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ON HYPERCYLINDERS IN CONFORMALLY SYMMETRIC MANIFOLDS

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Abstract. Hypercylinders in conformally symmetric manifolds are considered. The main result is the following theorem: Let (M, g) be a hypercylinder in a parabolic essentially conformally symmetric manifold $(N, \widetilde{g})$, $\dim N \geq 5$ and let $\widetilde{U}$ be the subset od N consisting of all points of N at which the Ricci tensor $\widetilde{S}$ of $(N, \widetilde{g})$ is not recurrent. If $\widetilde{U} \cap M$ is a dense subset of M, then (M, g) is a conformally recurrent manifold.

1. Introduction. Totally umbilical submanifolds in locally symmetric, recurrent, conformally flat, conformally symmetric and conformally recurrent manifolds were investigated by many authors (e.g. [**6**], [**10**], [**19**], [**21**], [**24**]–[**27**], [**29**], [**33**]). An important part of these investigation treats problems concerneing totally umbilical hypersurfaces in these classes of manifolds (e.g. [**7**], [**8**], [**20**], [**28**], [**30**]). On the other hand, totally umbilical hypersurfaces, as well as hypercylinders, are special examples of quasi-umbilical hypersurfaces. Certain results on quasi-umbilical hypersurfaces in locally symmetric, recurrent and conformally flat manifolds are presented in [**3**], [**34**] and [**22**] respectively. Moreover, hypercylinders in locally symmetric and conformally flat manifolds were studied in [**9**] and [**37**] (see also [**2**]) respectively. We shall continue study in this direction considering hypercylinders in conformally symmetric manifolds.

Let $(N, \widetilde{g})$ be an n-dimensional, $n \geq 4$, semi-Riemannian manifold with the metric tensor $\widetilde{g}$ and let $\widetilde{\nabla}$ be the Levi-Civita connection of $(N, \widetilde{g})$. Let $(N, \widetilde{g})$ be covered by a system of charts $\{\widetilde{U}; x^r\}$. We denote by $\widetilde{g}_{rs}$, $\left\{\begin{matrix} \tilde{r} \\ s\ t \end{matrix}\right\}$, $\widetilde{\nabla}_s$, $\widetilde{R}_{rstu}$, $\widetilde{C}_{rstu}$, $\widetilde{S}_{ts}$ and $\widetilde{K}$ the local components of the metric tensor $\widetilde{g}$, the Christoffel symbols, the operator of covariant differentiation, the Riemann-Christoffel curvature tensor $\widetilde{R}$, the Weyl conformal curvature tensor $\widetilde{C}$, the Ricci tensor $\widetilde{S}$ and the scalar curvature $\widetilde{K}$ of $(N, \widetilde{g})$ respectively, where $r, s, t, u, v, w \in \{1, 2, \dots, n\}$.

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We have

$$\widetilde{C}_{rstu} = \widetilde{R}_{rstu} + \frac{\widetilde{K}}{(n-1)(n-2)}(\widetilde{g}_{ru}\widetilde{g}_{st} - \widetilde{g}_{rt}\widetilde{g}_{su}) - \frac{1}{n-2}(\widetilde{g}_{ru}\widetilde{S}_{ts} + \widetilde{g}_{ts}\widetilde{S}_{ru} - \widetilde{g}_{rt}\widetilde{S}_{su} - \widetilde{g}_{su}\widetilde{S}_{rt}). \tag{1.1}$$

A $(0,k)$-tensor field T on N is said to be recurrent [**36**] if the condition

$$T(X_1,\dots,X_k)\widetilde{\nabla}T(Y_1,\dots,Y_k;Z) = T(Y_1,\dots,Y_k)\widetilde{\nabla}T(X_1,\dots,X_k;Z)$$

holds on N, where $X_1,\dots,X_k,Y_1,\dots,Y_k,Z \in \mathfrak{X}(N)$, $\mathfrak{X}(N)$ being the Lie algebra of vector fields on N. In particular, if $\widetilde{\nabla}T$ vanishes on N, then T is called parallel. A manifold $(N,\widetilde{g})$, $n \geq 4$, is said to be locally symmetric [**31**] (resp. conformally symmetric [**4**]) if its tensor $\widetilde{R}$ (resp. tensor $\widetilde{C}$) is parallel with respect to $\widetilde{\nabla}$. Further, a manifold $(N,\widetilde{g})$, $n \geq 4$, is said to be recurrent [**38**] (resp. conformally recurrent [**1**] or Ricci recurrent [**32**]) if its tensor $\widetilde{R}$ (resp. tensor $\widetilde{C}$ or tensor $\widetilde{S}$) is recurrent. A conformally symmetric manifold $(N,\widetilde{g})$ which is neither locally symmetric nor conformally flat is called essentially conformally symmetric or shortly e.c.s. manifold. Various examples of e.c.s. manifolds are given in [**35**], [**11**] and [**18**]. All e.c.s. metrics are indefinite ([**16**, Theorem 2]). Any e.c.s. manifold $(N,\widetilde{g})$ satisfies the following equation ([**18**], [**17**])

$$F\widetilde{C}(X,Y,Z,W) = \widetilde{S}(X,W)\widetilde{S}(Y,Z) - \widetilde{S}(X,Z)\widetilde{S}(Y,W)$$

for some function F, where $X,Y,Z,W \in \mathfrak{X}(M)$. F is called the fundamental function of $(N,\widetilde{g})$. All e.c.s. manifolds can be divided into the following five non-empty and mutually disjoint classes (according to the behaviour of the Ricci tensor and the fundamental function F [**12**]):

Class I. Ricci recurrent ones (they all satisfy $F = 0$).

