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L_p-APPROXIMATION BY ITERATIVE COMBINATION OF PHILLIPS OPERATORS

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Abstract. An estimate of error in L_p-approximation in terms of higher order integral modulus of smoothness is obtained using the device of Steklov means for an iterative combination, due to Micchelli, of Phillips operators.

1. Introduction. Phillips [**7**] introduced the following linear positive operators

$$S_\lambda(f,t)=\int_0^\infty W(\lambda,t,u)f(u)du,\quad f\in L_p[0,\infty)$$

where $p\geq 1$, $t\in[0,\infty)$ and

$$W(\lambda,t,u)=e^{-\lambda(t+u)}\left(\sum_{n=1}^\infty\frac{(\lambda^2t)^nu^{n-1}}{n!(n-1)!}+\delta(u)\right),$$

$\delta(u)$ being the Dirac–delta function.

It turns out that the order of approximation by the Phillips operator $S_\lambda(f,t)$ is at best $O(\lambda^{-1})$. With the aim of improving the order of approximation by the Phillips operators, May [**5**] applied the technique of linear combinations to S_λ. These combinations were introduced by Butzer [**2**] in order to improve the order of approximation by Bernstein polynomials. Micchelli [**6**] offered yet another approach for improving the order of approximation by Bernstein polynomials B_n by considering the iterative combinations $T_{n,k}=I-(I-B_n)^k$ and proved some direct and saturation results. Agrawal and Kasana [**1**] improved a result of Micchelli [**6**] and obtained a Voronovskaja type asymptotic formula for these operators.

In this paper, we consider Micchelli combination for the Phillips operator S_λ and prove some direct results in L_p-approximation. For $f\in L_p[0,\infty)$, we define the operator

$$S_{\lambda,k}(f(u),t)=[I-(I-S_\lambda)^k](f,t)=\sum_{r=1}^k(-1)^{r+1}\binom{k}{r}S_\lambda^r(f(u),t)\tag{1.1}$$

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where S_λ^r denotes the r-th iterative (superposition) of the operator S_λ.

In what follows, we suppose that

$$0 < a_1 < a_3 < a_2 < b_2 < b_3 < b_1 < \infty, \qquad I_i = [a_i, b_i], \quad i = 1, 2, 3.$$

and that $[\alpha]$ denotes the integral part of α.

2. Degree of approximation. We denote by $\omega_{2k}(f, p, I_1)$, $k = 0, 1, 2, \dots$, $1 \le p < \infty$, the $2k$-th order integral modulus of smoothness of f on I_1.

THEOREM 2.1. *If $f \in L_p[0,\infty)$, $p \ge 1$, then for all λ sufficiently large*

$$\|S_{\lambda,k}(f,\cdot) - f\|_{L_p(I_2)} \le M_k \left\{\omega_{2k}(f, \lambda^{-1/2}, p, I_1) + \lambda^{-k} \|f\|_{L_p[0,\infty)}\right\}$$

where M_k is a constant independent of f and λ.

The method of proof is first to approximate in a smooth subspace of $L_p[0,\infty)$ (Lemma 2.6 below) and then use Steklov means to obtain the degree of approximation in $L_p[0,\infty)$. The use of Steklov means has been a powerfull tool in the development of results as against the usual procedures exploiting Peetre's K-functional technique of Wood in [**9**].

First we define the Steklov means and then mention some results in the form of lemmas which will be used in the sequel. Let $f \in L_p[0,\infty)$, $1 \le p < \infty$. Then for sufficiently small $\eta > 0$, the Steklov mean $f_{\eta,m}$ of m-th order corresponding to f is defined by

$$f_{\eta,m}(u) = \eta^{-m} \int_{-\eta/2}^{\eta/2} \cdots \int_{-\eta/2}^{\eta/2} \left\{f(u) + (-1)^{m-1} \Delta^m_{\sum_{i=1}^m u_i} f(u)\right\} \prod_{i=1}^m du_i, \quad u \in I_1.$$

It is easy to check [**4,8**] that

(i) $f_{\eta,m}$ has derivatives up to order m, $f_{\eta,m}^{(m-1)} \in \mathrm{AC}(I_1)$ and $f_{\eta,m}^{(m)}$ exists a.e. and belongs to $L_p(I_1)$;

