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ON MEASURABILITY OF UNCOUNTABLE UNIONS OF MEASURABLE SETS

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Abstract. This paper deals exclusively with the question of Lebesgue measurability of subsets of the set **R** of all real numbers that are the unions of measurable subsets of **R**. Thus, in what follows every set is a set of real numbers and all references to measure are in the sense of Lebesgue.

It is well known that a countable union of measurable sets is measurable. However, there also exist uncountable unions of measurable sets which are measurable. For instance, the closed unit interval $[0,1]$ which is measurable is the union of uncountably, in fact $2^{\aleph_0}$ many of its singletons each of which is measurable.

On the other hand, in the setting of ZFC (Zermelo-Fraenkel Set Theory with the Axiom of Choice), there exist also nonmeasurable sets which are also unions of uncountably many measurable sets. For instance, the well known classical examples of nonmeasurable sets [**3**, p. 135] are unions of $2^{\aleph_0}$ many of their singletons.

However, it is also well known that there exist set-theoretical models [**5**] of ZF (with the negation of the Axiom of Choice) in which every set is measurable. Thus in these models any union of measurable sets is measurable. From the above it follows that the question of measurability of unions of measurable sets depends on the underlying Set Theory and consequently on the Set-theoretical models to which these sets belong.

Since the presence of the Axiom of Choice is desirable, in view of the existence of nonmeasurable sets which are unions of $2^{\aleph_0}$ many measurable sets, the question referred to in the above acquires the following form:

In which Set-theoretical models for ZFC is the following a valid statement:

(1) *For every cardinal $k < 2^{\aleph_0}$ the union $\bigcup_{u<k} M_u$ of k many measurable sets M_u is measurable.*

First however, we observe that by virtue of the following theorem (which is valid in ZF), it suffices to answer the above question exclusively for the case of sets of measure zero.

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Theorem 1. *For every cardinal k if the union of k many sets of measure zero is of measure zero, then the union of k many measurable sets is measurable.*

Proof. Assume that the union of k many sets of measure zero is of measure zero. We show that the set $M = \bigcup_{u<k} M_u$ where each M_u is a measurable set is measurable. It is well known [**1**] that M has a measurable kernel L, i.e., $L \subseteq M$ such that every measurable subset of $M - L$ is of measure zero. Clearly, $M_u - L$ (since it is the difference of two measurable sets) is a measurable subset of $M - L$. Thus for every $u < k$, it is the case that $M_u - L$ is of measure zero. Consequently, by our assumption $\bigcup_{u<k}(M_u - L)$ is of measure zero. But then $M = L \cup \left(\bigcup_{u<k}(M_u - L)\right)$ and thus M is the union of two measurable sets. Hence, the set M is measurable, as desired.

Now returning to statement (1), let us observe that it is valid in every model for ZFC + CH (where CH is the Continuum Hypothesis). Clearly in every model for ZFC + CH if $k < 2^{\aleph_0}$ then k is a countable cardinal. However, since every countable union of measurable sets is measurable, statement (1) is trivially valid in these models.

In view of the above, we are led to consider models for ZFC + $\neg$CH (i.e., ZFC with the negation of the Continuum Hypothesis). Indeed in Theorem 2 below, we prove that statement (1) is valid in every model for ZFC + $\neg$CH + MA, where MA is the Martin's Axiom. The latter states:

(2) *Let $(P, \leq)$ be a partially ordered set with the c.i.c. property and let $(D_u)_{u<k}$ with $k < 2^{\aleph_0}$ be a family of k many dense subsets D_u of P. Then there exists a filter F of P such that F has a nonempty intersection with every D_u.*

For the sake of completeness of the paper we recall the definitions of "c.i.c.", "dense", and "filter" mentioned in (2).

Let $(P, \leq)$ be a partially ordered set. Two elements X and Y od P are called *incompatible* iff *X and Y have no nonzero lower bound*, where by *zero* (0) we mean the minimum element of P (if P has a minimum element at all).

The partially ordered set $(P, \leq)$ has the *c.i.c.* property iff *every subset of P of pairwise incompatible elements is countable.*

A subset D of the partially ordered set $(P, \leq)$ is called a *dense* subset of P iff *for every $p \in P$ there exists a $d \in D$ such that $d \leq p$.*

A subset F of the partially ordered set $(P, \leq)$ is called a *filter* of P iff

(3) (i) *$0 \in F$ and every finite subset of F has a lower bound in F*

(ii) *if $x \in F$ and $y \geq x$ then $y \in F$.*

Property (ii) is not used in this paper.

