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APPROXIMATION OF CONTINUOUS FUNCTIONS BY MONOTONE SEQUENCES OF GENERALIZED POLYNOMIALS WITH RESTRICTED COEFFICIENTS

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Abstract. The problem of approximation of continuous functions by generalized polynomials with restricted coefficients was considered in [**2**–**3**] and [**4**–**6**]. In [**1**] we have obtained some results regarding the approximation by monotonous sequences of ordinary polynomials with restricted coefficients. The aim of this paper is to extend the results of [**1**] to the case of approximation by generalized polynomials with restricted coefficients.

1. Introduction

Replacing the ordinary polynomials by generalized polynomials, the results regarding the approximation by ordinary polynomials with restricted coefficients was firstly extended in [**2**–**3**] and [**5**].

Some important generalizations of those results were obtained in [**6**] in the following manner.

Let $-\infty < a < b < +\infty$ and $C_0([a,b];\mathbf{C}) = \{f : [a,b] \to \mathbf{C} : f$ continuous on $[a,b]$ with $f(a) = 0\}$, where $\mathbf{C}$ is the field of complex numbers. If $K = (K_k)$ is a sequence of functions $K_k \in C_0([a,b];\mathbf{C})$ and $D = (D_k)$ is a sequence of numbers $D_k > 0$, $k = 1, 2, \dots$, we define $P_{K,D}(\mathbf{C})$ to be the class of all linear combinations g, $g(t) = \sum_{k=1}^{N} a_k K_k(t)$ (a_k — complex) with the restrictions that $|a_k| \le D_k$, $k = 1, 2, \dots, N$. Also, if $K_k \in C_0([a,b];\mathbf{R}) = \{f : [a,b] \to \mathbf{R} : f$ continuous on $[a,b]$ and $f(a) = 0\}$, then we define $P_{K,D}(\mathbf{R})$ to be the class of all linear combinations g, $g(t) = \sum a_k K_k(t)$, with a_k real numbers such that $|a_k| \le D_k$.

In [**6**], among other results, the following two were proved:

THEOREM 1.1 [**6**, Theorem 3]. *If $K_k = t^{\lambda_k}$, $\lambda_k > 0$, $\lambda_{k+1} - \lambda_k \ge c > 0$, $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$ and $D_k = A_k^{\lambda_k}$ with $A_k > 0$ $(k = 1, 2, \dots)$, then for $f \in C_0([0,1];\mathbf{R})$*

AMS Subject Classification (1980): Primary 41 A 10, Secondary 41 A 29 *there exists a sequence* $g_n \in P_{K,D}(\mathbf{R})$, $n = 1, 2, \ldots$, *uniformly converging toward* f *on* $[0,1]$, *if and only if there exists a subsequence* (k_i) *of* (k) *such that*

$$\sum_{i=1}^{\infty} \lambda_{k_i}^{-1} = \infty \quad \text{and} \qquad A_{k_i} \to \infty \quad (i \to \infty). \tag{1}$$

THEOREM 1.2 [**6**, Theorem 5]. *If* $K_k(t) = t^{\lambda_k}$, $0 < \lambda_k \to b$ $(k \to \infty)$, $0 < b < \infty$ *with* $\lambda_i \neq \lambda_j$ $(i \neq j)$ *and* $D_k > 0$ $(k = 1, 2, \ldots,)$ *then, for any* $f \in C_0([0,1]; \mathbf{R})$, *there exists a sequence* $g_n \in P_{K,D}(\mathbf{R})$, $n = 1, 2, \ldots$, *uniformly converging toward* f *on* $[0,1]$, *if and only if*

$$\sum_{k=1}^{\infty} D_k \left|\lambda_k - b\right|^p = \infty, \qquad \text{for all } p = 0, 1, 2, \ldots. \tag{2}$$

Remark. In fact, in [**6**], those results were proved for $f \in C_0([0,1]; \mathbf{C})$, $g_n \in P_{K,D}(\mathbf{C})$ being complex function. But it is clear that, if $f \in C_0([0,1]; \mathbf{R})$, then $g_n \in P_{K,D}(\mathbf{C})$ are considered to be real-valued functions $\big(g_n(t) = \sum_{k=1}^{N_n} a_k^{(n)} t^{\lambda_k}$, with $a_k^{(n)} \in \mathbf{R}$, $k = 1, 2, \ldots, N_n\big)$; therefore, $g_n \in P_{K,D}(\mathbf{R})$.

