

ON A CLASS OF p -VALENT ANALYTIC FUNCTIONS
DEFINED BY RUSCHEWEYH DERIVATIVE

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Introduction. Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$, ($p \in \mathbf{N} = \{1, 2, 3, \dots\}$) which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by $f * g(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $A(p)$. Following Goel and Sohi [2] we put,

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z), \quad (n > -p)$$

for the $(n+p-1)$ -th order Ruscheweyh derivative of $f(z) \in A(p)$. Let h be convex univalent in E with $h(0) = 1$.

Definition 1. We say that a function $f(z) \in A(p)$ is in $T_{n,p}(h)$ if and only if $(D^{n+p} f(z))' / (p z^{p-1}) \prec h(z)$, $z \in E$.

We will prove that $T_{n,p}(h) \subset T_{n-1,p}(h)$ and that f is preserved under the Bernardi integral operator under certain conditions. Finally some coefficient estimates for the class will be also obtained.

We require the following theorems which provide a method for finding the best dominant for certain differential subordinations.

THEOREM A [1]. Let β and γ be complex constants and let h be convex (univalent) in E with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. If $p(z) = 1 + p_1 z + \dots$ is analytic in E , then $p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.

THEOREM B [1]. Let β and γ be complex constants and let h be convex in E with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. Let $p(z) = 1 + p_1 z + \dots$ be analytic in E and let it satisfy the differential subordination

$$(1) \quad p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z).$$

If the differential equation $q(z) + \frac{zp'(z)}{\beta q(z) + \gamma} = h(z)$ with $q(0) = 1$ has a univalent solution $q(z)$, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (1).

COROLLARY A [1]. Let $p(z)$ be analytic in E and let it satisfy the differential subordination,

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 - (1 - 2\delta)z}{1 + z} \equiv h(z) \quad \text{with } \beta > 0 \text{ and } -\operatorname{Re}(\gamma/\beta) \leq \delta < 1.$$

Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \quad q(0) = 1,$$

has a univalent solution $q(z)$. In addition $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant.

THEOREM 1. If $(n + p) > 0$, then $T_{n,p}(h) \subset T_{n-1,p}(h)$, where h is a convex univalent function in E with $h(0) = 1$.

Proof. Set $g = (D^{n+p-1}f)'/(pz^{p-1})$. Taking logarithmic derivatives and multiplying by z we get

$$(2) \quad \frac{zg'(z)}{g(z)} = z \frac{(D^{n+p-1}f)''}{(D^{n+p-1}f)'} - (p-1).$$

Using the fact

$$(3) \quad z(D^{n+p-1}f)' = (n+p)D^{n+p}f - nD^{n+p-1}f$$

(2) can be reduced to

$$(4) \quad \frac{zg'(z)}{(n+p)} + g(z) = \frac{(D^{n+p}f(z))'}{pz^{p-1}} \prec h(z),$$

since $f \in T_{n,p}(h)$. Now if $(n+p) > 0$, we can conclude, by Theorem A, that $g(z) \prec h(z)$, that is $f \in T_{n-1,p}(h)$.

Choosing $n = -p + 1$, we get the inclusion relations, $T_{n,p}(h) \subset T_{n-1,p}(h) \subset \dots \subset T_{-p+1,p}(h)$. So $f \in T_{-p+1,p}(h)$ implies $\frac{(D^0f)'}{pz^{p-1}} \prec h$ or $\frac{f'(z)}{pz^{p-1}} \prec h$. If $\operatorname{Re} h > 0$, it follows that f is p -valent, by a result due to Umezawa [6]. Hence we have the following

COROLLARY 1. Let $f \in T_{n,p}(h)$, $n+p > 0$, where $\operatorname{Re} h > 0$, $h(0) = 1$ and h is univalently convex in E . Then f is p -valent.