(ii) $\left\|f_{\eta,m}^{(r)}\right\|_{L_p(I_2)} \le M_r \eta^{-r} \omega_r(f, \eta, p, I_1)$, $r = 1\,(1)\,m$;

(iii) $\|f - f_{\eta,m}\|_{L_p(I_2)} \le M_{m+1} \omega_m(f, \eta, p, I_1)$;

(iv) $\|f_{\eta,m}\|_{L_p(I_2)} \le M_{m+2} \|f\|_{L_p(I_1)}$;

(v) $\left\|f_{\eta,m}^{(m)}\right\|_{L_p(I_2)} \le M_{m+3} \eta^{-m} \|f\|_{L_p(I_1)}$, where M_i's are certain constants depending on i but independent of f and η.

LEMMA 2.1, [**5**] *Let the function $\mu_{\lambda,m}(t)$, $m \in \mathbf{N}^0$ (the set of non-negative integers) be defined by $\mu_{\lambda,m}(t) = \int_0^\infty W(\lambda,t,u)(u-t)^m du$. Then $\mu_{\lambda,0}(t) = 1$, $\mu_{\lambda,1}(t) = 0$, $\mu_{\lambda,2}(t) = 2t/\lambda$, and the following recurrence relation holds*

$$\begin{aligned} &\frac{2t}{\lambda} D(\mu_{\lambda,m}(t)) + \frac{t}{\lambda^2} D^2(\mu_{\lambda,m}(t)) \\ &\quad = \mu_{\lambda,m+1}(t) - \frac{2tm}{\lambda}\mu_{\lambda,m-1}(t) - \frac{tm(m-1)}{\lambda^2}\mu_{\lambda,m-2}(t) - \frac{2tm}{\lambda^2} D^2(\mu_{\lambda,m-1}(t)). \end{aligned}$$

Consequently,

(i) $\mu_{\lambda,m}(t)$ is a polynomial in t and $1/\lambda$ for every $t \in [0,\infty)$.

(ii) $\mu_{\lambda,m}(t) = O\left(\lambda^{-\left[\frac{m+1}{2}\right]}\right)$, for every $t \in [0,\infty)$.

Moreover, by using Hölder's inequality we have

(2.1) $\quad S_\lambda(|u-t|^r, t) = O(\lambda^{-r/2}) \quad$ for each $r > 0$ and for every fixed $t \in [0,\infty)$.

For every $m \in \mathbf{N}^0$ the m-th moment $\mu_{\lambda,m}^{\{p\}}$ for the operator S_λ^p is defined by $\mu_{\lambda,m}^{\{p\}}(t) = S_\lambda^p((u-t)^p; t)$. Let $\mu_{\lambda,m}(t)$ denote $\mu_{\lambda,m}^{\{1\}}(t)$.

Lemma 2.2. *The following recurrence relation holds*

$$\mu_{\lambda,m}^{\{p+1\}}(t) = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{m-j} \frac{1}{i!} D^i \left(\mu_{\lambda,m-j}^{\{p\}}(t)\right) \mu_{\lambda,i+j}(t), \tag{2.2}$$

where D denotes the operator d/dt.

Proof. By the definition above, we have

$$\begin{aligned}
\mu_{\lambda,m}^{\{p+1\}}(t) &= S_\lambda(S_\lambda^p((\mu-t)^m; x); t) \\
&= \sum_{j=0}^{m} \binom{m}{j} S_\lambda((x-t)^j S_\lambda^p((u-x)^{m-j}; x); t) \\
&= \sum_{j=0}^{m} \binom{m}{j} S_\lambda \left(\sum_{i=0}^{m-j} \frac{(x-t)^{i+j}}{i!} D^i \left(\mu_{\lambda,m-j}^{\{p\}}(t)\right); t \right).
\end{aligned}$$

Now, (2.2) follows immediately.

