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SOME ESTIMATES OF THE INTEGRAL $\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|(2\pi)^{-1}\,d\theta$

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Abstract. We investigate some estimates of the integral $\displaystyle\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\frac{d\theta}{2\pi}$, if the polynomial $P(z)$ has a concentration at low degrees measured by the l_p-norm, $1 \le p \le 2$. We also obtain better estimates for some concentrations than those obtained in [**1**].

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients and let d be a real number such that $0 < d \le 1$. We say that $P(z)$ has a concentration d of degrees of at most k, measured by the l_p-norm $(p \ge 1)$, if

$$\Bigl(\sum_{j\le k} |a_k|^p\Bigr)^{1/p} \ge d\Bigl(\sum_{j\ge 0} |a_j|^p\Bigr)^{1/p}. \tag{1}$$

Polynomials with concentrations of low degrees were introduced by B. Beauzamy and P. Enflo; this plays an important role in the construction of an operator on a Banach space with no non-trivial invariant subspace. We investigate here the estimates of the integral $\displaystyle\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\frac{d\theta}{2\pi}$ of such polynomials. In the following, we shall normalize $P(z)$ under the l_p-norm and also assume that

$$\Bigl(\sum_{j\ge 0} |a_j|^p\Bigr)^{1/p} = 1. \tag{2}$$

Otherwise, the concentration of polynomials is measured by some of the well-known norms: $|P|_p$ $(p \ge 1)$, $|P|_2 = \|P\|_2$, $|P|_\infty$, $\|P\|_\infty, \ldots$. For details see [**1**].

Similarly, as in [**1**, Lemme 3] (case $p = 2$) and [**2**, Theorem 1] (case $p = 1$) we have the following results:

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Theorem 1. *Let* $P(z)=\sum_{j\ge 0} a_j z^j$ *be a polynomial which satisfies* (1) *and* (2). *Then*:

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\frac{d\theta}{2\pi} \ge \sup_{1<t\le 3} f_{d,k}(t), \quad \textit{where}$$

$$f_{d,k}(t)=\begin{cases} t\operatorname{Log} d\left(\dfrac{t-1}{t+1}\right)^{k+1}-\dfrac{1}{2}t^2, & 1<p\le 2\\[2ex] t\operatorname{Log} d\left(\dfrac{t-1}{t+1}\right)^{k+1}, & p=1\end{cases}$$

(see also [**3**, Lemma 3.2; p. 28, 29]).

Theorem 2. *Let* $P(z)$ *be a polynomial as in Theorem* 1. *Then*:

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\frac{d\theta}{2\pi} \ge \sup_{1<t<+\infty} f_{d,k,p}(t), \quad \textit{where}$$

$$f_{d,k,p}(t)=\begin{cases} \dfrac{t}{p}\operatorname{Log} d^p\dfrac{\left(\dfrac{t+1}{t-1}\right)^p-1}{\left(\dfrac{t+1}{t-1}\right)^{p(k+1)}-1}-\dfrac{1}{2}t^2, & 1<p\le 2\\[4ex] t\operatorname{Log}\dfrac{2d}{(t-1)\left[\left(\dfrac{t+1}{t-1}\right)^{k+1}-1\right]}, & p=1\end{cases}$$

(for the case $p=1$ see [**2**, Theorem 1]).

For proofs of the Theorems 1 and 2 we use (as in [**1**, Lemme 3] and [**2**, Theorem 1] (see also [**3**])) the following well known facts

1° $a_j=\displaystyle\int_0^{2\pi}\frac{P(re^{i\theta})}{r^j e^{ij\theta}}\cdot\frac{d\theta}{2\pi}$ if $0<r<1$.

2° $|a_j|\le |P(z_0)|\dfrac{1}{r^j}$, where $|P(z_0)|=\max\limits_{|z|=r}|P(z)|$.

3° The classical Jensen's inequality and the known transformation:

$$\operatorname{Log}|P(z_0)|\le\int_0^{2\pi}\operatorname{Log}\left|P\left(\frac{z_0+e^{i\theta}}{1+\bar z_0 e^{i\theta}}\right)\right|\frac{d\theta}{2\pi}=\int_0^{2\pi}\operatorname{Log}|P(e^{i\theta})|\frac{1-r^2}{|1-\bar z_0 e^{i\theta}|^2}\frac{d\theta}{2\pi},$$

where $|z_0|=r$.

4° If $0<r<1$ then $\dfrac{1-r}{1+r}\le\dfrac{1-r^2}{|1-\bar z_0 e^{i\theta}|^2}\le\dfrac{1+r}{1-r}$.

