PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 18–26

ON THE FEKETE-SZEGŐ THEOREM FOR CLOSE-TO-CONVEX FUNCTIONS

A. Chonweerayoot, D. K. Thomas and **W. Upakarnitikaset**

Abstract. Let $K(\beta)$ be the class of normalised close-to-convex functions with order $\beta \geq 0$, defined in the unit disc D by
$$\left|\arg e^{i\lambda}\frac{zf'(z)}{g(z)}\right| \leq \frac{\pi\beta}{2},$$
for $|\lambda| < \pi/2$ and g starlike in D. For $f \in K(\beta)$ with $f(z) = z + a_2z^2 + a_3z^3 + \dots$ and $z \in D$, sharp bounds are given for $|a_3 - \mu a_2^2|$ for real μ.

Let S denote the class of analytic univalent functions f, defined for $z \in D = \{z : |z| < 1\}$ by
$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$
Fekete and Szegö [**6**], showed that for $f \in S$, given by (1),
$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0 \\ 1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$
The inequalities are sharp in the sense that for each μ, there exists a function in S such that equality holds. Pfluger [**11**], [**12**] has recently considered the problem for complex μ.

Let S^* and K denote the classes of normalised starlike and close-to-convex functions respectively. Thus $f \in K$ if, and only if, there exists $g \in S^*$ and a real λ, with $|\lambda| < \pi/2$, such that for $z \in D$,
$$\operatorname{Re} e^{i\lambda}\frac{zf'(z)}{g(z)} > 0.$$

AMS Subject Classification (1991): Primary 30 C 45

We are grateful to the British Council for supporting this research and to the Forum for Theoretical Science at Chulalongkorn University for its initial and continuing encouragement.

Let K_0 be the subset of K when $\lambda = 0$. For $f \in K_0$, Keogh and Merkes [**8**] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 1/3, \\ 1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3, \\ 1, & \text{if } 2/3 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Again, for each μ, there are functions in K_0 such that equality holds in all cases.

Eenigenburg and Silvia [**5**] were able to extend the result of Keogh and Merkes to the whole class K, whilst Koepf [**9**], apparently unaware of [**5**] and [**8**], gave a proof for $\mu \in [0, 1]$.

Denote by $K(\beta)$ the class of close-to-convex functions of order $\beta \geq 0$. Thus $f \in K(\beta)$, if, and only if, for $\beta \geq 0$, there exists $g \in S^*$ and a real λ, with $|\lambda| < \pi/2$, such that for $z \in D$,

$$\left| \arg e^{i\lambda} \frac{zf'(z)}{g(z)} \right| \leq \frac{\beta\pi}{2}. \tag{2}$$

Clearly for $0 \leq \beta \leq 1$, $K(\beta)$ is a subset of S, whilst for $\beta > 1$, $K(\beta)$ can contain functions with infinite valence [**7**]. We also note that $K(0) = C$, the class of normalised convex functions. For C, the Fekete-Szegö problem has been solved in [**8**]. Let $K_0(\beta)$ be the subset of $K(\beta)$ when $\lambda = 0$. Then in [**1**] it was shown that the result of Keogh and Merkes extends to:

THEOREM A. *Let $f \in K_0(\beta)$ and be given by* (1). *Then for $0 \leq \beta \leq 1$,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu + \dfrac{\beta(2-3\mu)(\beta+2)}{3}, & \textit{if } \mu \leq \dfrac{2\beta}{3(\beta+1)}, \\ 1 - \mu + \dfrac{2\beta}{3} + \dfrac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]}, & \textit{if } \dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}, \\ \dfrac{2\beta+1}{3}, & \textit{if } \dfrac{2}{3} \leq \mu \leq \dfrac{2(\beta+2)}{3(\beta+1)}, \\ \mu - 1 + \dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \textit{if } \mu \geq \dfrac{2(\beta+2)}{3(\beta+1)}, \end{cases}$$

whilst for $\beta > 1$, the first two inequalities hold. For each μ there are functions in $K_0(\beta)$ such that equality holds in all cases.

Koepf [**10**] considered the problem for the class $K(\beta)$ and gave the solution when $\mu = 2/3$. He also showed that the first inequality in Theorem A extends to $K(\beta)$ in the case $\beta \geq 1$, for all $|\lambda| < \pi/2$, and established the sharp inequalities

$$|a_3 - a_2^2| \leq \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } 0 \leq \beta \leq 1 \\ \dfrac{\beta(\beta+2)}{3}, & \text{if } \beta \geq 1. \end{cases} \tag{3}$$

The purpose of this paper is to examine the question of extending Theorem A to $K(\beta)$.