Class II. Parabolic e.c.s. manifold [**15**] (satisfying $F = 0$ identically but not Ricci reccurent).

Class III. Elliptic ones [**14**] (F = constant $\neq 0$, semidefinite everywhere).

Class IV. Hyperbolic ones [**13**] (F = constant $\neq 0$, semidefinite nowhere)

Class V. Those with F non-constant.

Lemma 1 (Theorem 7, 8, 9 and formula (6) of [**17**] and Theorem 7 of [**18**]). *Let $(N,\widetilde{g})$ be an e.c.s. manifold and let $\{\widetilde{U};x^r\}$ be a chart on N. Then the following relations are satisfied on $\widetilde{U}$:*

$$\widetilde{\nabla}_v\widetilde{C}_{rstu} = 0, \tag{1.2}$$

$$F\widetilde{C}_{rstu} = \widetilde{S}_{ur}\widetilde{S}_{ts} - \widetilde{S}_{tr}\widetilde{S}_{us}, \tag{1.3}$$

$$\widetilde{\nabla}_r\widetilde{S}_{ts} = \widetilde{\nabla}_t\widetilde{S}_{rs}, \tag{1.4}$$

$$\widetilde{K} = 0, \tag{1.5}$$

$$\widetilde{S}^v{}_r \widetilde{C}_{vstu} = 0, \qquad \widetilde{S}^v{}_r = \widetilde{g}^{vt} \widetilde{S}_{tr}, \tag{1.6}$$

$$\widetilde{\nabla}_w \widetilde{\nabla}_v \widetilde{R}_{rstu} - \widetilde{\nabla}_v \widetilde{\nabla}_w \widetilde{R}_{rstu} = 0, \tag{1.7}$$

$$\widetilde{S}_{vw} \widetilde{C}_{rstu} + \widetilde{S}_{vr} \widetilde{C}_{swtu} + \widetilde{S}_{vs} \widetilde{C}_{wrtu} = 0. \tag{1.8}$$

2. Hypercylinders. Let M be a hypersurface in an n-dimensional, $n \geq 4$, semi-Riemannian manifold $(N, \widetilde{g})$ and let the tensor g, induced by the metric tensor $\widetilde{g}$, be the metric tensor of M. Moreover, let $x^r = x^r(y^a)$ be the local expression of M in N. Then we have $g_{ab} = \widetilde{g}_{rs} B^{rs}_{ab}$, where

$$B^{r_1 \ldots r_k}_{a_1 \ldots a_k} = B^{r_1}_{a_1} \ldots B^{r_k}_{a_k}, \qquad B^r_a = \partial_a x^r, \qquad \partial_a = \partial / (\partial y^a),$$

and g_{ab} are the local components of the tensor g. Further, we denote by $\left\{ \begin{matrix} a \\ b\ c \end{matrix} \right\}$, R_{abcd}, S_{ad}, C_{abcd} and K the local components of the Christoffel symbols, the curvature tensor R, the Ricci tensor S, the Weyl conformal curvature tensor C and the scalar curvature K of (M, g) respectively. Here and below, $a, b, c, d, e, f, h, i, j \in \{1, 2, \ldots, n-1\}$. Let N^r be the local components of a local unit vector field normal to M. Then we have the following relations

$$\widetilde{g}_{rs} N^r N^s = \varepsilon, \quad \widetilde{g}_{rs} N^r B^s_a = 0, \quad g^{ab} B^{rs}_{ab} = \widetilde{g}^{rs} - \varepsilon N^r N^s, \quad \varepsilon = \pm 1. \tag{2.1}$$

The hypersurface (M, g) is said to be a cylindrical hypersurface or shortly a hypercylinder (cf. [**5**, pp. 147–148], [**9**]) in $(N, \widetilde{g})$ if the second fundamental tensor H of (M, g) satisfies on M the condition $H = \beta u \otimes u$, where β is a function and u a 1-form on M, respectively. Let p be a point of the hypercylinder (M, g). Then the following equality

$$H_{ad} = \beta u_a u_d \tag{2.2}$$

holds on some neighbourhood $U \subset M$ of p, where H_{ad} and u_a are the local components of H and u on U, respectively. We denote by ∇ the operator of the van der Waerden-Bortolotti covariant derivative. Then, in virtue of (2.2), the Gauss and Weingarten formulas for (M, g) in $(N, \widetilde{g})$ take on U the following form

$$\nabla_a B^r_d = \varepsilon H_{ad} N^r = \varepsilon \beta u_a u_d N^r, \tag{2.3}$$

$$\nabla_a N^r = -H_{ac} g^{cd} B^r_d = -\beta u_a u^d B^r_d, \qquad u^d = g^{da} u_a, \tag{2.4}$$

