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POWER MOMENTS OF THE ERROR TERM FOR THE APPROXIMATE FUNCTIONAL EQUATION OF THE RIEMANN ZETA-FUNCTION

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Abstract. Let $\zeta(s)$ be the Riemann zeta-function, $d(n)$ the number of positive divisors of the integer n, and

$$R(s;t/2\pi)=\zeta^2(s)-\sum_{n\le t/2\pi}{}' \, d(n)n^{-s}-\chi^2(s)\sum_{n\le t/2\pi}{}' \, d(n)n^{s-1},$$

where

$$\chi(s)=2^s\pi^{s-1}\sin\left(\tfrac12\pi s\right)\Gamma(1-s).$$

We obtain the following power moment estimates:

$$\int_1^T\left|R\left(\tfrac12+it;t/2\pi\right)\right|^A dt \ll \begin{cases} T^{1-\frac14 A+\varepsilon}, & 0\le A\le 4,\\ 1, & A>4.\end{cases}$$

1. Statement of results. Let $d(n)$ denote the number of positive divisors of n, γ the Euler constant, and let

$$\Delta(x)=\sum_{n\le x}{}' d(n)-x(\log x+2\gamma-1)-\frac14 \tag{1}$$

where the symbol $\sum'$ indicates that the last term is to be halved if x is an integer. Kolesnik [**7**] proved the sharper estimate of (1):

$$\Delta(x)\ll x^{35/108+\varepsilon}. \tag{2}$$

Recently this was improved by Iwaniec and Mozzochi [**3**].

The asymptotic formula

$$\int_1^T\Delta^2(x)\,dx=\frac{1}{6\pi^2}\left\{\sum_{n=1}^{\infty}d^2(n)n^{-3/2}\right\}T^{3/2}+O(T\log^5 T) \tag{3}$$

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was proved by Tong [**10**], who improved an earlier result of Cramér (see [**2**, Theorem 13.5]). The error term of (3) has been improved to $O(T\log^4 T)$ by Preissmann [**9**]. Now we suppose that A is a fixed positive number (not necessarily an integer). Ivić [**1**] has shown the power moment estimates for $\Delta(x)$:

$$\int_1^T |\Delta(t)|^A\,dt \ll \begin{cases} T^{1+(A/4)+\varepsilon}, & 0\le A\le 35/4, \quad (4)\\ T^{19/54+35A/108+\varepsilon}, & A\ge 35/4, \quad (5)\end{cases}$$

by using Kolesnik's result (2).

Let $s=\sigma+it$ $(0\le\sigma\le 1,\, t\ge 1)$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. We now define

$$R(s;t/2\pi)=\zeta^2(s)-\sum_{n\le t/2\pi}{}' \, d(n)n^{-s}-\chi^2(s)\sum_{n\le t/2\pi}{}'\, d(n)n^{s-1},$$

where $\chi(s)=2^s\pi^{s-1}\sin\bigl(\frac12\pi s\bigr)\Gamma(1-s)$. It has been shown by Motohashi [**8**] that

$$\chi(1-s)R(s;t/2\pi)=-\sqrt2(t/2\pi)^{-1/2}\Delta(t/2\pi)+O(t^{-1/4}). \tag{6}$$

We note that Jutila [**4**] gives another proof of Motohashi's result (6). The asymptotic formula

$$\int_1^T \bigl|R\bigl(\tfrac12+it;t/2\pi\bigr)\bigr|^2\,dt=\sqrt{2\pi}\Bigl\{\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2}\Bigr\}T^{1/2}+O(T^{1/4}\log T)$$

was proved by Kiuchi and Matsumoto [**5**], and the error term has been improved to $O(\log^5 T)$ by Kiuchi [**6**], where

$$h(n)=(2/\pi)^{1/2}\int_0^\infty (y+\pi n)^{-1/2}\cos\bigl(y+\tfrac14\pi\bigr)\,dy.$$

The purpose of this paper is to prove the power moment estimates for $\bigl|R\bigl(\frac12+it;t/2\pi\bigr)\bigr|$. In view of the relation (6), to search analogues of (4) and (5) for $R(s;t/2\pi)$ is an interesting problem in itself and we can prove the following estimates:

THEOREM. *For $T\ge 1$, we have*

$$\int_1^T \bigl|R\bigl(\tfrac12+it;t/2\pi\bigr)\bigr|^A\,dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0\le A\le 4,\\ 1, & A>4.\end{cases}$$

2. Proof of Theorem. In case $\sigma=1/2$ from the inequality $(a+b)^A\ll a^A+b^A$ $(a>0,\, b>0)$, and (6), it follows that

$$\bigl|R\bigl(\tfrac12+it;t/2\pi\bigr)\bigr|^A \ll (t/2\pi)^{-A/2}|\Delta(t/2\pi)|^A+t^{-A/4} \tag{7}$$

where A is a fixed positive number.

From (7) and Schwarz's inequality, it follows that

$$\int_{T/2}^{T} \left|R\left(\tfrac{1}{2}+it; t/2\pi\right)\right|^{A} dt \ll \left\{\int_{T/2}^{T} (t/2\pi)^{-A}\, dt\right\}^{1/2} \left\{\int_{T/2}^{T} |\Delta(t/2\pi)|^{2A}\, dt\right\}^{1/2} + T^{1-A/4}.$$

From (4) and (5), we have the following estimates:

$$\int_{T/2}^{T} \left|R\left(\tfrac{1}{2}+it; t/2\pi\right)\right|^{A} dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0 \le A \le 35/8, \\ T^{(73-19A)/108+\varepsilon}, & A \ge 35/8. \end{cases}$$

Replacing T by $T/2$, $T/4$, and so on, and adding we have the theorem.

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