Remarks. However, we observe that h need not be the best dominant for g in Theorem 1. We proceed to find the best dominant for g using Theorem B. In fact if q is the best dominant for g in (4), then q should satisfy,

$$(5) \quad zq'/(n+p) + q = h$$

and q should be univalent. Hence, $q' + (n+p)z^{-1}q = (n+p)z^{-1}h$. Solving, we get the best dominant,

$$(6) \quad q(z) = \frac{(n+p)}{z^{n+p}} \int_0^z h(z) z^{n+p-1} dz;$$

h is univalently convex with $h(0) = 1$. We show that q is also univalently convex. Set $Q = 1 + zq''/q'$. Taking logarithmic derivative and multiplying by z we get,

$$(7) \quad \begin{aligned} \frac{zQ'}{Q-1} &= 1 + \frac{zq'''}{q''} - \frac{zq''}{q'} \\ Q + \frac{zQ'}{Q-1} &= 2 + \frac{zq'''}{q''}. \end{aligned}$$

From (5) we get,

$$\frac{zh''}{h'} = \frac{Q + \frac{zQ'}{Q-1} + (n+p)}{1 + \frac{(n+p+1)}{Q-1}}.$$

Using (7) this reduces to

$$1 + \frac{zh''}{h'} = Q + \frac{zQ'}{Q + (n+p)}.$$

Since h is convex, we have $(1 + zh''/h') \prec (1-z)/(1+z)$, therefore

$$\left(1 + \frac{zq''}{q'}\right) \prec \frac{1-z}{1+z}, \quad \text{if } \operatorname{Re}\left(\frac{1-z}{1+z} + (n+p)\right) > 0,$$

which is true; hence q is convex univalent.

Set $h(z) = [1 + (2\alpha - 1)z]/(1+z)$, $0 \leq \alpha < 1$, in (6) so that $\operatorname{Re} h(z) > \alpha$. The best dominant q for g in this case is given by,

$$q(z) = \frac{(n+p)}{z^{n+p}} \int_0^z z^{n+p-1} \frac{1 + (2\alpha - 1)z}{1+z} dz.$$

By integration we get,

$$(8) \quad q(z) = 1 - \frac{2(1-\alpha)(n+p)}{z^{n+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \dots - \frac{z^{n+p}}{n+p} \right) \right],$$

when $(n+p)$ is even,

$$(9) \quad q(z) = 1 + \frac{2(1-\alpha)(n+p)}{z^{n+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \dots + \frac{z^{n+p}}{n+p} \right) \right],$$

when $(n+p)$ is odd.

So we obtain the following

COROLLARY 2. *Set $p = 1$, $n = 0$ and $\alpha = 0$ in (9). Then the best dominant $q(z)$ reduces to $q(z) = 2z^{-1} \log(1+z) - 1$.*

We note that $\operatorname{Re} q(e^{i\theta}) = \operatorname{Re} q(e^{-i\theta})$ and $\operatorname{Im} q(e^{i\theta}) = -\operatorname{Im} q(e^{-i\theta})$. Furthermore $\operatorname{Re} q(-1) = +\infty$. Hence the curve given by the set of points $q(e^{i\theta})$, $0 \leq \theta \leq 2\pi$ is symmetrical about the real axis and since q is convex, $\operatorname{Re} q(e^{i\theta})$ is minimum at $\theta = 0$ and the minimum value is $\operatorname{Re} q(1) = 2 \log 2 - 1 = .38\dots$

THEOREM 2. *If $f(z) \in T_{n-1,p}(h)$, then*

$$(10) \quad F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \in T_{n-1,p}(h) \quad \text{for } c+p > 0.$$

Proof. From (10) we have,

$$(11) \quad z(D^{n+p-1}F)' = (c+p)(D^{n+p-1}f) - c(D^{n+p-1}F).$$

Differentiating (11) we get,

$$(12) \quad z(D^{n+p-1}f)'' + (D^{n+p-1}F)' = (c+p)(D^{n+p-1}f)' - c(D^{n+p-1}F)'.$$

Set $G = (D^{n+p-1}F)' / (pz^{p-1})$. Taking logarithmic derivatives and multiplying by z we get,

$$\frac{zG'(z)}{G(z)} = \frac{z(D^{n+p-1}F)''}{(D^{n+p-1}F)'} - (p-1).$$

Using (12) this reduces to,

$$(13) \quad \frac{zG'(z)}{(c+p)} + G(z) = \frac{(D^{n+p-1}f)'}{pz^{p-1}} \prec h.$$