Results

THEOREM 1. *Let* $f \in K(\beta)$ *and be given by* (1), *then for* $\beta \geq 0$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1-\mu+\dfrac{\beta(2-3\mu)(\beta+2)}{3}, & \text{if } \mu \leq \dfrac{2\beta}{3(\beta+1)}, \\ 1-\mu+\dfrac{2\beta}{3}+\dfrac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]}, & \text{if } \dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}, \end{cases}$$

provided $\cos^2\lambda \leq 1/2$ *or* $\lambda = 0$.

The inequalities are sharp in the sense that for each μ, *there exists a function in* $K(\beta)$, *such that equality holds.*

Proof. From (2) write

$$zf'(z) = g(z)\tilde{p}(z)^\beta, \tag{4}$$

for $g \in S^*$ given by $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and $\operatorname{Re} e^{i\lambda}\tilde{p}(z) > 0$ with $\tilde{p}(z) = 1 + \tilde{p}_1 z + \tilde{p}_2 z^2 + \cdots$. Thus for some p satisfying $\operatorname{Re} p(z) > 0$ and given by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, we have $\tilde{p}_n = p_n e^{-i\lambda}\cos\lambda$, so that $|\tilde{p}_n| = |p_n|\cos\lambda$ for $n \geq 1$.

Equating coefficients in (4) we have

$$\begin{aligned} 2a_2 &= b_2 + \beta\tilde{p}_1, \\ 3a_3 &= b_3 + \frac{\beta(\beta-1)}{2}\tilde{p}_1^2 + \beta\tilde{p}_2 + \beta\tilde{p}_1 b_2. \end{aligned}$$

and so

$$\begin{aligned} a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{\beta}{3}\left(\tilde{p}_2 + \left(\frac{\beta(2-3\mu)}{4} - \frac{1}{2}\right)\tilde{p}_1^2\right) \\ + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)\tilde{p}_1 b_2. \end{aligned} \tag{5}$$

Since $\dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}$, it follows from (5) that

$$\begin{aligned} |a_3 - \mu a_2^2| \leq 1 - \mu + \frac{\beta\cos\lambda}{3}\left(2 - \frac{|p_1|^2}{2}(1 - |\sin\lambda|)\right) \\ + \frac{\beta^2}{12}(2-3\mu)|p_1|^2\cos^2\lambda + \frac{\beta(2-3\mu)}{3}|p_1|\cos\lambda, \end{aligned} \tag{6}$$

where we have used the inequalities $|b_3 - \nu b_2^2| \leq \max\{1, |4\nu - 3|\}$ for $g \in S^*$ with ν real [**8**], $|b_2| \leq 2$ and

$$\left|\tilde{p}_2 - \frac{\tilde{p}_1^2}{2}\right| \leq \cos\lambda\left(2 - \frac{|p_1|^2}{2}(1 - |\sin\lambda|)\right),$$

proved in [**9**].

Now write $u = |p_1|$ and $v = \cos\lambda$. Then (6) can be written as $|a_3 - \mu a_2^2| \leq \phi(u,v)$, where

$$\begin{aligned}\phi(u,v) =& 1-\mu+\frac{\beta v}{3}\left(2-\frac{u^2}{2}\left(1-\sqrt{1-v^2}\right)\right)\\ &+\frac{\beta^2u^2v^2}{12}(2-3\mu)+\frac{\beta uv}{3}(2-3\mu),\end{aligned}$$

where, since $|p_1| \leq 2$, it follows that $(u,v) \in [0,2] \times [0,1]$.

Fix $v = v_0$ and assume first that $\phi(u, v_0)$ has a turning point at u. Then $\phi'(u, v_0) = 0$ implies that

$$2u\left(1-\sqrt{1-v_0^2}\right) = \beta u v_0 X + 2X, \tag{7}$$

where $X = 2 - 3\mu$, so that $0 \leq X \leq 2/(1+\beta)$.