Lemma 2.3. *We have*

$$\mu_{\lambda,m}^{\{p\}}(t) = O\left(\lambda^{-[(m+1)/2]}\right). \tag{2.3}$$

Proof. For $p = 1$, the result follows from Lemma 2.1. Suppose the result is true for p; we shall prove it for $p+1$. Now, $\mu_{\lambda,m-j}^{\{p\}}(t) = O\left(\lambda^{-[(m-j+1)/2]}\right)$ is a polynomial in t of degree $\leq m-j$; it follows that

$$D^i \left(\mu_{\lambda,m-j}^{\{p\}}(t)\right) = O\left(\lambda^{-[(m-j+1)/2]}\right).$$

using Lemma 2.2, we obtain

$$\begin{aligned}
\mu_{\lambda,m}^{\{p+1\}}(t) &= O\left(\sum_{j=0}^{m} \sum_{i=0}^{m-j} \lambda^{-[(m-j+1)/2]+[(i+j+1)/2]} \right) \\
&= O\left(\sum_{j=0}^{m} \sum_{i=0}^{m-j} \lambda^{-[(m+i+1)/2]} \right)
\end{aligned}$$

which implies (2.3), by induction hypothesis. $\diamondsuit$

LEMMA 2.4. *For l-th moment ($l \in \mathbf{N}$) of $S_{\lambda,k}$, we have*

$$S_{\lambda,k}((u-t)^l, t) = O(\lambda^{-k}). \tag{2.4}$$

Proof. For $k = 1$, the result follows from Lemma 2.1. Now, suppose that (2.4) holds for some k; then by using Lemma 2.2 and Lemma 2.3, we can infer it for $k+1$ (induction argument.) $\diamondsuit$

LEMMA 2.5 [**3**] *Let $1 \le p < \infty$, $f \in L_p[a,b]$, $f^{(k)} \in \mathrm{AC}[a,b]$ and $f^{(k+1)} \in L_p[a,b]$; then*

$$\left\| f^{(j)} \right\|_{L_p[a,b]} \le k_j \left(\left\| f^{(k+1)} \right\|_{L_p[a,b]} + \|f\|_{L_p[a,b]} \right), \qquad j = 1, 2, \ldots, k,$$

where k_j's are certain constants depending only on j, k, p, a and b.

LEMMA 2.6. *If $p > 1$, $f \in L_p[0,\infty)$, f has $2k$ derivatives on I_1 with $f^{(2k-1)} \in \mathrm{AC}(I_1)$ and $f^{(2k)} \in L_p(I_1)$, then for all λ sufficiently large*

$$\|S_{\lambda,k}(f,t) - f(t)\|_{L_p(I_2)} \le M_1 \lambda^{-k} \left\{ \left\| f^{(2k)} \right\|_{L_p(I_1)} + \|f\|_{L_p[0,\infty)} \right\}. \tag{2.5}$$

If $f \in L_1[0,\infty)$, f has $(2k-1)$ derivatives on I_1 with $f^{(2k-2)} \in \mathrm{AC}(I_1)$ and $f^{(2k-1)} \in \mathrm{BV}(I_1)$, then for all λ sufficiently large

$$\begin{aligned} &\|S_{\lambda,k}(f,t) - f(t)\|_{L_1(I_2)} \\ &\qquad \le M_2 \lambda^{-k} \left\{ \left\| f^{(2k-1)} \right\|_{\mathrm{BV}(I_1)} + \left\| f^{(2k-1)} \right\|_{L_1(I_2)} + \|f\|_{L_1[0,\infty)} \right\} \end{aligned} \tag{2.6}$$

where M_1 and M_2 are constants independent of f and λ.

Proof. First assume $p > 1$; then, by the hypothesis, for $t \in I_2$ and $u \in I_1$

$$f(u) = \sum_{j=0}^{2k-1} f^{(j)}(t) \frac{(u-t)^j}{j!} + \frac{1}{(2k-1)!} \int_t^u (u-w)^{2k-1} f^{(2k)}(w) dw.$$