Since $f \in T_{n-1,p}(h)$. If $(c+p) > 0$, we conclude, by Theorem A, that $G \prec h$, that is $(D^{n+p-1}F)' / (pz^{p-1}) \prec h$. If q is the best dominant for G in (13), then q should satisfy $zg'(z)/(c+p) + q(z) = h(z)$. Solving it we get the best dominant

$$q(z) = \frac{(c+p)}{z^{c+p}} \int_0^z h(z) z^{c+p-1} dz.$$

If we choose

$$h(z) = \frac{1 + (2\alpha - 1)z}{1+z}, \quad 0 \leq \alpha < 1,$$

then the best dominant in this case is given by,

$$(14) \quad q(z) = 1 - \frac{2(1-\alpha)(c+p)}{z^{c+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \dots - \frac{z^{c+p}}{c+p} \right) \right],$$

when $(c+p)$ is even,

$$(15) \quad q(z) = 1 + \frac{2(1-\alpha)(c+p)}{z^{c+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \dots + \frac{z^{c+p}}{c+p} \right) \right],$$

when $(c+p)$ is odd.

Evidently $q(z)$ is convex, since h is so.

COROLLARY 3. Taking $\alpha = 0$, $p = 1$ and $c = 1$, (14) reduces to $q(z) = 4z^{-1} - 4z^{-2} \log(1+z) - 1$. Here again we can show that $\operatorname{Re} q(e^{i\theta}) = \operatorname{Re} q(e^{-i\theta})$, $\operatorname{Im} q(e^{i\theta}) = -\operatorname{Im} q(e^{-i\theta})$ and $\operatorname{Re} q(-1) = +\infty$. Hence the curve extends to ∞ and since the curve is convex, it is minimum at $\theta = 0$ and the minimum value is $\operatorname{Re} q(1) = 3 - 4 \log 2 = .227\dots$

This is an improvement of the results of Goel and Sohi [3, Remarks (i) and (ii)].

THEOREM 3. The class $T_{n,p}(h)$ is closed with respect to convex combination, where h is univalently convex and $h(0) = 1$.

Proof. Let $f, g \in T_{n,p}(h)$. Therefore $(D^{n+p}f)/(pz^{p-1}) \prec h$ and $(D^{n+p}g)/(pz^{p-1}) \prec h$. Hence there exist points z_1, z_2 in E such that

$$\frac{(D^{n+p}f)'}{pz^{p-1}} = h(z_1) \quad \text{and} \quad \frac{(D^{n+p}g)'}{pz^{p-1}} = h(z_2).$$

Let $F = tf + (1-t)g$, $0 < t < 1$. Then

$$\frac{(D^{n+p}F)'}{pz^{p-1}} = t \frac{(D^{n+p}f)'}{pz^{p-1}} + (1-t) \frac{(D^{n+p}g)'}{pz^{p-1}} = th(z_1) + (1-t)h(z_2) = h(z_3)$$

for some z_3 in E , because h is convex. In other words $F \in T_{n,p}(h)$.

A connection between the classes $T_{n-1,p}(h)$ and $T_{m-1,p}(h)$. We now prove the following

THEOREM 4. Let $f \in T_{n-1,p}(h)$ and let

$$(16) \quad g(z) = (m+p+1)!z^{1-m} \times \int_0^z \int_0^{x_{m+p-1}} \dots \int_0^{x_2} \left[\frac{x_1^{n-1} f(x_1)}{(n+p-1)!} \right]^{(n+p-1)} dx_1 dx_2 \dots dx_{m+p-1}.$$

Then $g \in T_{m-1,p}(h)$.

Proof. From (16) we have,

$$\frac{g(z)z^{m-1}}{(m+p-1)!} = \int_0^z \int_0^{x_{m+p-1}} \dots \int_0^{x_2} \left[\frac{x_1^{n-1} f(x_1)}{(n+p-1)!} \right]^{(n+p-1)} dx_1 dx_2 \dots dx_{m+p-1}.$$

Differentiating $(m+p-1)$ times we get

$$\frac{(g(z)z^{m-1})^{(m+p-1)}}{(m+p-1)!} = \frac{(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!}.$$

Since

$$D^{n+p-1}f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!},$$

it follows that

$$\frac{(D^{m+p-1}g(z))'}{pz^{p-1}} = \frac{(D^{n+p-1}f(z))'}{pz^{p-1}}.$$

Therefore $f \in T_{n-1,p}(h)$ if and only if $g \in T_{m-1,p}(h)$.