respectively. Furthermore, by (2.2), the Gauss and Codazzi equations for (M, g) in $(N, \widetilde{g})$ can be expressed on U as follows:

$$R_{abcd} = \widetilde{R}_{rstu} B^{rstu}_{abcd} + \varepsilon (H_{ad} H_{bc} - H_{ac} H_{bd}) = \widetilde{R}_{rstu} B^{rstu}_{abcd}, \tag{2.5}$$

$$\begin{aligned} V_{bcd} = N^r \widetilde{R}_{rstu} B^{stu}_{bcd} &= \nabla_d H_{bc} - \nabla_c H_{bd} = u_b (\beta_d u_c - \beta_c u_d) \\ &+ \beta (u_c \nabla_d u_b - u_d \nabla_c u_b + u_b (\nabla_d u_c - \nabla_c u_d)), \quad \beta_c = \nabla_c \beta. \end{aligned} \tag{2.6}$$

From this, by contraction with g^{bc} and making use of (2.1), we obtain

$$\begin{aligned} v_d &= g^{bc} V_{bcd} = N^r \widetilde{S}_{rs} B^s_d \\ &= g^{bc} u_b u_c \beta_d - u^h \beta_h u_d + \beta (-u^h \nabla_h u_d - (\nabla_h u^h) u_d + 2 u^h \nabla_d u_h). \end{aligned} \tag{2.7}$$

Lemma 2. *Let (M, g) be a hypercylinder in a semi-Riemannian manifold $(N, \widetilde{g})$, $n \geq 4$. If p is a point of M such that the relations* (2.2) *and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p then the equality*

$$V_{bcd} = u_b u^h V_{hcd} + u_d(u^h V_{bch} + u_b v_c) - u_c(u^h V_{bdh} + u_b v_d) \tag{2.8}$$

holds on U.

Proof. From (2.5), by making use of (2.3) and (2.6), it follows that

$$\nabla_e R_{abcd} = B^{vrstu}_{eabcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} + \varepsilon\beta u_e(u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab}). \tag{2.9}$$

This, by contraction with g^{bc} and an application of (2.1), (2.7) and the identity

$$\widetilde{g}^{vr} \widetilde{\nabla}_v \widetilde{R}_{rstu} = \widetilde{\nabla}_u \widetilde{S}_{ts} - \widetilde{\nabla}_t \widetilde{S}_{us}, \tag{2.10}$$

yields

$$\nabla_e S_{ad} = B^{vts}_{eda} \widetilde{\nabla}_v \widetilde{S}_{ts} - \varepsilon N^s N^t B^{vru}_{ead} \widetilde{\nabla}_v \widetilde{R}_{rstu} + \varepsilon\beta u_e K_{ad}, \tag{2.11}$$

where

$$K_{ad} = u_a v_d + u_d v_a + u^h V_{bch} + u^h V_{cbh}, \tag{2.12}$$

$$k = g^{ad} K_{ad} = 4u^h v_h. \tag{2.13}$$

On the other hand, contracting (2.9) with g^{ae} and using (2.10) and (2.2), we find

$$\begin{aligned}\nabla_d S_{bc} - \nabla_c S_{bd} = {} & B^{uts}_{dcb}(\widetilde{\nabla}_u \widetilde{S}_{ts} - \widetilde{\nabla}_t \widetilde{S}_{us}) - \varepsilon N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} \\ & + \varepsilon\beta(V_{bcd} - u_b u^h V_{hcd} - u_c u^h V_{dbh} + u_d u^h V_{cbh}).\end{aligned} \tag{2.14}$$

The following equality follows immediately from the second Bianchi identity

$$N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} = N^v N^r B^{uts}_{dcb} \widetilde{\nabla}_u \widetilde{R}_{tvrs} - N^v N^r B^{tus}_{cdb} \widetilde{\nabla}_v \widetilde{R}_{uvrs},$$

which, in virtue of (2.11) and (2.12), turns into

$$\begin{aligned}\varepsilon N^v N^r B^{stu}_{bcd} \widetilde{\nabla}_v \widetilde{R}_{rstu} = {} & B^{vru}_{dcb}(\widetilde{\nabla}_v \widetilde{S}_{ru} - \widetilde{\nabla}_r \widetilde{S}_{vu}) + \nabla_c S_{db} - \nabla_d S_{cb} \\ & + \varepsilon\beta u_d(u^h V_{cbh} + u^h V_{bch} + u_b v_c) - u_c(u^h V_{dbh} + u^h V_{bdh} + u_b v_d).\end{aligned}$$

The last equation, together with (2.14), completes the proof.

