma\alpha\beta}^{*ca\ b} = C_{\gamma\alpha\beta}^{ca\ b} - C_{\beta\alpha\gamma}^{b\ ac} = g_{\alpha h}^a T_{\gamma}^{ch\ \beta} + g_{\alpha\delta}^a T_{\gamma\ d}^{\delta h}.$$

Proof. The proof follows from (6.7), (6.6), (6.3) and (5.7).

Definition 7.1. The recurrent K -Hamilton space, in which $T(X, Y) = 0$, for all $X, Y \in T(E^*)$, is called torsion free recurrent K -Hamilton space and is denoted by $(E^*, N, G, \nabla, 0, \lambda)$.

THEOREM 7.2. *In the torsion-free recurrent K -Hamilton space $(E^*, N, G, \nabla, 0, \lambda)$, the connection coefficients are determined by*

$$\begin{aligned}
(7.9) \quad 2F_{ijk} &= (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}) - (\lambda_k g_{ij} + \lambda_i g_{jk} - \lambda_j g_{ki}), \\
2F_{i\alpha k}^a &= (\delta_k g_{i\alpha}^a + \delta_i g_{\alpha k}^a - \partial_\alpha^a g_{ik}) - (\lambda_k g_{i\alpha}^a + \lambda_i g_{\alpha k}^a - \lambda_\alpha^a g_{ik}) \\
&\quad - (g_{i\delta}^d \partial_\alpha^a N_{dk}^\delta + g_{k\delta}^d \partial_\alpha^a N_{di}^\delta) + g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta), \\
2F_{\alpha ik}^a &= (\delta_k g_{\alpha i}^a + \partial_\alpha^a g_{ik} - \delta_i g_{\alpha}^a) - (\lambda_k g_{\alpha i}^a + \lambda_\alpha^a g_{ik} - \lambda_i g_{k\alpha}^a) \\
&\quad + (g_{i\delta}^d \partial_\alpha^a N_{dk}^\delta + g_{k\delta}^d \partial_\alpha^a N_{di}^\delta) - g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta) \\
2F_{\alpha\beta k}^{ab} &= (\delta_k g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta k}^b - \partial_\beta^b g_{k\alpha}^a) - (\lambda_k g_{\alpha\beta}^{ab} + \lambda_\alpha^a g_{\beta k}^b - \lambda_\beta^b g_{k\alpha}^a) \\
&\quad + g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta - g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta, \\
2C_{ij\alpha}^a &= (\partial_\alpha^a g_{ij} + \delta_i g_{j\alpha}^a - \delta_j g_{\alpha i}^a) - (\lambda_\alpha^a g_{ij} + \lambda_i g_{j\alpha}^a - \lambda_j g_{\alpha i}^a) \\
&\quad + g_{\alpha\delta}^a (\delta_i N_{dj}^\delta - \delta_j N_{di}^\delta) + g_{i\delta}^d \partial_\alpha^a N_{dj}^\delta - g_{j\delta}^d \partial_\alpha^a N_{di}^\delta, \\
2C_{k\alpha\beta}^{ab} &= (\partial_\beta^b g_{k\alpha}^a + \delta_k g_{\alpha\beta}^{ab} - \partial_\alpha^a g_{\beta k}^b) - (\lambda_\beta^b g_{k\alpha}^a + \lambda_k g_{\alpha\beta}^{ab} - \lambda_\alpha^a g_{\beta k}^b) \\
&\quad - (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta), \\
2C_{\alpha k\beta}^{ab} &= (\partial_\beta^b g_{\alpha k}^a + g_{\alpha\beta}^{ab} - \delta_k g_{\beta\alpha}^a) - (\lambda_\beta^b g_{\alpha k}^a - \lambda_\alpha^a g_{k\beta}^b - \lambda_k g_{\beta\alpha}^a) \\
&\quad + (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta), \\
2C_{\alpha\beta\gamma}^{abc} &= (\partial_\gamma^c g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta\gamma}^{bc} - \partial_\beta^b g_{\gamma\alpha}^{ca}) - (\lambda_\gamma^c g_{\alpha\beta}^{ab} + \lambda_\alpha^a g_{\beta\gamma}^{bc} - \lambda_\beta^b g_{\gamma\alpha}^{ca}).
\end{aligned}$$

Definition 7.2. The metric K -Hamilton space in which $T(X, Y) = 0$ for all $X, Y \in T(E^*)$, is called torsion-free metric K -Hamilton space and denoted by $(E^*, N, G, \nabla, 0, 0)$.