Before proving Theorem 2, we give some examples to show that the restrictions on the partially ordered set $(P, \leq)$ to have the c.i.c. property and for the cardinal k to be $< 2^{\aleph_0}$ are essential in (2).

First, we show that the c.i.c. property in (2) cannot be dropped.

To this end, we consider the following example. Let $(P, \leq)$ be the set of all finite functions from ω into ω_1 (i.e., $f \in P$ iff Domain of f is a finite subset of ω) and let $f \leq g$ in P iff f is an extension of g i.e., $f \supseteq g$.

We observe that $(P, \leq)$ does not have the c.i.c. property. This is because $\{\{(0, u)\} : u \in \omega_1\}$ is an uncountable set of pairwise incompatible elements of P.

Now, we show that in this example, statement (2) does not hold even though $k = \omega_1 < 2^{\aleph_0}$. Indeed, for every $u \in \omega_1$ it can be readily seen that

$$D_u = \{p : p \in P \textit{ and } u \in \text{Range of } p\} \tag{4}$$

is a dense subset of P. Now consider the family $(D_u)_{u<\omega_1}$ of dense subsets of P, where $\omega_1 < 2^{\aleph_0}$ since (2) refers to ZFC + $\neg$CH + MA.

We show that there cannot be a filter F in P which has a nonempty intersection with every dense subset D_u of P. This is because otherwise, $\bigcup F$ by (4) would be a function from ω onto ω_1 which is a contradiction. Therefore, we have shown that the c.i.c. property is essential in (2).

In the next example, we show also that $k < 2^{\aleph_0}$ in (2) cannot be dropped, even though the partial order involved has the c.i.c. property. To this end, let $(P, \leq)$ be the set of all finite dyadic sequences (i.e., finite sequences whose terms consist only of 0's or 1's) and let $s \leq q$ in P iff "*s is an extension of q*". Clearly, $(P, \leq)$ has the c.i.c. property since P itself is countable. Moreover, $(P, \leq)$ has $2^{\aleph_0}$ dense subsets since $(P, \leq)$ has an infinite descending chain [**2**]. But then $(P, \leq)$ cannot have a filter F which intersects every dense subset of P. This is because, as is easily verified, the complement $P - F$ of F is a dense subset of P. Therefore, the condition $k < 2^{\aleph_0}$ in (2) cannot be dropped.

Next, we prove the main preliminary Theorem.

THEOREM 2. *In every set-theoretical model for* ZFC + MA, *for every cardinal* $k < 2^{\aleph_0}$ *the union* $\bigcup_{u<k} Z_u$ *of* k *many sets* Z_u *of measure zero is of measure zero.*

Proof. Let $Z = \bigcup_{u<k} Z_u$ be a union of k many sets of measure zero. To show that Z is of measure zero, it suffices to show that for every $\varepsilon > 0$ there exists an open set V such that $Z \subseteq V$ and $m(V) \leq \varepsilon$ (where m is the Lebesgue measure). We show this based on Martin's Axiom in connection with the partially ordered set $(P, \leq)$ described below:

Let P be the set of all open sets X of $\mathbf{R}$ each of measure $< \varepsilon$ i.e.,

$$P = \{X : X \subseteq \mathbf{R} \textit{ and } m(X) < \varepsilon\}. \tag{5}$$

We partially order P by reverse inclusion, i.e.

$$X \leq Y \quad \textit{if} \quad X \supseteq Y \quad \textit{for every} \quad X, Y \in P. \tag{6}$$

Thus for every two elements X and Y of P

$$X \textit{ is incompatible with } Y \textit{ iff } m(X \cup Y) \geq \varepsilon. \tag{7}$$

We first observe that $(P, \supseteq)$ has the c.i.c. property. Let I be a set of pairwise incompatible elements of P and let

$$I_n = \{X : X \in I \textit{ and } m(X) < (1 - \tfrac{1}{n})\varepsilon\} \quad \textit{with} \quad n \geq 2. \tag{8}$$

Clearly,

$$I = \bigcup_{n=2}^{\infty} I_n. \tag{9}$$

To prove that $(P, \supseteq)$ has the c.i.c. property, in view of (9) it suffices to show that I_n as given in (8) is a countable set for every $n \geq 2$.