Hence, we can write

$$\begin{aligned} f(u) = {} & \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t) \\ & + \frac{1}{(2k-1)!} \int_t^u (u-w)^{2k-1} \phi(u) f^{(2k)}(w) dw \\ & + F(u,t)(1-\phi(u)), \end{aligned} \tag{2.7}$$

where $\phi(u)$ is the characteristic function of I_1 and for all $u \in [0,\infty)$ and $t \in I_2$

$$F(u,t) = f(u) - \sum_{j=0}^{2k-1} \frac{(u-t)^j}{j!} f^{(j)}(t),$$

Using (2.7) in (1.1), we have

$$\begin{aligned} S_{\lambda,k}(f,t)-f(t) &= \sum_{j=1}^{2k-1}\frac{f^{(j)}(t)}{j!}S_{\lambda,k}((u-t)^j,t) \\ &\quad+\frac{1}{(2k-1)!}S_{\lambda,k}\left(\varphi(u)\int_t^u (u-w)^{2k-1}f^{(2k)}(w)dw,t\right) \\ &\quad+S_{\lambda,k}\left(F(u,t)(1-\phi(u)),t\right) \\ &= \Sigma_1+\Sigma_2+\Sigma_3, \quad \text{say} \end{aligned}$$

In view of Lemma 2.4 and [**3**]

$$\|\Sigma_1\|_{L_p(I_2)} \le C_1\lambda^{-k}\left(\sum_{j=1}^{2k-1}\left\|f^{(j)}\right\|_{L_p(I_2)}\right) \le C_2\lambda^{-k}\left(\|f\|_{L_p(I_2)}+\left\|f^{(2k)}\right\|_{L_p(I_2)}\right).$$

To estimate Σ_2, let h_f be the Hardy–Littlewood majorant [**9**] of $f^{(2k)}$ on I_1. Use of Hölder's inequality and (2.1) leads to:

$$\begin{aligned} J_1 &= \left|S_\lambda(\varphi(u)\int_t^u (u-w)^{2k-1}f^{(2k)}(w)dw,t)\right| \\ &\le S_\lambda\left(\varphi(u)\left|\int_t^u |u-w|^{2k-1}|f^{(2k)}(w)|dw\right|,t\right) \\ &\le S_\lambda\left(\varphi(u)(u-t)^{2k}|h_f(u)|,t\right) \\ &\le \left\{S_\lambda\left(|u-t|^{2kq}\varphi(u),t\right)\right\}^{1/q}\cdot\left\{S_\lambda\left(|h_f(u)|^p\varphi(u),t\right)\right\}^{1/p} \\ &\le C_3\lambda^{-k}\left(\int_{a_1}^{b_1} W(\lambda,t,u)|h_f(u)|^p du\right)^{1/p} \end{aligned}$$

Fubini's theorem and [**10**, Ch. 2] imply that

$$\begin{aligned} \|J_1\|^p_{L_p(I_2)} &\le C_3\lambda^{-kp}\int_{a_2}^{b_2}\int_{a_1}^{b_1} W(\lambda,t,u)|h_f(u)|^p du\,dt \\ &\le C_3\lambda^{-kp}\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} W(\lambda,t,u)dt\right)|h_f(u)|^p du \\ &\le C_4\lambda^{-kp}\left\|f^{(2k)}\right\|^p_{L_p(I_1)} \end{aligned}$$

Consequently $\|\Sigma_2\|_{L_p(I_2)} \le C_5\lambda^{-k}\left\|f^{(2k)}\right\|_{L_p(I_1)}$. For $u\in[0,\infty)\setminus[a_1,b_1]$, $t\in I_2$ there exists a $\delta>0$ such that $|u-t|\ge\delta$. Thus

$$\begin{aligned} &|S_\lambda(F(u,t)(1-\varphi(u)),t)| \\ &\qquad\le \delta^{-2k}S_\lambda(|F(u,t)|(u-t)^{2k},t) \\ &\qquad= \delta^{-2k}\left[S_\lambda(|f(u)|(u-t)^{2k},t)+\sum_{j=0}^{2k-1}\frac{|f^{(j)}(t)|}{j!}S_\lambda(|u-t|^{2k+j},t)\right] \\ &\qquad= J_2+J_3, \quad \text{say.} \end{aligned}$$

Hölder's inequality and (2.1) get us:

$$\begin{aligned}|J_2| &\le \delta^{-2k}(S_\lambda(|f(u)|^p,t)^{1/p}(S_\lambda(|u-t|^{2kq},t))^{1/q}\\ &\le C_6\lambda^{-k}\left[S_\lambda(|f(u)|^p,t)\right]^{1/p}.\end{aligned}$$