THEOREM 7.3. *In the torsion free metric K -Hamilton space $(E^*, N, G, \nabla, 0, 0)$, the connection coefficients are given by (7.9) if we substitute in them $\lambda_i = 0$, $\lambda_j = 0$, $\lambda_k = 0$, $\lambda_\alpha^a = 0$, $\lambda_\beta^b = 0$ and $\lambda_\gamma^c = 0$.*

Definition 7.3. The torsion-free, metric K -Hamilton space, in which $[g_{k\alpha}^a] = 0$ for all $\alpha = \overline{1, K}$ (i.e. $T_H(E^*)$ is orthogonal to ${}_{(\alpha)}T_V(E^*)$, for all $\alpha = \overline{1, K}$, or equivalently, $T_H(E^*)$ is orthogonal to $T_V(E^*)$), will be denoted by $(E^*, N, G_H, G_V, \nabla, 0, 0)$.

THEOREM 7.4. *In $(E^*, N, G_H, G_V, \nabla, 0, 0)$ the connection coefficients are determined by*

$$\begin{aligned}
(7.10) \quad 2F_{ijk} &= \delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}, \\
2F_{i\alpha k}^a &= -\partial_\alpha^a g_{ik} + g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta), \\
2F_{\alpha ik}^a &= \partial_\alpha^a g_{ik} - g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta), \\
2F_{\alpha\beta k}^{ab} &= \delta_k g_{\alpha\beta}^{ab} + g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta - g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta,
\end{aligned}$$

$$\begin{aligned}
2C_{ij\alpha}^a &= \partial_\alpha^a g_{ij} + g_{\alpha\delta}^a (\delta_i N_{dj}^\delta - \delta_j N_{di}^\delta), \\
2C_{k\alpha\beta}^{ab} &= \delta_k g_{\alpha\beta}^{ab} - (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta), \\
2C_{\alpha k\beta}^{ab} &= -\delta_k g_{\beta\alpha}^{ab} + (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta), \\
2C_{\alpha\beta\gamma}^{abc} &= \partial_\gamma^c g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta\gamma}^{bc} - \partial_\beta^b g_{\gamma\alpha}^{ca}.
\end{aligned}$$

It is obvious that in (7.10), $F_{\alpha ik}^a = -F_{i\alpha k}^a$, $C_{k\alpha\beta}^{ab} = -C_{\alpha k\beta}^{ab}$. In the recurrent K -Hamilton space in which $T_H(E^*)$ is orthogonal to $T_V(E^*)$, i.e. where the metric tensor has the property $[g_{i\alpha}^a] = 0$ for all $\alpha \in \overline{1, K}$ (in space $(E^*, N, G_H, G_V, \nabla, T, \lambda)$) we obtain from (6.6):

$$\begin{aligned}
(7.11) \quad F_i^h{}_k &= g^{hj} F_{ijk}, & F_{idk}^\delta &= g_{ad}^{\alpha\delta} F_{i\alpha k}^a, & F_{\alpha k}^a{}^h &= g^{hi} F_{\alpha ik}^a, \\
F_{\alpha dk}^{\delta} &= g_{db}^{\delta\beta} F_{\alpha\beta k}^b, & C_i^h{}_{\gamma}{}^c &= g^{hj} C_{ij\gamma}^c, & C_{id\gamma}^{\delta c} &= g_{da}^{\delta\alpha} C_{i\alpha\gamma}^{ac}, \\
C_{\alpha k\gamma}^{akc} &= g^{ki} C_{\alpha i\gamma}^{ac}, & C_{\alpha d\gamma}^{a\delta c} &= g_{db}^{\delta\beta} C_{\alpha\beta\gamma}^{abc}.
\end{aligned}$$

The connection coefficients, which appear on the right-hand side of (7.11) for $(E^*, N, G_H, G_V, \nabla, 0, 0)$, are determined by (7.10).

THEOREM 7.5. *The necessary and sufficient conditions that in $(E^*, N, G_H, G_V, \nabla, 0, 0)$ the connection coefficients satisfy the relations*

$$\begin{aligned}
(7.12) \quad F_{i\alpha k}^a = 0 &\Leftrightarrow F_{idk}^\delta = 0, & F_{\alpha ik}^a = 0 &\Leftrightarrow F_{\alpha k}^a{}^h = 0, \\
C_{\alpha i\gamma}^{ac} = 0 &\Leftrightarrow C_{\alpha k\gamma}^{ac} = 0, & C_{i\alpha\gamma}^{ac} = 0 &\Leftrightarrow C_{id\gamma}^{\delta c} = 0,
\end{aligned}$$

(for the nonlinear connection N) are

$$(7.13) \quad \partial_\alpha^a g_{ik} - g_{\alpha\delta}^a (\delta_i N_{dk}^\delta - \delta_k N_{di}^\delta) = 0, \quad \delta_k g_{\beta\alpha}^{ab} - (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta) = 0.$$

Equations (7.12) and (7.13) should be satisfied for all $a, b, d \in \overline{1, m}$, all $h, i \in \overline{1, n}$, all $\alpha, \beta, \delta \in \overline{1, K}$.