Lemma 3. *Let (M, g) be a hypercylinder in a semi-Riemannian manifold $(N, \widetilde{g})$, $n \geq 4$. If p is a point of M such that the relations* (1.7) *and* (2.2) *are fulfilled at every point of some neighbourhood $U \subset M$ of p then the equalities*

$$v^h V_{hef} = 0, \tag{2.15}$$

$$u^h v_h \omega_{abc} = 0 \tag{2.16}$$

hold on U, where

$$\omega_{acd} = \beta(u_a(\nabla_d u_c - \nabla_c u_d) + u_c(\nabla_a u_d - \nabla_d u_a) + u_d(\nabla_c u_a - \nabla_a u_c)). \tag{2.17}$$

Proof. Transvecting (1.7) with B^{rstuvw}_{abcdef} and using the Ricci identity, (2,1), (2.5) and (2.6), we find

$$\begin{aligned}&\nabla_f\nabla_e R_{abcd} - \nabla_e\nabla_f R_{abcd}\\&\quad + \varepsilon(-V_{bcd}V_{aef} + V_{acd}V_{bef} - V_{dab}V_{cef} + V_{cab}V_{def}) = 0\,.\end{aligned} \tag{2.18}$$

Next, contracting the above equation with g^{ad} and g^{bc} and applying (2.7) we get (2.15). Finally, (2.16) is an immediate consequence of (2.15) and the following identity

$$u_aV_{bcd} + u_cV_{bda} + u_dV_{bac} = u_b\omega_{acd}\,. \tag{2.19}$$

Our lemma is thus proved.

Remark. In the next sections we shall consider hypercylinders satisfying certain additional conditions. Let (M,g) be a hypercylinder in a manifold $(N,\widetilde{g})$ and let p be a point of M such that the relation (2.2) is fulfilled at every point of some neighbourhood $U \subset M$ of p. We assume that at every point of U one of the following relations is satisfied:

$$g^{ad}u_au_d = 1, \tag{2.20}$$

$$g^{ad}u_au_d = -1, \tag{2.21}$$

$$g^{ad}u_au_d = 0. \tag{2.22}$$

Thus the scalar $g^{ad}u_au_d$ is a constant on U. This fact implies

$$u^h\nabla_a u_h = 0. \tag{2.23}$$

Transvecting now (2.6) with u^b, u^c and u^d respectively and applying (2.20)–(2.23) we easily get

$$u^hV_{hcd} = \eta(\beta_d u_c - \beta_c u_d + \beta(\nabla_d u_c - \nabla_c u_d)), \tag{2.24}$$

$$u^hu^jV_{hjd} = \eta^2\beta_d - \eta(u^h\nabla_h u_d + \beta u^h\nabla_h u_d), \tag{2.25}$$

$$u^hV_{bch} = (u^h\beta_h u_c - \eta\beta_c)u_b + (u_cu^h\nabla_h u_b - \eta\nabla_c u_b + u_bu^h\nabla_h u_c) \tag{2.26}$$

respectively, where $\eta \in \{-1,0,1\}$.

3. Hypercylinders in conformally symmetric manifolds.

LEMMA 4. *Let (M,g) be a hypercylinder in a conformally symmetric manifold $(N,\widetilde{g})$, $n \geq 4$ and let p be a point of M such that the conditions* (2.2) *and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p. Then we have*:

(i) *the equality*

$$\begin{aligned}\nabla_e C_{abcd} = \varepsilon\beta u_e\bigg(&\frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd})\\&+ \left(u_bu_c - \frac{g_{bc}}{n-3}\right)K_{ad} + \left(u_au_d - \frac{g_{ad}}{n-3}\right)K_{bc}\end{aligned}$$

$$- \left(u_a u_c - \frac{g_{ac}}{n-3}\right) K_{bd} - \left(u_b u_d - \frac{g_{bd}}{n-3}\right) K_{ac} \Big) \tag{3.1}$$

holds on U.

(ii) *If at every point of U (2.21) is fulfilled then $\nabla C = 0$ holds on U.*

Proof. (i): Transvecting (1.2) with $\varepsilon N^s N^t B^{vru}_{ead}$ and using (1.1) and (2.1) we get

$$-\varepsilon N^s N^t B^{vru}_{ead} \widetilde{\nabla}_v \widetilde{R}_{rstu} = -\frac{B^{vur}_{eda} \widetilde{\nabla}_v \widetilde{S}_{ur} + \varepsilon (\widetilde{\nabla}_v \widetilde{S}_{ur}) N^r N^u B^v_e g_{ad}}{n-2} + \frac{B^v_e (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)} .$$

Substituting this in (2.11) we find

$$\begin{aligned} \frac{B^{vur}_{eda}(\widetilde{\nabla}_v \widetilde{S}_{ur})}{n-2} &= \frac{\nabla_e S_{ad}}{n-3} - \frac{\varepsilon \beta u_e K_{ad}}{n-3} \\ &+ \frac{\varepsilon N^r N^u B^v_e (\widetilde{\nabla}_v \widetilde{S}_{ur}) g_{ad}}{(n-1)(n-2)} - \frac{B^v_e (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)} . \end{aligned} \tag{3.2}$$

Contracting (3.2) with g^{ad} and using (2.1) and (2.13) we obtain

$$\frac{2\varepsilon N^r N^u B^v_e \widetilde{\nabla}_v \widetilde{S}_{ur}}{n-3} = -\frac{\nabla_e K}{n-3} + \frac{\varepsilon \beta k u_e}{n-3} + \frac{2 B^v_e \widetilde{\nabla}_v \widetilde{K}}{n-1} .$$