Coefficient estimates. THEOREM 5. *If $f \in T_{n-1,p}(h)$, where h is univalently convex in E with $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$, then*

$$(17) \quad |a_{p+k}| \leq \frac{|h_1|p}{(p+k) \binom{n+p+k-1}{k}}, \quad k = 1, 2, 3, \dots$$

Proof. Our hypothesis implies

$$(18) \quad \frac{(D^{n+p-1}f)'}{pz^{p-1}} = 1 + \sum_{k=1}^{\infty} \binom{n+p+k-1}{k} \frac{(p+k)}{p} a_{p+k} z^k \prec h(z).$$

Let $h(z) = 1 + h_1 z + h_2 z^2 + \dots$, $z \in E$. From (18) we have, for $k = 1, 2, 3, \dots$

$$\frac{p+k}{p} \binom{n+p+k-1}{k} |a_{p+k}| \leq |h_1|$$

using a result due to Rogosinski [5]. Choosing $h(z) = [1 + (2\alpha - 1)z]/(1 + z)$, $0 \leq \alpha < 1$, we get the sharp coefficient estimate

$$|a_{p+k}| \leq \frac{2p(1-\alpha)}{(p+k) \binom{n+p+k-1}{k}}$$

attained for

$$f(z) = z^p + 2p(1-\alpha) \sum_{k=1}^{\infty} \frac{(-1)^k z^{p+k}}{(p+k) \binom{n+p+k-1}{k}}.$$

THEOREM 6. *If $f \in T_{n-1,p}(h)$, where h is convex univalent in E , with $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$, then for any complex number γ ,*

$$(19) \quad |a_{p+2} - \gamma a_{p+1}^2| \leq \frac{2p|h_1| \max(1, |\mu|)}{(n+p+1)(n+p)(p+2)}$$

where

$$(20) \quad \mu = \left(\frac{\gamma(n+p+1)(p+2)ph_1^2}{2(n+p)(p+1)^2} - h_2 \right).$$

The result is sharp.

Proof. Our hypothesis on f enables us to write

$$(21) \quad (D^{n+p-1}f)'/(pz^{p-1}) = h(\omega(z)),$$

where ω is analytic and $|\omega(z)| \leq |z|$ in $|z| < 1$. Let $\omega = \sum_{j=1}^{\infty} c_j z^j$; then,

$$\left\{ 1 + \sum_{k=1}^{\infty} \binom{n+p+k-1}{k} a_{p+k} \frac{(p+k)}{p} z^k \right\} = \{1 + h_1(c_1 z + c_2 z^2 + \dots) + h_2(c_1 z + c_2 z^2 + \dots)^2 + \dots\}.$$

Equating the same powers of z we get

$$(22) \quad c_1 = \frac{(n+p)(p+1)}{h_1 p} a_{p+1}$$

$$(23) \quad c_2 = \frac{1}{h_1} \left[\frac{(n+p+1)(n+p)(p+2)}{2p} a_{p+2} - \frac{h_2}{h_1^2 p^2} (p+1)^2 (n+p)^2 a_{p+1}^2 \right].$$

Define μ by (20). Then we have,

$$(24) \quad |c_2 - \mu c_1^2| = \frac{1}{|h_1|} \left| \frac{(n+p+1)(n+p)(p+2)}{2p} a_{p+2} - \frac{(h_1 \mu + h_2)}{h_1^2} \frac{(p+1)^2 (n+p)^2}{p^2} a_{p+1}^2 \right| \\ = \frac{(n+p+1)(n+p)(p+2)}{|h_1|^{2p}} |a_{p+2} - \gamma a_{p+1}^2|.$$

Using the coefficient inequality,

$$(25) \quad |c_2 - \mu c_1^2| \leq \max(1, |\mu|)$$

due to Keogh and Merkes [4], in (24) we obtain (19).

The equality is attained in (19) for the function $f(z)$ given by (21) when we choose $\omega(z) = z$ or $\omega(z) = z^2$.

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