In this paper we shall extend the results in [**1**] to the case of Theorems 1.1 and 1.2, using in their proofs an important remark, communicated to me by Professor D. Leviatan.

2. Basic Results

In the following, for $a > 0$, let us denote by $\langle a \rangle$ the least integer such that $a \leq \langle a \rangle$ and let us denote by $C_0^{\langle a \rangle}([0,1]; \mathbf{R}) = \{f : [0,1] \to \mathbf{R} : f$ continuous on $[0,1]$ and $f(0) = f'(0) = \ldots = f^{(\langle a \rangle)}(0) = 0\}$, where $f^{(\langle a \rangle)}(0)$ denotes the derivative of order $\langle a \rangle$ of f at the point 0.

Let (λ_k), (A_k) be two sequences or real numbers satisfying

$$1 \leq A_k, \quad k = 1, 2, \ldots, \qquad A_k \xrightarrow{k} \infty, \tag{3}$$

$$0 < \lambda_k, \quad \lambda_{k+1} - \lambda_k \geq c > 0, \quad (k = 1, 2, \ldots), \quad \sum_{k=1}^{\infty} \lambda_k^{-1} = \infty. \tag{4}$$

Regarding the approximation by monotone sequences, to Theorem 1.1 there corresponds

THEOREM 2.1. *Assume that* (3) *and* (4) *hold. For any* $f \in C_0^{\langle () \lambda_1 \rangle}([0,1]; \mathbf{R})$ *there exists a sequence of generalized polynomials* (P_n),

$$P_n(t) = \sum_{k=1}^{i_n} b_k^{(n)} t^{\lambda_k}, \quad \text{with } b_k^{(n)} \in \mathbf{R},\ n = 1, 2, \ldots,\ t \in [0,1]$$

such that $P_n \to f$ *uniformly on* $[0,1]$, $|b_k^{(n)}| \le A_k^{\lambda_k}$, $k = \overline{1, i_n}$, $n = 1, 2, \dots$, *and*

$$f(t) < P_{n+1}(t) < P_n(t) \quad \text{for all } t \in (0,1], \quad P_n(0) = 0, \ n = 1, 2, \dots .$$

Proof. Take $F(t) = f(t)/t^{\lambda_1}$, $t \in (0,1]$, $F(0) = 0$. Since $f \in C_0^{\langle \lambda_1 \rangle}([0,1]; \mathbf{R})$ we obtain:

$$\lim_{t \to 0} \frac{f(t)}{t^{\lambda_1}} = \lim_{t \to 0} \frac{f'(t)}{\lambda_1 t^{\lambda_1 - 1}} = \dots = \lim_{t \to 0} \frac{1}{M_0} \cdot f^{(\langle \lambda_1 \rangle)}(t) t^{\lambda_1 - \langle \lambda_1 \rangle} = 0$$

(where $M_0 = \lambda_1(\lambda_1 - 1) \cdot \ldots \cdot (\lambda_1 - \langle \lambda_1 \rangle + 1)$), and therefore $F \in C_0([0,1]; \mathbf{R})$.

Now let us denote by $\mu_k = \lambda_{k+1} - \lambda_1$ and $L = (L_k)$, $L_k(t) = t^{\mu_k}$. Using an idea of D. Leviatan, communicated to me through a personal letter, let us denote by $B_k = A_{k+1}^{k/(k+1)}$, $C = (C_k)$, $C_k = B_k^{\mu_k}$. Because of (3) it is obvious that $0 < B_k$ and $B_k \xrightarrow{k} +\infty$.

