PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 1–4

FRAGMENTS OF COMPLETE EXTENSIONS OF PA AND McDOWELL-SPECKER'S THEOREM

Ilijas Farah

Abstract. We generalise Theorem 1.4 of [**2**] and prove that for every complete extension **T** of **PA** and any $n \in \omega$ there exists a model for Σ_n–fragment of **T** that is not extendable (that is, a model with no proper strong elementary end-extension.) This is accomplished using a model called Σ_n-atomic. This result can be interpreted as "McDowell-Specker's Theorem does not hold for Σ_n-fragments of **PA**".

Basic definitions and notation. The notation is the same as in [**2**]. **PA** stands for the axiom system of Peano arithmetic (e.g. as described in [**1**, p. 40]). A formula is Σ_n (Π_n) iff the string of quanitifiers in one of its prenex normal forms, begins with $\exists$ ($\forall$), and has no more than $n-1$ quantifier alternations. A sentence is Δ_n iff it is both Σ_n and Π_n. $\mathbf{T_n}$ stands for the Σ_n-fragment of the theory **T**, that is a theory consisting of all the consequences of **T** that are Σ_n sentences.

The notations $\mathfrak{A}$ (A), $\mathfrak{B}$ (B), ... denote models (their universes), and the notations $\mathfrak{M}$ (M) and $\mathfrak{N}$ (N) denote models of **PA** (their universes.) The letters $x, y, z, \ldots$ denote variables, while the letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ denote constants. For a model $\mathfrak{A}$ of some language $\mathfrak{L}$, the theory of $\mathfrak{A}$ (denoted $\mathrm{Th}(\mathfrak{A})$) is the set of all the sentences φ of $\mathfrak{L}$ such that $\mathfrak{A} \models \varphi$. Models $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent iff $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$ (denoted $\mathfrak{A} \equiv \mathfrak{B}$.) For some set of sentences Φ (some model $\mathfrak{A}$), $\mathfrak{L}_\Phi$ ($\mathfrak{L}_\mathfrak{A}$) denotes the language of Φ (of $\mathfrak{A}$.) If z codes an ordered pair $\langle x, y\rangle$ we write $(z)_0$ for x and $(z)_1$ for y.

Definition 1. A model $\mathfrak{A}$ is said to be a Σ_n-elementary extension of a model $\mathfrak{B}$ ($\mathfrak{B} \prec_n \mathfrak{A}$) iff for any Σ_n-formula φ with m free variables and any m-tuple $\mathbf{a} \in B^m$,

$$\mathfrak{A} \models \varphi(\mathbf{a}) \qquad \text{iff} \qquad \mathfrak{B} \models \varphi(\mathbf{a}).$$

A Σ_n-elementary extension is a Σ_n-elementary end-extension iff it is also an end-extension. We say that a complete theory **T** contains some formula schemata iff **T** contains it as a set of formulas.

AMS Subject Classification (1990): Primary 03 C 20, 03 C 62

In [**2**, Theorem 1.6] it is shown that for some complete theory $\mathbf{T}$ with binary relational symbol ρ the existence of ω_1-like models for a first-order theory $\mathbf{T}$ and extendability of all (or any) countable models for $\mathbf{T}$ are equivalent to the same first-order property, certain scheme denoted by $\mathcal{R}^+$. It consists of the following sentences:

$$\mathbf{C1.}\quad \forall x \exists y \neg\rho(x,y)$$

$$\mathbf{C2.}\quad \forall x \forall y \exists z(\rho(x,z) \wedge \rho(y,z)),$$

and for all formulas $\varphi(x,u)$ of $\mathfrak{L}$:

$$\begin{aligned}\mathbf{C3.}\quad & \forall v[\forall x \exists y \forall u(\rho(x,v) \to (\varphi(x,u) \to \rho(u,y))) \to \\ & \exists y \forall x \forall u(\rho(x,v) \to (\varphi(x,u) \to \rho(u,y)))].\end{aligned}$$

These axioms first appeared in [**3**], where a proof is given that every countable model satisfying **C1**, **C2** and **C3** is extendable. In [**2**, Theorem 1.4] it is shown that for every $n \in \omega$ the theory $\mathbf{PA}_n$ does not contain the scheme $\mathcal{R}^+$. Actually, a somewhat stronger result is given — any Σ_n-fragment of True Arithmetic (that is, $\mathrm{Th}(\omega, S, +, \cdot, 0)$) does not contain $\mathcal{R}^+$.