From (6) and (7) one obtains

$$\begin{aligned}\phi(u,v_0) &= 1-\mu+\frac{2\beta v_0}{3}+\frac{\beta u v_0}{6}X\\ &\leq 1-\mu+\frac{2\beta v_0}{3}+\frac{\beta v_0 X}{3}\\ &\leq 1-\mu+\frac{2\beta}{3}+\frac{\beta X^2}{3[2-\beta X]},\end{aligned}$$

if

$$v_0 \leq \frac{2(2-\beta X)+X^2}{(2+X)(2-\beta X)} = \Psi(\beta, X) \qquad \text{say.}$$

An elementary argument shows that $\Psi(\beta, X)$ has a minimum value of $2\sqrt{2}-2$ when $\beta \geq 0$. Next suppose that $u = 0$. Then $\phi(0,v) = 1-\mu+2\beta v/3 \leq 1-\mu+2\beta/3$. Finally let $u = 2$. Then if $v \leq 1/\sqrt{2}$,

$$\begin{aligned}\phi(2,v) &= 1-\mu+\frac{2\beta v}{3}\sqrt{1-v^2}+\frac{\beta^2v^2}{3}X+\frac{2\beta v}{3}X,\\ &\leq 1-\mu+\frac{\beta}{3}+\frac{\beta^2}{6}X+\frac{\sqrt{2}\beta}{3}X,\\ &\leq 1-\mu+\frac{2\beta}{3}+\frac{\beta X^2}{3[2-\beta X]},\end{aligned}$$

since $0 \leq X \leq 2/(1+\beta)$.

Thus in all cases, the second inequality in Theorem 1 is established, provided $v \leq 1/\sqrt{2}$.

Choosing $\lambda = 0$, $b_2 = p_2 = 2$, $b_3 = 3$ and $p_1 = \dfrac{2(2-3\mu)}{2-\beta(2-3\mu)}$ shows that the inequality is sharp on the interval $\dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}$, since $|p_1| \leq 2$.

Next consider the case $\mu \leq \dfrac{2\beta}{3(\beta+1)}$. Then

$$\begin{aligned}|a_3 - \mu a_2^2| &\leq \left|a_3 - \frac{2\beta}{3(\beta+1)}a_2^2\right| + \left(\frac{2\beta}{3(\beta+1)} - \mu\right)|a_2|^2, \\ &\leq 1 + \frac{2\beta}{3} + \left(\frac{2\beta}{3(\beta+1)} - \mu\right)(\beta+1)^2 \\ &= 1 - \mu + \frac{\beta(2-3\mu)(\beta+2)}{3},\end{aligned}$$

for $\beta \geq 0$, where we have used the result already proved in the case $\mu = 2\beta/3(\beta+1)$, and the fact that for $f \in K(\beta)$, the inequality $|a_2| \leq \beta+1$ holds [**2**], [**3**], [**4**]. Equality is attained on choosing $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$.

Remark 1. As mentioned above, Koepf [**10**] established the first inequality of Theorem 1 for all λ, such that $|\lambda| < \pi/2$, provided $\beta \geq 1$ and $\mu \geq 0$. We note that maximising the expression for $H_\beta(y)$ on page 424 gives another proof of the same inequality when $0 \leq \beta \leq 1$, provided $\cos^2\lambda \leq 1/2$ or $\lambda = 0$.

We now consider the case $\mu \geq 2/3$. We prove:

THEOREM 2. *Let $f \in K(\beta)$ and be given by* (1). *Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } \dfrac{2}{3} \leq \mu \leq \dfrac{2(\beta+2)}{3(\beta+1)}, \\ \mu - 1 + \dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \text{if } \mu \geq \dfrac{2(\beta+2)}{3(\beta+1)}, \end{cases}$$

for $0 \leq \beta \leq 1$ if $\cos^2\lambda \leq 1/2$ or if $\lambda = 0$. For $\beta \geq 1$, the inequalities hold if $\cos^2\lambda \leq (3-\sqrt{5})/2$. The inequalities are sharp.

Proof. We first deal with the case when $\mu = \dfrac{2(\beta+2)}{3(\beta+1)}$. In [**8**] it was shown that for $g \in S^*$ given by $g(z) = z + b_2z^2 + b_3z^3 + \cdots$

$$\left|b_3 - \frac{3\mu}{4}b_2^2\right| \leq 1 + \frac{|b_2^2|}{4}(3|\mu - 1| - 1). \tag{8}$$

Also, since $\mathrm{Re}\, p(z) > 0$, it follows that (see e.g. [**8**])