Since

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k+1} - \lambda_k} > \sum_{k=1}^{\infty} \frac{1}{\lambda_{k+1}} = +\infty,$$

we obtain $\sum_{k=1}^{\infty} 1/\mu_k = +\infty$. Also, $0 < \mu_k$, $\mu_{k+1} - \mu_k = \lambda_{k+2} - \lambda_{k+1} \ge c > 0$, $k = 1, 2, \dots$, and, therefore, taking into account Theorem 1.1, the set $P_{L,C}(\mathbf{R})$ is dense in $C_0([0,1]; \mathbf{R})$ in the sense of the uniform norm.

Then, for $F \in C_0([0,1]; \mathbf{R})$, there exists a sequence $R_n \in P_{L,C}(\mathbf{R})$, $R_n(t) = \sum_{k=1}^{j_n} a_k^{(n)} t^{\mu_k}$ such that $|F(t) - R_n(t)| < 1/[n(n+1)]$, for all $t \in (0,1]$ and all $n = 1, 2, \dots$, where

$$|a_k^{(n)}| \le B_k^{\mu_k}, \qquad k = 1, 2, \dots, j_n, \quad n = 1, 2, \dots . \tag{5}$$

Hence

$$|f(t) - t^{\lambda_1} R_n(t)| < t^{\lambda_1}/[n(n+1)], \qquad \forall t \in (0,1], \ n = 1, 2, \dots . \tag{6}$$

Take $Q_n(t) = t^{\lambda_1} R_n(t)$ and $S_n(t) = Q_n(t) + 2t^{\lambda_1}/n$, $t \in [0,1]$, $n = 1, 2, \dots$. From (6) it is evident that $Q_n \xrightarrow{n} f$ uniformly on $[0,1]$ and, therefore, $S_n \to f$, uniformly on $[0,1]$. Then, by (6), we obtain

$$\begin{aligned} |Q_n(t) - Q_{n+1}(t)| &\le |Q_n(t) - f(t)| + |f(t) - Q_{n+1}(t)| \\ &< \frac{t^{\lambda_1}}{n(n+1)} + \frac{t^{\lambda_1}}{(n+1)(n+2)} < 2 \cdot \frac{t^{\lambda_1}}{n(n+1)}, \end{aligned}$$

for all $t \in (0,1]$ and all $n = 1, 2, \dots$, and, therefore,

$$S_n(t) - S_{n+1}(t) = Q_n(t) - Q_{n+1}(t) + 2t^{\lambda_1}/[n(n+1)] > 0$$

for all $t \in (0,1]$ and $S_n(0) = S_{n+1}(0) = 0$, for all $n = 1, 2, \dots$. But

$$\begin{aligned} S_n(t) &= \frac{2t^{\lambda_1}}{n} + t^{\lambda_1} \sum_{k=1}^{j_n} a_k^{(n)} t^{\mu_k} = \frac{2t^{\lambda_1}}{n} + \sum_{k=1}^{j_n} a_k^{(n)} t^{\lambda_{k+1}} \\ &= \frac{2t^{\lambda_1}}{n} + \sum_{k=2}^{j_n+1} a_{k-1}^{(n)} t^{\lambda_k} = \sum_{k=1}^{i_n} b_k^{(n)} t^{\lambda_k}, \end{aligned}$$

where $i_n = j_n + 1$, $b_1^{(n)} = 2/n$ and $b_k^{(n)} = a_{k-1}^{(n)}$, $k = 2, \dots, i_n$.

Taking now into account (3), (4) and (5), we obtain: there exists an $n_0 \in \mathbf{N}$, such that $b_1^{(n)} = 2/n \le A_1^{\lambda_1}$ for all $n \ge n_0$ and then

$$|b_k^{(n)}| = |a_{k-1}^{(n)}| \le B_{k-1}^{\mu_{k-1}} = A_k^{\mu_{k-1}(k-1)/k} \le A_k^{\mu_{k-1}} = A_k^{\lambda_k - \lambda_1} \le A_k^{\lambda_k},$$

$k = 2, \dots, i_n$, $n = 1, 2, \dots$. Hence, it is evident that $P_n(t) = S_{n+n_0}(t)$, $n = 1, 2, \dots$, satisfies the conclusions of Theorem 2.1.