Now, by the above equation, (3.2) takes the form

$$\begin{aligned} \frac{B^{vur}_{eda} \widetilde{\nabla}_v \widetilde{S}_{ur}}{n-2} &= \frac{\nabla_e S_{ad}}{n-3} - \frac{\varepsilon \beta u_e K_{ad}}{n-3} \\ &+ \frac{B^v_e (\widetilde{\nabla}_v \widetilde{K}) g_{ad}}{(n-1)(n-2)} + \frac{\varepsilon \beta k u_e g_{ad}}{2(n-2)(n-3)} - \frac{(\nabla_e K) g_{ad}}{2(n-1)(n-2)} . \end{aligned}$$

But this, together with (2.9), (1.1) and (1.2) gives

$$\begin{aligned} \nabla_e C_{abcd} = \varepsilon \beta u_e \Big(& u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} \\ & - \frac{g_{ad} K_{bc} + g_{bc} K_{ad} - g_{ac} K_{bd} - g_{bd} K_{ac}}{n-3} + \frac{k (g_{ad} g_{bc} - g_{ac} g_{bd})}{(n-2)(n-3)} \Big) . \end{aligned} \tag{3.3}$$

On the other hand, (2.8), (2.12) and (2.13) yield

$$\begin{aligned} & u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} \\ & \quad = u_a u_d K_{bc} + u_b u_c K_{ad} - u_a u_c K_{bd} - u_b u_d K_{ac} . \end{aligned} \tag{3.4}$$

Now (3.3) turns into (3.1), completing the proof of (i).

(ii): Identity (2.8), in virtue of (2.24), (2.26), (2.7) and (2.21), reduces to $V_{bcd} = 0$. Now (3.3) completes the proof.

From Lemma 4(i) the following proposition follows easily.

Proposition 1. *Let (M,g) be a hypercylinder in a conformally symmetric manifold $(N,\widetilde{g})$, $n \geq 5$ and let p be a point of M such that the conditions* (2.2), (2.20) *and $\beta \neq 0$ are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then the condition $\nabla C = 0$ is satisfied on U if and only if the relation*

$$K_{ad} = \frac{k}{2(n-2)}((n-3)u_a u_d + g_{ad})$$

holds on U.

Lemma 5. *Let (M,g) be a hypercylinder in a conformally symmetric manifold $(N,\widetilde{g})$, $n \geq 5$ and let p be a point of M such that the relation $H_{ad} = 0$ holds at p. Then the tensor ∇C vanishes at p.*

Proof. We note that the equality

$$\begin{aligned}\nabla_e(\widetilde{C}_{rstu}B^{rstu}_{abcd}) = {} & B^{vrstu}_{eabcd}\widetilde{\nabla}_v\widetilde{C}_{rstu} + \widetilde{C}_{rstu}B^{stu}_{bcd}\nabla_e B^r_a - \widetilde{C}_{srtu}B^{rtu}_{acd}\nabla_e B^r_b \\ & + \widetilde{C}_{turs}B^{urs}_{dab}\nabla_e B^t_c - \widetilde{C}_{utrs}B^{trs}_{cab}\nabla_e B^u_d\end{aligned}$$

holds on some neighbourhood U of p. This, by (1.1), (2.5), (1.2) and (2.3) reduces at p to

$$\begin{aligned}&\nabla_e R_{abcd} + \frac{(\nabla_e\widetilde{K})(g_{ad}g_{bc} - g_{ac}g_{bd})}{(n-1)(n-2)} \\ &- \frac{1}{n-2}\left(g_{bc}B^{vru}_{ead}\widetilde{\nabla}_v\widetilde{S}_{ru} + g_{ad}B^{vst}_{ebc}\widetilde{\nabla}_v\widetilde{S}_{st} - g_{bd}B^{vrt}_{eac}\widetilde{\nabla}_v\widetilde{S}_{rt} - g_{ac}B^{vsu}_{ebd}\widetilde{\nabla}_v\widetilde{S}_{su}\right).\end{aligned} \tag{3.5}$$

Next, contracting (3.5) with g^{bc} we obtain

$$\frac{B^{vru}_{ead}\widetilde{\nabla}_v\widetilde{S}_{ru}}{n-2} = \frac{\nabla_e S_{ad}}{n-3} + \frac{(\nabla_e\widetilde{K})g_{ad}}{(n-1)(n-3)} - \frac{(\widetilde{\nabla}_v\widetilde{S}_{st})B^{vst}_{ebc}g^{bc}g_{ad}}{(n-2)(n-3)}.$$

Substituting this into (3.5) we get our assertion.

Lemma 6. *Let (M,g) be a hypercylinder in a conformally symmetric manifold $(N,\widetilde{g})$, $n \geq 5$ and let p be a point of M such that the relations* (2.2) *and $\beta \neq 0$ are satisfied at every point of some neighbourhood $U \subset M$ of p. If the equality* (2.22) *is fulfilled at p then the tensor ∇C vanishes at p.*

Proof. Transvecting (2.6) with u^b and u^d and using (2.22) we get

$$\begin{aligned}u^h V_{bch} &= u^h\beta_h u_b u_c + \beta(u_c u^h\nabla_h u_b + u_b u^h\nabla_h u_c - u_b u^h\nabla_c u_h), \\ u^h V_{hcd} &= \beta(u_c u^h\nabla_d u_h - u_d u^h\nabla_c u_h)\end{aligned}$$

respectively. Moreover, from (2.7), by (2.22), we obtain

$$v_d = -u^h\beta_h u_d + \beta(2u^h\nabla_d u_h - (\nabla_h u^h)u_d - u^h\nabla_h u_d).$$

Now we can verify that the identity (2.8), by the above three relations, reduces to $V_{bcd} = 0$. Finally, (3.3) completes the proof.