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1^2|}{2}. \tag{9}$$

Thus with $\mu = \dfrac{2(\beta+2)}{3(\beta+1)}$ we have from (8) that if $0 \leq \beta \leq 1$,

$$\left|b_3 - \frac{3\mu}{4}b_2^2\right| \leq 1 - \frac{\beta|b_2^2|}{2(1+\beta)}, \tag{10}$$

and so from (5), (9) and (10) we obtain

$$\begin{aligned}
\left|a_3-\frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| &\le \frac{1}{3}\left(1-\frac{\beta}{2(1+\beta)}|b_2^2|\right)+\frac{\beta\cos\lambda}{3}\left(2-\frac{|p_1^2|}{2}\right)\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1b_2|\cos\lambda}{3(1+\beta)}\\
&=\frac{2\beta\cos\lambda+1}{3}-\frac{\beta}{6(1+\beta)}\left(|b_2|-|p_1|\cos\lambda\right)^2-\frac{\beta|p_1^2|\cos\lambda}{6}\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1^2|\cos^2\lambda}{6(1+\beta)}\\
&\le\frac{2\beta\cos\lambda+1}{3}+\frac{\beta|p_1^2|\cos\lambda}{6}\left[-1+\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\cos\lambda}{1+\beta}\right]\\
&\le\frac{2\beta+1}{3},
\end{aligned}$$

if $\cos^2\lambda\le(1+\beta)/2$, or if $\cos^2\lambda=1$, where we have used the inequality $|p_1|\le 2$. Since $(1+\beta)/2$ increases for $0\le\beta\le 1$, the above inequality is valid for $\cos^2\lambda\le 1/2$.

For $\beta\ge 1$ and $\mu=\dfrac{2(\beta+2)}{3(\beta+1)}$, it follows from (8) that

$$\left|b_3-\frac{3\mu}{4}b_2^2\right|\le 1-\frac{|b_2^2|}{2(1+\beta)},$$

and so again using (5) and (9) we obtain

$$\begin{aligned}
\left|a_3-\frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| &\le \frac{1}{3}\left(1-\frac{|b_2^2|}{2(1+\beta)}\right)+\frac{\beta\cos\lambda}{3}\left(2-\frac{|p_1^2|}{2}\right)\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1b_2|\cos\lambda}{3(1+\beta)}\\
&=\frac{2\beta\cos\lambda+1}{3}-\frac{1}{6(1+\beta)}\left(|b_2|-\beta|p_1|\cos\lambda\right)^2-\frac{\beta|p_1^2|\cos\lambda}{6}\\
&\quad+\frac{\beta|p_1^2|\cos^2\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta^2|p_1^2|\cos^2\lambda}{6(1+\beta)}\\
&\le\frac{2\beta\cos\lambda+1}{3}+\frac{\beta|p_1^2|\cos\lambda}{6}\left[-1+\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta\cos\lambda}{1+\beta}\right]\\
&\le\frac{2\beta+1}{3},
\end{aligned}$$

if $\cos^2\lambda\le\left(1+3\beta-\sqrt{(5\beta+3)(\beta-1)}\right)/\left(2(1+\beta)\right)$, again since $|p_1|\le 2$. Since $\left(1+3\beta-\sqrt{5(\beta+3)(\beta-1)}\right)/\left(2(1+\beta)\right)$ decreases for $\beta\ge 1$, the inequality is valid for $\cos^2\lambda\le(3-\sqrt{5})/2$.

Next suppose that $\frac{2}{3} \leq \mu \leq \frac{2(\beta+2)}{3(\beta+1)}$. Then writing

$$a_3 - \mu a_2^2 = \frac{(\beta+1)(3\mu-2)}{2}\left(a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right) + \frac{3(\beta+1)}{2}\left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right)\left(a_3 - \frac{2}{3}a_2^2\right),$$

the result follows on using the Theorem already proved at the end points $\mu = 2/3$ and $\mu = \frac{2(\beta+2)}{3(\beta+1)}$.

Finally let $\mu \geq \frac{2(\beta+2)}{3(\beta+1)}$. Then, since

$$a_3 - \mu a_2^2 = \left(a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right) + \left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right)a_2^2,$$

the result follows again on using the case $\mu = \frac{2(\beta+2)}{3(\beta+1)}$ already established and the inequality $|a_2| \leq 1+\beta$, proved in [**7**]. Equality is attained when $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$.

Remark 2. An examination of the proof of Theorem 2 in the case $0 \leq \beta \leq 1$ when $\mu = \frac{2(\beta+2)}{3(\beta+1)}$ shows that

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \leq \frac{1}{3} + \frac{2\beta}{3}\psi_1(\cos\lambda),$$

where

$$\psi_1(t) = t\left[\sqrt{1 - \left(\frac{1+2\beta}{(1+\beta)^2}\right)t^2} + \frac{t}{1+\beta}\right].$$