Remarks. 1°. If, in the previous proof, we consider $S_n(t) = Q_n(t) - 2t^{\lambda_1}$, then it can easily be seen that $(S_n)_{n \ge n_0}$ is a monotonously increasing sequence in $(0, 1]$.

2°. For $\lambda_k = k$, $k = 1, 2, \dots$, we obtain a more general version of Theorem 2.1 in [**1**] in the sense that the monotonicity condition on the sequence A_k in [**1**] is completely unnecessary.

3°. Suppose that $\lambda_1 \ge 1$ is an integer. Then, as it was also pointed out by D. Leviatan (in the case of $\lambda_1 = 1$, see M.R.90d – 41010) the condition $f \in C^{\langle \lambda_1 \rangle}([0, 1]; \mathbf{R})$ in Theorem 2.1 can be replaced by

$$f \in \left\{ f \in C[0,1] : f(0) = \dots = f^{(\lambda_1 - 1)}(0) = 0,\ |f^{(\lambda_1)}(0)/(\lambda_1!)| < A_1^{\lambda_1} \right\}.$$

Indeed, denote

$$F(x) = f(x) - f'(0)x - f''(0)x^2/2! - \dots - f^{(\lambda_1)}x^{\lambda_1}/\lambda_1!.$$

Then, since obviously $F(0) = F'(0) = \dots = F^{(\lambda_1)}(0) = 0$, following the proof of Theorem 2.1, there is a generalized polynomial sequence (F_n) satisfying $F_n \to f$ uniformly on $[0, 1]$,

$$F(x) < F_{n+1}(x) < F_n(x), \quad F_n(0) = 0, \quad x \in (0, 1], \quad n \ge n_0,$$

where

$$F_n(x) = \frac{2x^{\lambda_1}}{n} + \sum_{k=2}^{i_n} b_k^{(n)} x^{\lambda_k} \quad \text{and} \quad |b_k^{(n)}| \le A_k^{\lambda_k}, \quad k = \overline{2, i_n}.$$

Hence, we obtain,

$$\begin{aligned} f(x) - f'(0)x - \dots - f^{(\lambda_1)}(0)\frac{x^{\lambda_1}}{\lambda_1!} &< \frac{2x^{\lambda_1}}{n+1} + \sum_{k=2}^{i_{n+1}} b_k^{(n+1)} x^{\lambda_k} \\ &< \frac{2x^{\lambda_1}}{n} + \sum_{k=2}^{i_n} b_k^{(n)} x^{\lambda_k}, \end{aligned}$$

that is

$$\begin{aligned} f(x) &< f'(0)x + \dots + f^{(\lambda_1)}(0)\frac{x^{\lambda_1}}{\lambda_1!} + \frac{2x^{\lambda_1}}{n+1} + \sum_{k=2}^{i_{n+1}} b_k^{(n+1)} x^{\lambda_k} \\ &< f'(0)x + \dots + f^{(\lambda_1)}(0)\frac{x^{\lambda_1}}{\lambda_1!} + \frac{2x^{\lambda_1}}{n} + \sum_{k=2}^{i_n} b_k^{(n)} x^{\lambda_k}. \end{aligned}$$

Denoting now by

$$S_n(x) = f'(0)x + \cdots + f^{(\lambda_1)}(0)\frac{x^{\lambda_1}}{\lambda_1!} + \frac{2x^{\lambda_1}}{n} + \sum_{k=2}^{i_n} b_k^{(n)} x^{\lambda_k},$$

it is obvious that if $f(0) = \ldots = f^{(\lambda_1-1)}(0) = 0$ and $|f^{(\lambda_1)}(0)/(\lambda_1!)| < A_1^{\lambda_1}$, for all $n \geq n_1$. As a conclusion, the sequence (P_n) in Theorem 2.1 can be chosen by $P_n(x) = S_{n+n_1}(x)$.