4. Hypercylinders in non-Ricci-recurrent parabolic e.c.s. manifolds.

LEMMA 7 [**15**, Lemmas 1 and 4]. *Let $(N, \widetilde{g})$, $n \geq 4$, be a parabolic e.c.s. manifold. If p is a point of N such that*

$$(\widetilde{S}_{ur}\widetilde{\nabla}_v\widetilde{S}_{ts} - \widetilde{S}_{ts}\widetilde{\nabla}_v\widetilde{S}_{ur})(p) \neq 0, \tag{4.1}$$

then there exists a neighbourhood $\widetilde{U}$ of p with two vector fields A and B which are unique (up to a sign of A) determined by the following two conditions

$$\widetilde{S}_{rs} = eA_rA_s, \qquad e = \pm 1, \tag{4.2}$$

$$\widetilde{\nabla}_u\widetilde{S}_{rs} = B_u\widetilde{S}_{rs} + B_r\widetilde{S}_{su} + B_s\widetilde{S}_{ur}, \tag{4.3}$$

where A_r and B_r are the local components of A and B respectively. The vector fields A and B satisfy on $\widetilde{U}$ the following relations:

$$\widetilde{g}^{rs}A_rA_s = 0, \qquad \tilde{g}^{rs}A_rB_s = 0, \tag{4.4}$$

$$\widetilde{\nabla}_sA_r = (1/2)A_rB_s + A_sB_r, \tag{4.5}$$

$$\widetilde{\nabla}_sB_r = B_rB_s + 3\lambda B_rA_s + \lambda A_rB_s + \sigma A_rA_s, \tag{4.6}$$

where λ and σ are some functions on $\widetilde{U}$. Moreover, we have

$$\widetilde{C}_{rstu} = -\Phi(A_rB_s - A_sB_r)(A_tB_u - A_uB_t) \tag{4.7}$$

for a certain (uniquely determined) function Φ.

LEMMA 8. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$, $n \geq 4$ and let p be a point of M such that the conditions:* (2.2), (2.20), (4.1) *and*

$$N^rA_r \neq 0 \tag{4.8}$$

are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then the equality

$$\omega_{abc} = 0 \tag{4.9}$$

holds on U.

Proof. The equality (2.16), in virtue of (2.7), (4.2) and (4.8), give

$$u^hA_h\omega_{abc} = 0, \tag{4.10}$$

where

$$A_h = A_rB^r_h. \tag{4.11}$$

Suppose that at a point q of U we have

$$\omega_{abc}(q) \neq 0. \tag{4.12}$$

Then, by (4.10), the equality

$$u^hA_h = 0 \tag{4.13}$$

holds on some open subset $U' \subset U$. From this we obtain

$$A_h \nabla_c u^h + u^h \nabla_c A_h = 0. \tag{4.14}$$

Using (4.5) and (2.3), we can easily verify that the following equality is fulfilled on U

$$\nabla_c A_a = A_a B_c/2 + A_c B_a + \varepsilon\beta N^r A_r u_a u_c, \tag{4.15}$$

$$B_c = B_r B^r_c. \tag{4.16}$$

Substituting (4.15) into (4.14) and applying (4.13) we get

$$A_h \nabla_c u^h + u^h B_h A_c + \varepsilon\beta N^r A_r u_c = 0. \tag{4.17}$$

The formula (2.17), because of (2.7), (4.2) and (4.8), yields

$$A^h V_{hef} = 0, \tag{4.18}$$

where $A^h = g^{ah} A_a$. Thus (4.18), by (2.6) and (4.13), gives

$$A^h (u_v \nabla_d u_h - u_d \nabla_c u_h) = 0,$$

whence, by (4.17), it follows that

$$u^h B_h (u_c A_d - u_d A_c) = 0.$$

If $(u_c A_d - u_d A_c)(q) = 0$ then also $A_d(q) = 0$. The last equation, in virtue of the relation

$$g^{ad} A_a A_d + \varepsilon (N^r A_r)^2 = 0, \tag{4.19}$$

which follows immediately from (4.4) and (2.1), gives $(N^r A_r)(q) = 0$. But this contradicts (4.8). If $(u^h B_h)(q) = 0$ then from (4.7), by transvection with $N^r B^{stu}_{bcd} u^b$ and the use of (1.1), (2.1), (4.2), (4.11), (4.16), (1.5) and (4.13), we obtain

$$(u^h V_{hcd} + (e/(n-2)) N^r A_r (A_c u_d - A_d u_c))(q) = 0. \tag{4.20}$$

On the other hand, transvecting (2.19) with u^a we get

$$u_a u^h V_{hcd} + u_c u^h V_{hda} + u_d u^h V_{hac} = \omega_{acd}.$$

This, by (4.20), gives $\omega_{acd}(q) = 0$, a contradiction. Our lemma is thus proved.