5° $\displaystyle\int_0^{2\pi}\operatorname{Log}|P(e^{i\theta})|\frac{d\theta}{2\pi}=\int_{\operatorname{Log}|P|<0}+\int_{\operatorname{Log}|P|>0}$, and

$$\begin{aligned}\int_{\operatorname{Log}|P|>0}&=\frac{1}{2}\int_{\operatorname{Log}|P|>0}\operatorname{Log}|P|^2<\frac{1}{2}\int_{\operatorname{Log}|P|>0}|P|^2<\frac{1}{2}\int_0^{2\pi}|P|^2\frac{d\theta}{2\pi}\\ &=\frac{1}{2}\|P\|_2^2=\frac{1}{2}|P|_2^2<\frac{1}{2}|P|_p^2=\frac{1}{2}\end{aligned}$$

because the l_p-norm decreases with p.

Finally, we get the functions $f_{d,k}(t)$ and $f_{d,k,p}(t)$ after the change of variables $t=(1+r)/(1-r)$.

Taking $t=2$ and $1<p\le 2$, we have the Beauzamy-Enflo's estimate from [**1**]:

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \ge 2\operatorname{Log}\frac{d}{e\cdot 3^{k+1}}.$$

From the following proposition and Corollaries 1 and 3, it follows that this is not the best possible estimate.

Proposition 1. *Let $P(z)$ be a polynomial as in Theorem 1. Then there exists a $t_k\in\,]1,3]$ such that*

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \ge f_{d,k}(t_k) \ge \begin{cases} 2\operatorname{Log}\dfrac{d}{e\cdot 3^{k+1}}; & 1<p\le 2;\\ 2\operatorname{Log}\dfrac{d}{3^{k+1}}; & p=1.\end{cases}$$

Proof. First observe that $\lim\limits_{t\to 1+} f_{d,k}(t)=-\infty$ and the function $f_{d,k}(t)$ has the form

$$f_{d,k}(t)=t\operatorname{Log}d+t(k+1)\operatorname{Log}(t-1)-t(k+1)\operatorname{Log}(t+1)-t^2/2,\quad 1<p\le 2.$$

We find derivatives:

$$\begin{aligned} f'_{d,k} &= \operatorname{Log}d+(k+1)\operatorname{Log}(t-1)-(k+1)\operatorname{Log}(t+1)\\ &\quad -t+t(k+1)\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\\ f''_{d,k} &= \frac{2(k+1)}{t-1}-\frac{2(k+1)}{t+1}-1+t(k+1)\left(\frac{1}{(t+1)^2}-\frac{1}{(t-1)^2}\right)\\ f'''_{d,k} &= \frac{3(k+1)}{(t+1)^2}-\frac{3(k+1)}{(t-1)^2}+2t(k+1)\left(\frac{1}{(t-1)^3}-\frac{1}{(t+1)^3}\right). \end{aligned}$$

It is clear that $\lim\limits_{t\to 1+} f''_{d,k}=-\infty$ and $f''_{d,k}(3)<0$. Since $f'''_{d,k}(t)>0$, $t\in\,]1,3]$, it follows that $f''_{d,k}(t)<0$, hence $f'_{d,k}(t)$ decreases. We also observe that $\lim\limits_{t\to 1+} f'_{d,k}(t)=+\infty$. Hence, there exists exactly one $t_k\in\,]1,3]$ such that $f'_{d,k}(t_k)=0$ or $f'_{d,k}(t)>0$ for each $t\in\,]1,3]$. This proves the proposition. The case $p=1$ can be treated similarly.

Corollary 1. *Let $P(z)$ be a polynomial as in Theorem 1. Then for every $d\in\,]0,1]$ and $k\in\{0,1,2,3,4,5,6,7\}$ there exists a $t_k\in\,]1,2[$, such that*

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \ge f_{d,k}(t_k) > 2\operatorname{Log}\frac{d}{e\cdot 3^{k+1}},\quad 1<p\le 2.$$

For the case $p=1$ a similar result does not hold.