In that proof a kind of definable ultraproduct is used. It is Σ_n-definable ultrapower of a model $\mathfrak{N}$ of $\mathbf{PA}$, a model that consists of Σ_n-definable functions modulo some ultrafilter G of Σ_n-definable sets. This model is denoted by $F_{\Sigma_n}(\mathfrak{N})/G$, and for such models a variant of Fundamental Theorem for Ultraproducts holds, namely $\mathfrak{N} \prec_n F_{\Sigma_n}(\mathfrak{N})/G$.

We will prove that the scheme $\mathcal{R}^+$ is not contained in any Σ_n-fragment of $\mathbf{T}$ (from now on, $\mathbf{T}$ stands for some (fixed) complete extension of $\mathbf{PA}$.) From this we have our

Main Theorem. *For any theory $\mathbf{T}_n$ ($\mathbf{T}$ is some complete extension of $\mathbf{PA}$) there is a model that is not extendable.*

Let $\mathfrak{M}$ be a model for $\mathbf{PA}$. An element $\mathbf{a} \in M$ is said to be Σ_n-definable in the model $\mathfrak{M}$ iff there exists a Σ_n-formula $\varphi_{\mathbf{a}}$ of $\mathfrak{L}_{\mathbf{PA}}$ such that the following holds:

$$\mathfrak{M} \models \varphi_{\mathbf{a}}(\mathbf{a}) \wedge \forall x(\varphi_{\mathbf{a}}(x) \to x = \mathbf{a})$$

(We will usually say "Σ_n-definable" instead of "Σ_n-definable in a model $\mathfrak{M}$" when no ambiguity occurs.)

By $\Sigma_n^{\mathfrak{M}}$ we denote a countable submodel of $\mathfrak{M}$ that consists of exactly those elements $\mathbf{a}$ that are Σ_n-definable in $\mathfrak{M}$. It is easily verified that $\Sigma_n^{\mathfrak{M}}$ is closed under the operations $+$ and $\cdot$.

Lemma 1. *For any model $\mathfrak{M} \models \mathbf{PA}$ the following holds:*

$$\mathfrak{M} \prec_n \Sigma_n^{\mathfrak{M}}$$

so $\Sigma_n^{\mathfrak{M}} \models \mathrm{Th}_n(\mathfrak{M})$.

Proof. Suppose that $\varphi(x,y)$ is Σ_n-formula and $\mathbf{a} \in \Sigma_n^{\mathfrak{M}}$ is such that

$$\mathfrak{M} \models \exists x \varphi(x, \mathbf{a}).$$

It is enough to show that there is a Σ_n-definable $\mathbf{b} \in M$ such that $\mathfrak{M} \models \varphi(\mathbf{b}, \mathbf{a})$. Note that for some Σ_n-formula $\varphi_{\mathbf{a}}(x)$ the following holds

$$\mathfrak{M} \models \exists x \exists y (\varphi(x,y) \land \varphi_{\mathbf{a}}(y))$$

If we encode the pair $\langle x, y \rangle$ by z, then the formula above becomes:

$$\mathfrak{M} \models \exists z (\varphi((z)_0, (z)_1) \land \varphi_{\mathbf{a}}((z)_1))$$

It is clear that the formula $\psi(z)$ defined as

$$\varphi((z)_0, (z)_1) \land \forall x < z \neg \varphi((x)_0, (x)_1)$$

is again Σ_n, and that $\mathfrak{M} \models \exists y \psi(y)$. So there is some Σ_n-definable $\mathbf{c} \in M$ such that $\mathfrak{M} \models \psi(\mathbf{c})$ and $\mathfrak{M} \models \varphi((\mathbf{c})_1, \mathbf{a})$, and we just set $\mathbf{b} = (\mathbf{c})_1$. $\square$

Remark. In $\Sigma_n^{\mathfrak{M}}$ every element is Σ_n-definable, so we may say that $\Sigma_n^{\mathfrak{M}}$ is Σ_n-atomic. It can easily be shown that this model is also Σ_n-prime (that is, Σ_n-elementarily embeddable in every model for $\mathbf{T}_n$), so this construction might be of interest in its own right.