LEMMA 9. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$ and let p be a point of M such that the conditions* (2.2), (2.20), (4.1) *and*

$$N^r A_r = 0 \tag{4.21}$$

are fulfilled. Then the equality (4.9) *holds at p.*

Proof. First of all we note that at p the following relation

$$A_d \neq 0 \tag{4.22}$$

is satisfied. In fact, if we had $A_d = 0$ the, by (4.21), we get $A(p) = 0$ and $\widetilde{S}(p) = 0$, which contradicts (4.1). Transvecting now (1.8) with $N^w B^{vrstu}_{eabcd}$ and making use of (4.2), (4.22), (4.21), (2.1) and (2.6), we find

$$A_a V_{bcd} = A_b V_{acd}. \tag{4.23}$$

Multiplying (4.23) by u_f and summing the resulting equality cyclically in f, c, d and applying (2.19) we obtain $(A_a u_b - A_b u_a)\omega_{fcd} = 0$. Assume that $A_a u_b - A_b u_a$ vanishes at p. Then we have

$$A_a = u^h A_h u_a. \tag{4.24}$$

So, (4.23) turns into $u^h A_h (u_a V_{bcd} - u_b V_{acd}) = 0$. Summing this cyclically in a, c, d and using again (2.19), we get $u^h A_h \omega_{acd} = 0$. From (4.24), in virtue of (4.22), it follows that $u^h A_h$ is non-zero at p. Thus the last equality completes the proof.

Proposition 2. *Let (M, g) be a hypercylinder in a parabolic e.c.s. manifold $(N, \widetilde{g})$, $n \geq 5$, and let p be a point of M such that the conditions*: (2.2), $\beta \neq 0$, (2.20) *and* (4.1) *are fulfilled at every point of some neighbourhood $U \subset M$ of p. Then C is a recurrent tensor on U.*

Proof. The identity (2.19), in view of Lemmas 8 and 9 reduces to

$$u_a V_{bcd} + u_c V_{bda} + u_d V_{bac} = 0. \tag{4.25}$$

This, by transvection with u^a, yields

$$V_{bcd} = u_d u^h V_{bch} - u_c u^h V_{bdh}. \tag{4.26}$$

On the other hand, transvecting (4.7) with $N^r B^{stu}_{bcd}$ and using (1.1), (2.6), (4.2), (2.1), (1.5) and (4.7), we find

$$V_{bcd} = \frac{eN^r A_r}{n-2}(A_d g_{bc} - A_c g_{bd}) + \phi D_b (A_c B_d - A_d B_c), \tag{4.27}$$

where $D_b = N^r B_r A_b - N^r A_r B_b$. Substituting (4.27) into (4.26) and (4.25) respectively, we obtain

$$\begin{aligned} V_{bcd} &= \Phi D_b (u_d C_c - u_c C_d) \\ &+ ((eN^r A_r)/(n-2))(u^h A_h (u_d g_{bc} - u_c g_{bd}) - u_b (A_c u_d - A_d u_c)), \end{aligned} \tag{4.28}$$

$$\begin{aligned} &\Phi D_b (u_a (A_c B_d - A_d B_c) + u_c (A_d B_a - A_a B_d) + u_d (A_a B_c - A_c B_a)) \\ &\frac{eN^r A_r}{n-2}((u_a A_d - u_d A_a) g_{bc} \\ &+ (u_c A_a - u_a A_c) g_{bd} + (u_d A_c - u_c A_d) g_{ab}) = 0 \end{aligned} \tag{4.29}$$

resepectively, where $C_c = u^h B_h A_c - u^h A_h B_c$. From (4.29), by transvection with $u^a u^b$, we get

$$\Phi u^h D_h (A_c B_d - A_d B_c) = u_d C_c - u_c C_d. \tag{4.30}$$

Further, asumming (4.28) cyclically in b, c, d, we obtain

$$D_b (u_d C_c - u_c C_d) + D_c (u_b C_d - u_d C_b) + D_d (u_c C_b - u_b C_c) = 0,$$

which, by multiplication with u_f and antisymmetrization in b, f, gives

$$(u_f D_b - u_b D_f)(u_d C_c - u_c C_d) = (C_b u_f - C_f u_b)(u_d D_c - u_c D_d).$$

But this implies

$$C_c(u_f D_b - u_b D_f) = (D_c - u^h D_h u_c)(u_f C_b - u_b C_f). \tag{4.31}$$

If $C_c(p) = 0$ then (4.28) turns into

$$V_{bcd} = \frac{eN^r A_r}{n-2}(u^h A_h (u_d g_{bc} - u_c g_{bd}) - u_b(A_c u_d - A_d u_c)).$$