Proof. Since

$$f'_{d,k}(3) = \frac{4}{3}k - \frac{2}{3} - (k+1)\operatorname{Log} 3 + \operatorname{Log} d = 0.235k - 1.773 + \operatorname{Log} d, \quad \operatorname{Log} 3 = 1.098$$

it follows that $f'_{d,k}(2) < 0$, for each $d \in]0,1]$ and $k \in \{0,1,\dots,7\}$. Hence,

$$\max_{1<t\le 3} f_{d,k}(2) > f_{d,k}(2) = 2\operatorname{Log}\frac{d}{e\cdot 3^{k+1}}.$$

If $p=1$, we have

$$f'_{d,k}(2) = \big(4/3 - \operatorname{Log} 3\big)k + \big(4/3 - \operatorname{Log} 3\big) + \operatorname{Log} d = 0.235k + 0.235 + \operatorname{Log} d \gtrless 0.$$

COROLLARY 2. *Let $P(z)$ be a polynomial as in Theorem 1. Then for every $d \in]0,1]$ and $k > 7$ for which $\operatorname{Log}(3^{k+1}/d)$ is a rational number, there exists a $t_k \in]1,3]$, $t_k \ne 2$, such that*

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \ge f_{d,k}(t_k) > f_{d,k}(2), \qquad 1 \le p \le 2.$$

Proof. In both cases ($1 < p \le 2$, $p = 1$) we have that $f'_{d,k}(2) = 0$ iff $\frac{4}{3}k - \frac{2}{3} = \operatorname{Log}\frac{3^{k+1}}{d}$, that is $\frac{4}{3}k + \frac{4}{3} = \operatorname{Log}\frac{3^{k+1}}{d}$.

COROLLARY 3. *Let $P(z)$ be a polynomial as in Theorem 1. Then for every $d \in]0,1]$ there exists a $k_1 \in \mathbb{N}$ such that for $k > k_1$:*

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \ge f_{d,k}(3) = \begin{cases} 3\operatorname{Log}\dfrac{d}{e^{3/2}2^{k+1}}, & 1 < p \le 2 \\ 3\operatorname{Log}\dfrac{d}{2^{k+1}}, & p = 1 \end{cases}$$
$$> \begin{cases} 2\operatorname{Log}\dfrac{d}{e\cdot 3^{k+1}}, & 1 < p \le 2 \\ 2\operatorname{Log}\dfrac{d}{3^{k+1}}, & p = 1. \end{cases}$$

Proof. Since

$$f'_{d,k}(3) = \frac{3}{4}k - \frac{9}{4} - (k+1)\operatorname{Log} 2 + \operatorname{Log} d = 0.057k - 2.943 + \operatorname{Log} d,$$

we have that $\max_{1<t\le 3} f_{d,k}(t) = f_{d,k}(3)$ ($1 < p \le 2$) iff $f'_{d,k}(3) \ge 0$. Hence, it follows that

$$k_1 = \left[\frac{(9/4) + \operatorname{Log} 2 - \operatorname{Log} d}{(3/4) - \operatorname{Log} 2}\right] = [51.634 - 17.543\operatorname{Log} d].$$

Similarly, for $p = 1$ there exists the corresponding number k_1.

COROLLARY 4. *Let $P(z)$ be a polynomial as in Theorem 1. Then, for every $d \in\,]0,1]$ and $k \in \{0,1,2,\dots,51\}$, there exists a $t_k \in\,]1,3]$, such that*

$$\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi} \geq f_{d,k}(t_k) > f_{d,k}(3) = 3\operatorname{Log}\frac{d}{e^{3/2}2^{k+1}}, \quad 1 < p \leq 2.$$

Proof. This is clear from the equality

$$f'_{d,k}(3) = \frac{3}{4}k - \frac{9}{4} - (k+1)\operatorname{Log}2 = 0.057k - 2.943 + \operatorname{Log}d, \qquad 1 < p \leq 2.$$

Since for $p = 1$ we have that $f'_{d,k}(3) = 0.057k + 0.057 + \operatorname{Log}d$, it follows that the conclusion is not the same as in the case $1 < p \leq 2$.

We shall now analyse the estimate of the integral $\displaystyle\int_0^{2\pi} \operatorname{Log}|P(e^{i\theta})|\,\frac{d\theta}{2\pi}$ with the function $f_{d,k,p}(t)$ as in Theorem 2. The following results can be compared with [**2**, Th. 2, Lemmas 3 and 4]. Firstly, we represent $f_{d,k,p}(t)$ in the form:

$$f_{d,k,p} = h_{d,p}(t) + g_k(t) - \frac{1}{p}\cdot t\cdot\operatorname{Log}\left[1 - \left(\frac{t-1}{t+1}\right)^{p(k+1)}\right],$$

where (see [**2**])

$$\begin{aligned} h_{d,p} &= t\operatorname{Log}d - \frac{1}{2}t^2 + \frac{t}{p}\operatorname{Log}[(t+1)^p - (t-1)^p] \\ g_k(t) &= kt\operatorname{Log}(t-1) - (k+1)t\operatorname{Log}(t+1). \end{aligned}$$

It is clear that $f_{d,k,p}(t) > h_{d,p}(t) + g_k(t)$, $t > 1$. We shall now prove the following.