An elementary argument shows that ψ_1 attains its maximum at $t_0 \in (0,1)$ when

$$t_0^2 = \frac{2(1+\beta)^2 + (1+\beta)\sqrt{2(1+\beta)}}{4(1+2\beta)},$$

and that

$$\psi_1(t_0) = \frac{(1+\beta)[\sqrt{2(1+\beta)}+1]}{2(1+2\beta)}.$$

Thus if $0 \leq \beta \leq 1$ and $|\lambda| < \pi/2$,

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \leq \frac{1}{3} + \frac{\beta(1+\beta)[\sqrt{2(1+\beta)}+1]}{3(1+2\beta)}. \tag{11}$$

Similarly, for $\beta \geq 1$, one obtains

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \leq \frac{1}{3} + \frac{2\beta}{3}\psi_2(\cos\lambda),$$

where

$$\psi_2(t) = t\left[\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)t^2}+\frac{\beta t}{1+\beta}\right].$$

It is easy to see that ψ_2 increases on $[0,1]$ and so for $|\lambda| < \pi/2$,

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \le \frac{1}{3} + \frac{4\beta^2}{3(1+\beta)}. \tag{12}$$

It is unlikely that either of (11) or (12) is sharp.

Finally, it is easy to see that, using (3), the following result obtains:

Theorem 3. *Let $f \in K(\beta)$ and be given by* (1). *Then if* $0 \le \beta \le 1$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } \dfrac{2}{3} \le \mu \le 1 \\ \dfrac{2\beta+1}{3} + (\mu-1)(1+\beta)^2, & \text{if } \mu \ge 1, \end{cases}$$

and if $\beta \ge 1$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2\beta+1}{3} + \dfrac{(\beta^2-1)(3\mu-2)}{3}, & \text{if } \dfrac{2}{3} \le \mu \le 1 \\ \mu - 1 + \dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \text{if } \mu \ge 1. \end{cases}$$

We note that if $0 \le \beta \le 1$, the inequality for $2/3 \le \mu \le 1$ is sharp when $\lambda = 0$, $b_2 = 0$, $b_3 = 1$, $p_1 = 0$ and $p_2 = 2$. When $\beta \ge 1$, the inequality for $\mu \ge 1$ is sharp for $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$. The inequality for $0 \le \beta \le 1$ and $\mu \ge 1$ appears sharp only when $\mu = 1$, and the inequality for $\beta \ge 1$ when $2/3 \le \mu \le 1$ appears sharp only at the end points $\mu = 2/3$ and $\mu = 1$. However, in view of Theorem A, splitting the real line at $\mu = 1$ is probably not optimum, unless $\beta = 1$.

REFERENCES

[**1**] H.R. Abdel-Gawad and D.K. Thomas, *The Fekete-Szegő problem for strongly close-to-convex functions*, Proc. Amer. Math. Soc. **114** (1992), 345–349.

[**2**] D. Aharonov and S. Friedland, *On an inequality connected with the coefficient conjecture for functions of bounded boundary rotation*, Ann. Acad. Sci. Fenn. **A1 524** (1972), 14pp.

[**3**] D.A. Brannan, *On coefficient problems for certain power series*, London Math. Soc. Lecture Series Notes **12** (1974), 17–27.

[**4**] D.A. Brannan, J.G. Clunie and W.E. Kirwan, *On the coefficient problem for functions of bounded boundary rotation*, Ann. Acad. Sci. Fenn. **A1 523** (1973), 1–18.

[**5**] P.J. Eenigenburg. and E.M. Silvia, *A coefficient inequality for Bazilevič functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. **A 27** (1973), 5–12.

[**6**] Fekete and Szegő, *Eine Bermerkung über ungerade schlichte Funktionen*, J. London Math. Soc. **8** (1933), 85–89.

[**7**] A.W. Goodman, *On close-to-convex functions of higher order*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **15** (1972), 17–30.

[**8**] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc. **20** (1969), 8–12.

[**9**] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions*, Proc. Amer. Math. Soc. **101** (1987), 89–95.

[**10**] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions 2*, Arch. Math. **49** (1987), 420–433.

[**11**] A. Pfluger, *The Fekete-Szegő inequality for complex parameters*, Complex Variables **7** (1986), 149–160.

[**12**] A. Pfluger, *On the functional* $|a_3 - \lambda a_2^2|$ *in the class* S, Complex Variables **10** (1988), 83–95.

D. K. Thomas,
Department of Mathematics and Computer Science,
University of Wales,
Swansea SA2 8PP, Wales, U.K.

(Received 20 01 1992)

A. Chonweerayoot and W. Upakarnitikaset
Department of Mathematics, Faculty of Science,
Chulalongkorn University,
Bangkok 10330, Thailand