Proof. The proof follows from (7.10) and (7.11).

THEOREM 7.6. *In $(E^*, N, G_H, G_V, \nabla, 0, 0)$ the generalized connection ∇ , defined by (5.1), reduces to the Miron's d -connection defined by*

$$\begin{aligned}
(7.14) \quad \nabla_{\delta_i} \delta_j &= F_j^k{}_i \delta_k, & \nabla_{\delta_i} \partial_\alpha^a &= F_{\alpha c}^{\gamma i} \partial_\gamma^c, \\
\nabla_{\partial_\alpha^a} \delta_j &= C_j^k{}_{\alpha} \delta_k, & \nabla_{\partial_\alpha^a} \partial_\beta^b &= C_{\beta c}^{\gamma a} \partial_\gamma^c,
\end{aligned}$$

iff the nonlinear connection N and the metric tensor G are connected by (7.13). The connection coefficients of the d -connection are determined by

$$\begin{aligned}
(7.15) \quad 2F_i^h{}_k &= g^{kj} (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}), \\
2F_{\alpha c}^{\gamma k} &= g_c^{\gamma\beta} \delta_k g_{\alpha\beta}^{ab} + g_c^{\gamma\beta} (g_{\beta\delta}^b \partial_\alpha^a N_{dk}^\delta - g_{\alpha\delta}^a \partial_\beta^b N_{dk}^\delta), \\
2C_i^h{}_{\alpha} &= g^{hj} (\partial_\alpha^a g_{ij} + g_{\alpha\delta}^a (\delta_i N_{dj}^\delta - \delta_j N_{di}^\delta)), \\
2C_{\alpha b\gamma}^{abc} &= g_b^{\beta\delta} (\partial_\gamma^c g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta\gamma}^{bc} - \partial_\beta^b g_{\gamma\alpha}^{ca}).
\end{aligned}$$

Proof. (7.14) follows from Theorems (7.5) and (5.1). (7.15) follows from (7.10), (7.11) and (7.13).

From (7.14) it is obvious that the d -connection is the linear connection, for which $\nabla_X: T_H(E^*) \rightarrow T_H(E^*)$ and $\nabla_X: T_V(E^*) \rightarrow T_V(E^*)$ for all $X \in T(E^*)$.

THEOREM 7.7. *The space $(E^*, N, G_H, G_V, 0, 0)$, with integrable nonlinear connection N :*

$$(7.16) \quad \delta_i N_{dk}^\delta - \delta_k N_{di}^\delta = 0,$$

allows a d -connection iff

$$(7.17) \quad (a) \quad \partial_\alpha^a g_{ik} = 0, \quad (b) \quad \delta_k g_{\beta\alpha}^{ba} = g_{\beta\delta}^{bd} \partial_\alpha^a N_{dk}^\delta + g_{\alpha\delta}^{ad} \partial_\beta^b N_{dk}^\delta.$$

((7.17a) means that the horizontal metric tensor is a function only of x).

The connection coefficients of such a d -connection are given by

$$\begin{aligned} F_i^h{}_k &= 2^{-1} g^{hj} (\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ki}), & F_{\alpha c}^a{}_{\gamma k} &= \partial_\alpha^a N_{ck}^\gamma, \\ C_{\alpha d}^a{}_{\gamma} &= 2^{-1} g_b^{\beta\delta} (\partial_\gamma^c g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta\gamma}^{bc} - \partial_\beta^b g_{\gamma\alpha}^{ca}). & C_i^h{}_\alpha &= 0, \end{aligned}$$

Proof. From Theorem 7.6, it follows that the space $(E^*, N, G_H, G_V, 0, 0)$ allows a d -connection iff the relations (7.13) are satisfied. (7.13) and (7.16) result (7.17). Relation (7.17b) can be obtained if in (5.7d) we substitute $T_{HVV} = 0$ (torsion equal to zero) and $C_i^{\gamma b}{}_\beta = 0$ (for the case of a d -connection).

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