Using this we can rewrite (3.3) in the following form

$$\begin{aligned}\nabla_e C_{abcd} &= \varepsilon\beta u_e \frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd})\\ &+ \frac{2eN^r A_r u^h A_h}{n-2}(u_a u_d g_{bc} + u_b u_c g_{ad} - u_a u_c g_{bd} - u_b u_d g_{ac})\\ &- \frac{1}{n-3}(g_{ad}K_{bc} + g_{bc}K_{ad} - g_{ac}K_{bd} - g_{bd}K_{ac}),\end{aligned} \tag{4.32}$$

which reduces to $\nabla C = 0$. If $C_c(p) \neq 0$ then (4.31) and (4.30) yield

$$u_f D_b - u_b D_f = \tau(A_b B_f - A_f B_b), \qquad \tau \in \mathbf{R}. \tag{4.33}$$

Moreover, using (4.28) and (4.33) we obtain

$$\begin{aligned}&u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} = ((eN^r A_r)/(n-2))((u_a A_d + u_d A_a)g_{bc}\\ &+ (u_c A_b + u_b A_c)g_{ad} - (u_a A_c + u_c A_a)g_{bd} - (u_b A_d + u_d A_b)g_{ac})\\ &- 2\Phi\tau(A_a B_b - A_b B_a)(A_c B_d - A_d B_c).\end{aligned}$$

This, by an application of $\widetilde{C}_{rstu}B^{rstu}_{abcd} = -\Phi(A_a B_b - A_b B_a)(A_c B_d - A_d B_c)$, (1.1), (2.5), (1.5) and (4.2), turns into

$$\begin{aligned}&u_a V_{bcd} - u_b V_{acd} + u_c V_{dab} - u_d V_{cab} = 2\tau R_{abcd}\\ &+ \frac{eN^r A_r}{n-2}((u_a A_d + u_d A_a)g_{bc} + (u_c A_b + u_b A_c)g_{ad}\\ &- (u_a A_c + u_c A_a)g_{bd} - (u_b A_d + u_d A_b)g_{ac})\\ &+ \frac{2e\tau}{n-2}(A_b A_c g_{ad} + A_a A_d g_{bc} - A_a A_c g_{bd} - A_b A_d g_{ac}).\end{aligned} \tag{4.34}$$

We have now two possibilities: (a) $\tau(p) = 0$ and (b) $\tau(p) \neq 0$. (a) In this case (3.3), by (4.34), becomes

$$\begin{aligned}\nabla_e C_{abcd} = \varepsilon\beta u_e\big(&g_{ad}L_{bc} + g_{bc}L_{ad} - g_{ac}L_{bd} - g_{bd}L_{ac}\\ &+ \big(k(g_{ad}g_{bc} - g_{ac}g_{bd})\big)/\big((n-2)(n-3)\big)\big),\end{aligned} \tag{4.35}$$

where $L_{bc} = (eN^r A_r)(u_b A_c + u_c A_b)/(n-2) - K_{bc}/(n-3)$. From (4.35) it folows that $\nabla C = 0$ at p. (b) Using (4.34), we can present (3.3) in the form

$$\begin{aligned}\nabla_e C_{abcd} = \varepsilon\beta u_e\bigg(&\frac{k}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd}) + 2\tau R_{abcd}\\
&+ \frac{eN^r A_r}{n-2}(g_{ad}(u_c A_b + u_b A_c) + g_{bc}(u_a A_d + u_d A_a) - g_{ac}(u_b A_d + u_d A_b)\\
&- g_{bd}(u_a A_c + u_c A_a) - \frac{1}{n-3}(g_{ad}K_{bc} + g_{bc}K_{ad} - g_{ac}K_{bd} - g_{bd}K_{ac})\\
&+ \frac{2e\tau}{n-2}(g_{ad}A_b A_c + g_{bc}A_a A_d - g_{ac}A_b A_d - g_{bd}A_a A_c)\bigg),\end{aligned}$$

which can be rewritten in the following form

$$\begin{aligned}\nabla_e C_{abcd} = \varepsilon\beta u_e\bigg(&\frac{k - 2\tau K}{(n-2)(n-3)}(g_{ad}g_{bc} - g_{ac}g_{bd})\\
&+ 2\tau C_{abcd} + g_{ad}L_{bc} + g_{bc}L_{ad} - g_{ac}L_{bd} - g_{bd}L_{ac}\bigg)\end{aligned}\tag{4.36}$$

where

$$L_{bc} = \frac{2\tau}{n-3}S_{bc} - \frac{1}{n-3}K_{bc} + \frac{eN^r A_r}{n-2}(u_c A_b + u_b A_c) + \frac{2e\tau}{n-2}A_b A_c.$$

But from (4.36) we obtain $\nabla_e C_{abcd} = 2\tau\beta u_e C_{abcd}$, which states that C is recurrent. The last remark completes the proof.

Finally, combining Lemmas 5,6,4(ii) with Proposition 2 we immediately get the following theorem.

THEOREM 1. *Let (M,g) be a hypercylinder in a parabolic essentially conformally symmetric manifold $(N,\widetilde{g})$, $n \geq 5$ and let $\widetilde{U}$ be the subset of N consisting of all points of N at which the Ricci tensor $\widetilde{S}$ of $(N,\widetilde{g})$ is not recurrent. If $\widetilde{U} \cap M$ is a dense subset of M then (M,g) is a conformally recurrent manifold.*

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