LEMMA 2 (cf. [**2**, Lemma 1.1]). *For any $n \in \omega$ there exists a Δ_{n+1}-formula $\varphi(x,y)$ in $\mathfrak{L}_{\mathbf{PA}}$ and a model $\mathfrak{M}_1 \models \mathbf{T}_n \cup \{\neg \mathcal{R}(\varphi)\}$.*

Proof. Let $\mathfrak{M}$ be a model for $\mathbf{T}$. By Lemma 1 we have $\Sigma_n^{\mathfrak{M}} \models \mathbf{T}_n$. Let G be a nonprincipal ultrafilter in $\mathcal{D}_n(\Sigma_n^{\mathfrak{M}})$, the set of Σ_n-definable subsets of $\Sigma_n^{\mathfrak{M}}$. Now we construct a model $\mathfrak{M}_1 = F_{\Sigma_n}(\Sigma_n^{\mathfrak{M}})/G$. By Lemma 1 this is also a model for $\mathbf{T}_n$. Fix some $\mathbf{b} \in M_1$. It is a $=_G$-equivalence class of some function f that is Σ_n-definable without parameters (remember that $\Sigma_n^{\mathfrak{M}}$ is Σ_n-atomic) in $\Sigma_n^{\mathfrak{M}}$. So we have a Σ_n-formula $\psi_f(x,y)$ such that $fm = n$ iff $\Sigma_n^{\mathfrak{M}} \models \psi_f(m,n)$ for all $m, n \in \Sigma_n^{\mathfrak{M}}$. And now,

$$\begin{aligned} B &= \{n \in \Sigma_n^{\mathfrak{M}} | \Sigma_n^{\mathfrak{M}} \models \mathrm{SAT}_{\Sigma_n}(\ulcorner \theta \urcorner, n, fn)\} \\ &= \{n \in \Sigma_n^{\mathfrak{M}} | \Sigma_n^{\mathfrak{M}} \models \theta(n, fn)\} \\ &= \Sigma_n^{\mathfrak{M}}, \end{aligned}$$

and $\mathfrak{M}_1 \models \mathrm{SAT}_{\Sigma_n}(\ulcorner \theta \urcorner, i_G, \mathbf{b})$, where i_G stands for the $=_G$-equivalence class of the diagonal i of $\Sigma_n^{\mathfrak{M}}$. We conclude that for every $\mathbf{b}$ in M_1 there exists some $\mathbf{e} \in \omega$ such that

$$\mathfrak{M}_1 \models \mathrm{SAT}_{\Sigma_n}(\mathbf{e}, i_G, \mathbf{b}).$$

It remains to show that wanted Δ_{n+1} formula is $\psi(x,y)$, defined as

$$\mathrm{SAT}_{\Sigma_n}(x, i_G, y) \wedge (\forall z < x)\neg \mathrm{SAT}_{\Sigma_n}(z, i_G, y),$$

"x is the least Gödel's number of a formula that defines y". From the previous discussion it is evident that $\mathfrak{M}_1 \models \psi(x,y)$ only if x is standard, and that the set of all $x \in \omega$ such that $\mathfrak{M}_1 \models \exists y \psi(x,y)$ is cofinal in ω. Now we check that ψ is not regular in $\mathfrak{M}_1$, i.e. that the following holds:

$$\begin{aligned}\mathfrak{M} \models \exists v[\forall x \exists y \forall u(x < v \rightarrow (\phi(x,u) \rightarrow u < y))\wedge\\ \forall y \exists x \exists u(x < v \wedge \phi(x,u) \wedge y \leq u)].\end{aligned}$$

For v we fix some nonstandard element $\mathbf{v}$ of M_1. To prove the first part of the statement, fix any $\mathbf{x} < \mathbf{v}$. The set $\{u \in M_1 | \mathfrak{M}_1 \models \psi(\mathbf{x}, u)\}$ has at most one element, thus it is bounded by some $\mathbf{y}$. To check the second part, note that for any $\mathbf{y} \in M_1$ there is an $\mathbf{x} \in \omega$ (thus $\mathbf{x} < \mathbf{v}$) and $\mathbf{u} > \mathbf{y}$ such that $\mathfrak{M}_1 \models \psi(\mathbf{x}, \mathbf{u})$. □

Note that the minor modification of the proof that ψ is not regular in $\mathfrak{M}_1$ gives the following semantical characterization of **C3** for every model $\mathfrak{M}$ with built-in Skolem functions:

$\mathfrak{M} \models$**C3** *iff there is no definable* (*in* $\mathfrak{M}$) *function mapping a bounded subset of* M *cofinally into* M.

Proof of the Main Theorem. The model $\mathfrak{M}_1$ from Lemma 2 is not extendable. □

REFERENCES

[**1**] C. C. Chang, H. J. Keisler, *Model Theory*, North-Holland, Amsterdam, 1973.

[**2**] I. Farah, *Existence and nonexistence of some elementary end-extensions*, Math. Logic Quarterly, to appear.

[**3**] Ž. Mijajlović, *A note on elementary end extension*, Publ. Inst. Math. (Beograd) **21** (1977), 141–144.

Matematički fakultet
11 001 Beograd, p.p. 550
Jugoslavija

(Received 17 02 1992)