In the following, let (λ_k), (D_k) be two sequences satisfying

$$\lambda_k \in \mathbf{R}, \quad 0 < \lambda_k \uparrow b, \quad 0 < b < +\infty \tag{7}$$

$$\begin{gathered} D_k \in \mathbf{R}, \quad 0 < D_k, \quad k = 1, 2, \ldots, \\ \sum_{k=1}^{\infty} D_k (b - \lambda_k)^p = +\infty, \quad \text{for all } p = 0, 1, \ldots. \end{gathered} \tag{8}$$

Regarding the approximation by monotone sequences, to Theorem 1.2 there corresponds

Theorem 2.2. *Assume that* (7) *and* (8) *hold. For any* $f \in C_0^{\langle\lambda_1\rangle}([0,1];\mathbf{R})$ *there exists a sequence of generalized polynomials* (P_n), $P_n(t) = \sum_{k=1}^{i_n} b_k^{(n)} t^{\lambda_k}$, $b_k^{(n)} \in \mathbf{R}$, *such that* $P_n \to f$ *uniformly on* $[0,1]$, $|b_k^{(n)}| \leq D_k$, $k = \overline{1, i_n}$, $n = 1, 2, \ldots$, *and* $f(t) < P_{n+1}(t) < P_n(t)$ *for all* $t \in (0,1]$, $P_n(0) = 0$, $n = 1, 2, \ldots$.

Proof. Taking $F(t)/t^{\lambda_1}$, $t \in (0,1]$, $F(0) = 0$, as in proof of Theorem 2.1, we have $F \in C_0([0,1];\mathbf{R})$. Now, let us denote by $\mu_k = \lambda_{k+1} - \lambda_1$ and $L = (L_k)$, $C = (C_k)$, defined by $L_k(t) = t^{\mu_k}$, $C_k = D_{k+1}$, $k = 1, 2, \ldots$. Since $\mu_k \uparrow b - \lambda_1 = b_1 > 0$ (from (7)) and

$$\sum_{k=1}^{\infty} C_k (b_1 - \mu_k)^p = \sum_{k=1}^{\infty} D_{k+1} (b - \lambda_{k+1})^p = +\infty, \quad \text{for } p = 0, 1, \ldots,$$

(from (8)), taking into account Theorem 2.1, we get that the set $P_{L,C}(\mathbf{R})$ is dense in $C_0([0,1];\mathbf{R})$ in the uniform norm. Then, for $F \in C_0([0,1];\mathbf{R})$, there exists a sequence $R_n(t) = \sum_{k=1}^{j_n} a_k^{(n)} t^{\mu_k} \in P_{L,C}(\mathbf{R})$, such that $|F(t) - R_n(t)| < 1/[n(n+1)]$, for all $t \in (0,1]$ and all $n = 1, 2, \ldots$, where

$$|a_k^{(n)}| \leq C_k = D_{k+1}, \qquad k = 1, 2, \ldots, j_n, \quad n = 1, 2, \ldots. \tag{9}$$

Taking $S_n(t) = t^{\lambda_1} R_n(t) + 2t^{\lambda_1}/n$ and using the same arguments as in the proof of Theorem 2.1, we obtain that $S_n \to f$ uniformly on $[0,1]$, $S_n(t) - S_{n+1}(t) > 0$, for all $t \in (0,1]$, $S_n(0) = 0$ for all $n = 1, 2, \ldots$, and $S_n(t) = \sum_{k=1}^{i_n} b_k^{(n)} t^{\lambda_k}$, where $i_n = j_n + 1$, $b_1^{(n)} = 2/n$ and $b_k^{(n)} = a_{k-1}^{(n)}$, $k = 2, \ldots, i_n$.

Taking into account (9), we obtain that $|b_k^{(n)}| = |a_{k-1}^{(n)}| \leq D_k$, $k = 2, \ldots, i_n$, $n = 1, 2, \ldots$. Also, from $D_1 > 0$, there obviously exists an $n_0 \in \mathbf{N}$ such that $b_1^{(n)} = 2/n < D_1$ for all $n > n_0$; therefore it is self-evident that $P_n(t) = S_{n+n_0}(t)$, $n = 1, 2, \ldots$, satisfies the conclusions of Theorem 2.2.

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