PROPOSITION 2. *The function $h_{d,p}(t) + g_k(t)$ takes its maximum value at a point (unique) t_k such that $t_k \to +\infty$, when $k \to +\infty$.*

Proof. We essentially use the same argument as in [**2**]. From [**2**] it follows that $g''_k(t) < 0$, $t > 1$. Now, we find derivatives for $h_{d,p}(t)$

$$\begin{aligned} h'_{d,p}(t) &= \operatorname{Log}d - t + \frac{1}{p}\operatorname{Log}[(t+1)^p - (t-1)^p] + t\cdot\frac{(t+1)^{p-1} - (t-1)^{p-1}}{(t+1)^p - (t-1)^p}; \\ h''_{d,p}(t) &= -1 + 2\cdot\frac{(t+1)^{p-1} - (t-1)^{p-1}}{(t+1)^p - (t-1)^p} \\ &\quad + \frac{t(p-1)}{A^2(t)}\Big([(t+1)^{p-2} - (t-1)^{p-2}]A(t) - p[(t+1)^{p-1} - (t-1)^{p-1}]^2\Big), \end{aligned}$$

where $A(t) = (t+1)^p - (t-1)^p$.

Since $p \in\,]1,2]$, $t > 1$, it is clear that

$$h''_{d,p}(t) < 0 \qquad \text{iff} \qquad -1 + 2\cdot\frac{(t+1)^{p-1} - (t-1)^{p-1}}{A(t)} < 0.$$

But, this is true iff $\varphi_p(t) < 0$, where $\varphi_p(t) = 2(t+1)^{p-1} - 2(t-1)^{p-1} - (t+1)^p + (t-1)^p$. Hence, we find that

$$\varphi_p'(t) = 2(p-1)[(t+1)^{p-2} - (t-1)^{p-2}] + p[(t-1)^{p-1} - (t+1)^{p-1}] < 0.$$

This shows that $h_{d,p}''(t) + g_k''(t) < 0$. Since

$$\lim_{t\to 1+} \big(h_{d,p}'(t) + g_k'(t)\big) = +\infty \qquad \text{and} \qquad \lim_{t\to +\infty} \big(h_{d,p}'(t) + g_k'(t)\big) = -\infty,$$

equation $h_{d,p}'(t) + g_k'(t) = 0$ has exactly one solution t_k. From the equality $h_{d,p}'(t) + g_k'(t) = 0$ we get with $t = t_k$,

$$k = \frac{(t^2-1)\operatorname{Log}(t+1) + t^2 - t - (t^2-1)h_{d,p}'(t)}{2t + (t^2-1)\operatorname{Log}(t-1) - (t^2-1)\operatorname{Log}(t+1)},$$

wherefrom we easily deduce that $t_k \to +\infty$.

Remark 1. From the Proposition 1 it follows that the function $f_{d,k,p}(t)$ $(1 < p \le 2)$ has the same behaviour as the function $f_{d,k}(t)$ from [**2**]. If $p = 2$ we get

$$f_{d,k,2}(t) = t\operatorname{Log}\frac{2d}{t-1}\sqrt{\frac{t}{((t+1)/(t-1))^{2k+2} - 1}} - \frac{1}{2}t^2,$$

which is the answer to the remark from [**2**, p. 223].

For the function $f_{d,k,2}(t)$ we have the following results

PROPOSITION 3. *Let $f_{d,k,2}(t)$ be the function from Theorem 2 $(p = 2)$. Then, when $k \to +\infty$*

1° $\dfrac{4}{3}\dfrac{k}{t_k^4} \to 1$;

2° $t_k \operatorname{Log}\left[1 - \left(\dfrac{t_k - 1}{t_k + 1}\right)^{2(k+1)}\right] \to 0$;

3° *$f_{d,k,2}(t_k)$ and $h_{d,2}(t_k) + g_k(t_k)$ are asymptotically equivalent.*

Namely, $f_{d,k,2}(t) = t\operatorname{Log} d - \dfrac{1}{2}t^2 + \dfrac{t}{2}\operatorname{Log} 4t + g_k(t)$, where $g_k(t)$ is same as in [**2**]. The proof is similar as in [**2**], i.e. it uses the Taylor expansion of $\log(1 \pm x)$, $x \to 0$.

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