To this end, let us observe that to every $X \in I_n$ there corresponds a subset $\overline{X}$ of X such that $\overline{X}$ is a finite union of open intervals with rational endpoints and

$$m(X - \overline{X}) < \varepsilon/n. \tag{10}$$

This is because $X \in I_n$ is an open set of finite measure say, q, and hence X is a countable union of pairwise disjoint open intervals H_i each of finite measure. Consequently, q is the sum of a convergent series of real numbers and therefore has a partial sum p such that $(q-p) < \varepsilon/(2n)$ and where p corresponds to the measure of the union of some finitely many, say, $H_1, \ldots, H_k$ open sets H_i. But then each H_i with $i = 1, \ldots, k$ can be replaced by a subinterval W_i of H_i with rational endpoints in such a way that their union $\overline{X}$ satisfies (10).

Next, based on (8) and (10), we observe that for every two elements X and Y of I_n

$$X \neq Y \quad \textit{implies} \quad \overline{X} \neq \overline{Y}. \tag{11}$$

To prove (11), let us assume to the contrary that $\overline{X} = \overline{Y}$. But then since $\overline{X} = \overline{Y}$ and $\overline{Y} \subseteq Y$, we have $\overline{X} \subseteq Y$ and therefore $Y \cup X = Y \cup (X - \overline{X})$, wherefrom it readily follows that $m(X \cup Y) \leq m(Y) + m(X - \overline{X})$ and since $Y \in I_n$ from (8) and (10) we obtain $m(Y \cup X) < (1 - \frac{1}{n})\varepsilon + \frac{1}{n}\varepsilon = \varepsilon$ which contradicts (7).

From (11) it follows, that for every element X in I_n the correspondence X to $\overline{X}$ is one to one. Since every $\overline{X}$ is a finite union of open intervals with rational endpoints, we see that I_n as given in (8) is a countable set. Therefore, I as given in (9) is also a countable set. Thus $(P, \supseteq)$ as given by (5) and (6) has the c.i.c. property, as desired.

Next, for every set Z_u of measure zero, we define the set D_u given by

$$D_u = \{X : X \in P \textit{ and } X \supseteq Z_u\} \quad \textit{with} \quad u < k < 2^{\aleph_0}. \tag{12}$$

We show that for every $u < k$ it is the case that D_u is a dense subset of P. To this end, we show that if $Y \in P$, then there exists an $X \in D_u$ such that

$$X \supseteq Y. \tag{13}$$

Since Y is an open set in P, it follows from (5) that $m(Y) < \varepsilon$ and since $m(Z_u) = 0$, we see that Z_u can be covered with an open set H such that $m(H \cup Y) < \varepsilon$. But then we let $X = H \cup Y$. Clearly, this X satisfies (13). Thus D_u as given in (12) is a dense subset of P for every $u < k < 2^{\aleph_0}$.

Now we invoke Martin's Axiom and assert the existence of a filter F of $(P, \supseteq)$ which has a nonempty intersection with every D_u for $u < k < 2^{\aleph_0}$, where for convenience we let X_u stand for an element X of D_u mentioned in (12). From (12) it follows that $X_u \supseteq Z_u$ and therefore $\bigcup_{u<k} X_u \supseteq Z = \bigcup_{u<k} Z_u$. Thus $\bigcup_{u<k} X_u$ is an open cover of Z and hence by the Lindelöf property it has a countable subcover, say $\bigcup_{u<\omega} X_u \supseteq Z$. But the X_u's are elements of the filter F and therefore, by (3) for every natural number n, we see that $(X_u)_{u<n}$ has a nonempty lower bound, say, B in F. Thus, $B \supseteq X_u$ for every $u < n$ and $m(B) < \varepsilon$ since $B \in P$ as given in (5). But then $m\big(\bigcup_{u<n} X_n\big) < \varepsilon$ from which it follows that

$$m\Big(\bigcup_{u=1}^{\infty} X_u\Big) = \lim_{n\to\infty} m\Big(\bigcup_{u<n} X_u\Big) \le \varepsilon. \tag{14}$$

Clearly, (14) shows that the open set $V = \big(\bigcup_{u=1}^{\infty} X_u\big)$ covers Z and from (14) it follows that $m(V) \le \varepsilon$. Hence, Z is of measure zero, as desired. Thus Theorem 2 is proved. Combining Theorem 2 with Theorem 1, we have the following theorem which answers our question (1).

THEOREM 3. *In every Set-theoretical model for* ZFC + MA, *any union of* $k < 2^{\aleph_0}$ *many measurable sets is measurable.*

The proofs given above are adaptations from Schoenfield [**4**].

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