Again applying Fubini's theorem, we get $\|J_2\|_{L_p(I_2)} \le C_7\lambda^{-k}\|f\|_{L_p[0,\infty)}$. Moreover, using (2.1) and [**3**], we obtain

$$\|J_3\|_{L_p(I_2)} \le C_8\lambda^{-k}\sum_{j=0}^{2k-1}\left\|f^{(i)}\right\|_{L_p(I_2)} \le C_8\lambda^{-k}\left(\|f\|_{L_p(I_2)}+\left\|f^{(2k)}\right\|_{L_p(I_2)}\right).$$

Combining the estimates of J_2 and J_3, we are led to:

$$\|\Sigma_3\|_{L_p(I_2)} \le C_9\lambda^{-k}\left[\|f\|_{L_p[0,\infty)}+\left\|f^{(2k)}\right\|_{L_p(I_2)}\right].$$

Hence the result (2.5) follows.

Now assume $p=1$; then by the assumption on f for almost all $t\in I_2$ and for all $u\in I_1$,

$$f(u)=\sum_{i=0}^{2k-1}\frac{(u-t)^i}{i!}f^{(i)}(t)+\frac{1}{(2k-1)!}\int_t^u (u-w)^{2k-1}df^{(2k-1)}(w).$$

We can write

$$\begin{aligned}f(u)=\sum_{i=0}^{2k-1}\frac{(u-t)^i}{i!}f^{(i)}(t)+\frac{1}{(2k-1)!}\int_t^u (u-w)^{2k-1}df^{(2k-1)}(w)\varphi(u)&\\ +F(u,t)(1-\varphi(u)),&\end{aligned}$$

where $\varphi(u)$ denotes the characteristic function on I_1 and $F(u,t)$ is defined as earlier for almost all $t\in I_2$ and for all $u\in[0,\infty)$. Thus

$$\begin{aligned}S_{\lambda,k}(f,t)-f(t) &= \sum_{i=1}^{2k-1}\frac{f^{(i)}(t)}{i!}S_{\lambda,k}((u-t)^i,t)\\ &\quad+\frac{1}{(2k-1)!}S_{\lambda,k}\left(\int_t^u(u-w)^{2k-1}df^{(2k-1)}(w)\varphi(u),t\right)\\ &\quad+S_{\lambda,k}(F(u,t)(1-\varphi(u)),t)\\ &= J_1+J_2+J_3,\quad \text{say.}\end{aligned}$$

Applying Lemma 2.2 and [**3**], we obtain

$$\|J_1\|_{L_1(I_2)} \le C_1\lambda^{-k}\left[\|f\|_{L_1(I_2)}+\left\|f^{(2k-1)}\right\|_{L_1(I_2)}\right].$$

Furthermore

$$\begin{aligned}K &\equiv \left\|S_\lambda\left(\int_t^u(u-w)^{2k-1}df^{2k-1}(w)\varphi(u),t\right)\right\|_{L_1(I_2)}\\ &\le \int_{a_2}^{b_2}\int_{a_1}^{b_1}W(\lambda,t,u)|u-t|^{2k-1}\left|\int_t^u\left|df^{(2k-1)}(w)\right|\right|du\,dt.\end{aligned}$$

For each λ there exists a non-negative integer $r = r(\lambda)$ such that

$$r\lambda^{-1/2} < \max(b_1 - a_2, b_2 - a_1) \leq (r+1)\lambda^{-1/2}.$$

Then, we have

$$\begin{aligned} K \leq \sum_{l=0}^{r} \int_{a_2}^{b_2} \Bigg\{ & \int_{t+l\lambda^{-1/2}}^{t+(l+1)\lambda^{-1/2}} \varphi(u) W(\lambda, t, u) |u-t|^{2k-1} \\ & \cdot \left(\int_{t}^{t+(l+1)\lambda^{-1/2}} \varphi(w) \left| df^{(2k-1)}(w) \right| \right) du \\ & + \int_{t-(l+1)\lambda^{-1/2}}^{t-l\lambda^{-1/2}} \varphi(u) W(\lambda, t, u) |u-t|^{2k-1} \\ & \cdot \left(\int_{t-(l+1)\lambda^{-1/2}}^{t} \varphi(w) \left| df^{(2k-1)}(w) \right| \right) du \Bigg\} \, dt. \end{aligned}$$

Let $\varphi_{t,c,d}(w)$ denote the characteristic function of the interval

$$\left[t - c\lambda^{-1/2}, t + d\lambda^{-1/2} \right]$$

where c, d are non-negative integers. Now proceeding along the lines of [**9**, p. 70] we obtain, after using Lemma 2.1 and Fubini's theorem:

$$\begin{aligned} K &\leq C_2 \lambda^{-(2k+1)/2} \Bigg\{ \sum_{l=1}^{r} l^{-4} \left(\int_{a_1}^{b_1} \left(\int_{w-(l+1)\lambda^{-1/2}}^{w} dt \right) \left| df^{(2k-1)}(w) \right| \right. \\ &\quad \left. + \int_{a_1}^{b_1} \left(\int_{w}^{w+(l+1)\lambda^{-1/2}} dt \right) \left| df^{(2k-1)}(w) \right| \right) \\ &\quad + \int_{a_1}^{b_1} \left(\int_{w-\lambda^{-1/2}}^{w+\lambda^{-1/2}} dt \right) \left| df^{(2k-1)}(w) \right| \Bigg\} \\ &\leq C_3 \lambda^{-k} \left\| f^{(2k-1)}(w) \right\|_{\mathrm{BV}(I_1)}. \end{aligned}$$

Hence, $\|J_2\|_{L_1(I_2)} \leq C_4 \lambda^{-k} \left\| f^{(2k-1)} \right\|_{\mathrm{BV}(I_1)}$, where C_4 is a constant which depends on k.

For all $u \in [0, \infty) \setminus [a_1, b_1]$ and all $t \in I_2$, we can choose a $\delta > 0$ such that $|u - t| \geq \delta$. Therefore

$$\begin{aligned} \|S_\lambda(F(u,t)(1-\varphi(u)), t)\|_{L_1(I_2)} &\leq \int_{a_2}^{b_2} \int_0^\infty W(\lambda, t, u) |f(u)| (1 - \varphi(u)) du \, dt \\ &\quad + \sum_{i=0}^{2k-1} \frac{1}{i!} \int_{a_2}^{b_2} \int_0^\infty W(\lambda, t, u) |f^{(i)}(t)| \, |u-t|^i (1 - \varphi(u)) du \, dt \\ &= J_4 + J_5, \quad \text{say}. \end{aligned}$$

For sufficiently large u there exist positive constants R_0 and C_6 such that

$$\frac{(u-t)^{2k}}{u^{2k}+1} > C_6 \quad \text{for all } u \geq R_0,\, t \in I_2.$$

By Fubini's theorem

$$\begin{aligned} J_4 &= \left(\int_0^{R_0} \int_{a_2}^{b_2} + \int_{R_0}^{\infty} \int_{a_2}^{b_2} \right) W(\lambda,t,u)|f(u)|(1-\varphi(u))dt\,du \\ &= J_6 + J_7, \quad \text{say.} \end{aligned}$$

Next, by using Lemma 2.1, we have

$$\begin{aligned} J_6 &\leq C_7 \lambda^{-k} \left(\int_0^{R_0} |f(u)|du \right), \\ J_7 &\leq \frac{1}{C_6} \int_{R_0}^{\infty} \int_{a_2}^{b_2} W(\lambda,t,u) \frac{(u-t)^{2k}}{(u^{2k}+1)} |f(u)|dt\,du \leq C_8 \lambda^{-k} \left(\int_{R_0}^{\infty} |f(u)|du \right) \end{aligned}$$

Hence, $J_4 \leq C_9 \lambda^{-k} \|f\|_{L_1([0,\infty))}$. Further, using (2.1) and [**3**] we get

$$J_5 \leq C_{10} \lambda^{-k} \left(\|f\|_{L_1(I_2)} + \left\| f^{(2k-1)} \right\|_{L_1(I_2)} \right).$$

Combining the estimates of J_4 and J_5 we have

$$\|J_3\|_{L_1(I_2)} \leq C_{11} \lambda^{-k} \left(\|f\|_{L_1[0,\infty)} + \left\| f^{(2k-1)} \right\|_{L_1(I_2)} \right).$$

The result (2.6) follows.

Proof of Theorem 2.1. Let $f_{\eta,2k}(u)$ be the Steklov mean of $2k$-th order corresponding to $f(u)$ where $\eta > 0$ is sufficiently small and $f(u)$ is defined to be zero outside $[0,\infty)$. Then we have

$$\begin{aligned} &\|S_{\lambda,k}(f,\cdot) - f\|_{L_p(I_2)} \\ &\qquad \leq \|S_{\lambda,k}(f - f_{\eta,2k},\cdot)\|_{L_p(I_2)} + \|S_{\lambda,k}(f_{\eta,2k},\cdot)\|_{L_p(I_2)} + \|f_{\eta,2k} - f\|_{L_p(I_2)} \\ &\qquad = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad \text{say.} \end{aligned}$$

To estimate Σ_1, let $\varphi(u)$ be the characteristic function of I_3. Then

$$\begin{aligned} S_\lambda((f - f_{\eta,2k})(u),t) &= S_\lambda(\varphi(u)(f - f_{\eta,2k})(u),t) + S_\lambda((1-\varphi(u))(f - f_{\eta,2k})(u),t) \\ &= \Sigma_4 + \Sigma_5, \quad \text{say.} \end{aligned}$$

The following is true for $p = 1$; the truth for $p > 1$ follows from Hölder's inequality.

$$\int_{a_2}^{b_2} |\Sigma_4|^p dt \leq \int_{a_2}^{b_2} \int_{a_3}^{b_3} W(\lambda,t,u)|(f - f_{\eta,2k})(u)|^p du\,dt$$

Now, applying Fubini's theorem, we get

$$\int_{a_2}^{b_2} |\Sigma_4|^p dt \le \int_{a_3}^{b_3}\int_{a_2}^{b_2} W(\lambda,t,u)|(f-f_{\eta,2k})(u)|^p dt\,du \le \|f-f_{\eta,2k}\|^p_{L_p(I_3)}\,.$$

Hence, $\|\Sigma_4\|_{L_p(I_2)} \le \|f-f_{\eta,2k}\|_{L_p(I_3)}$. Using Hölder's inequality (2.1) and Fubini's theorem we get the following for $p \ge 1$:

$$\|\Sigma_5\|_{L_p(I_2)} \le C_1\lambda^{-k}\,\|f-f_{\eta,2k}\|_{L_p[0,\infty)}\,.$$

Now, using Jensen's inequality and Fubini's theorem we obtain $\|f_{\eta,2k}\|_{L_p[0,\infty)} \le C_2\,\|f\|_{L_p[0,\infty)}$. Hence $\|\Sigma_5\|_{L_p(I2)} \le C_3\lambda^{-k}\,\|f\|_{L_p[0,\infty)}$. Again by the property of Steklov means, we get

$$\Sigma_1 \le C_4\left\{\omega_{2k}(f,\eta,p,I_1) + \lambda^{-k}\,\|f\|_{L_p[0,\infty)}\right\}.$$

It is well known that

$$\left\|f^{(2k-1)}_{\eta,2k}\right\|_{\mathrm{BV}(I_3)} \le \left\|f^{(2k)}_{\eta,2k}\right\|_{L_1(I_3)}.$$

Therefore by virtue of Lemma 2.6 (for $p \ge 1$) and Lemma 2.5 we have

$$\begin{aligned}\Sigma_2 &\le C_5\lambda^{-k}\left\{\left\|f^{(2k)}_{\eta,2k}\right\|_{L_p(I_3)} + \|f_{\eta,2k}\|_{L_p[0,\infty)}\right\}\\ &\le C_6\lambda^{-k}\left\{\eta^{-(2k)}\omega_{2k}(f,\eta,p,I_1) + \|f\|_{L_p[0,\infty)}\right\},\end{aligned}$$

in view of the properties of Steklov means.

To estimate Σ_3, we use the Steklov means property (iii) and obtain that $\Sigma_3 \le C_6\omega_{2k}(f,\eta,p,I_1)$. The result follows. $\Diamond$

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