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HYPERPLANES OF LOCALLY CONVEX GROUPS

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Abstract. We consider finite codimensional subgroups of some classes of locally convex groups. We have obtained also a new class of locally convex groups.

0. Introduction. In this paper (E, t) will denote a Hausdorff locally convex group (further LCG), i.e. a topological vector group which has convex subsets as a base of neighbourhoods of zero (see [**8**]).

Similarly as in locally convex spaces (further LCS) we can introduce the notions of a barrel, of a bornivorous disc, of an absorbing barrel and of a bounded set (see: [**3**], [**7**]). So, for example, an absolutely convex, closed subset with the open linear hull is said to be a barrel, and a bornivorous disc is an absolutely convex subset absorbing all the t-bounded subsets [**7**]. Let us mention, that the family of t-bounded subsets can be very poor in LCG (the only bounded set can be $\{0\}$), i.e. that finite subsets are not necessary bounded (see: [**3**], [**4**], [**7**], as well as [**5**, §10]).

In [**6**] and [**7**] the classes of barrelled, bornological and quasibarrelled LCGs are introduced, in accordance with the corresponding classes in LCSs. So (E, t) is barrelled (resp. bornological, quasibarrelled [**7**]) if every barrel (resp. bornivorous disc, bornivorous barrel) is a t-neighbourhood of zero. From [**6**] it is known that every LCG (E, t) has the associated LCS $(E, \operatorname{loc} t)$, where $\operatorname{loc} t$ is the strongest locally convex topology weaker than t. Some properties of LCG (E, t) are equivalent to the same properties of the space $(E, \operatorname{loc} t)$, (for example, the property of being "barrelled" [**6**, Proposition 3.1]). It is easy to verify that a hyperplane of LCG (E, t) is t-dense (t-closed) if and only if it is $\operatorname{loc} t$-dense ($\operatorname{loc} t$-closed).

1. Hyperplanes of bornological and quasibarrelled LCGs. It is known from [**2**] and [**9**] that in the class of LCSs the properties of being barrelled, bornological and quasibarrelled are inherited on the finite codimensional subspaces with respect to the induced topology. It has been shown, moreover, that a barrel,

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bornivorous disk and a bornivorous barrel of a finite codimensional subspace have their extensions in the space.

It is natural to consider the corresponding questions in the class of LCGs. Before we give the results, let us just mention that a connected component of zero in every LCG (E,t) has the form $L = \bigcap_{U\in\mathcal{U}} \bigcup_{n\in\mathbf{N}} nU$, where $\mathcal{U}$ is a fundamental system of neighbourhoods of zero [**4**, Proposition 2].

THEOREM 1.1. *Let (E,t) be a bornological LCG and let F be a finite codimensional subgroup. Then F is a bornological LCG with respect to the induced topology.*

This theorem is a consequence of the following result, which should be compared with [**2**].

LEMMA 1.1. *If (E,t) is a LCG, F is a finite codimensional subgroup, and T is a bornivorous disc in F, then there exists a bornivorous disc T' in E, such that $T'\cap F = T$.*

Proof. It is sufficient to prove the lemma in the case when F is a hyperplane, and then to use the principle of mathematical induction.

If L is the connected component of zero in E, then either $L\subset F$ of $L\not\subset F$. Since L contains all bounded subsets, then, in the case when $L\subset F$, we can take T for T'. If $L\not\subset F$, the proof is not trivial. Then $L\cap F$ is a hyperplane in L (t induces a locally convex topology on L) and $T\cap L$ is a bornivorous disk on it. According to [**2**], there exists a bornivorous disc T_1 in L, such that $T_1\cap(L\cap F) = T_1\cap F = T\cap L = T\cap(L\cap F)\subset T$. Let us take that $T' = \langle T\cup T_1\rangle$ where $\langle T\cup T_1\rangle$ is a convex hull of $T\cup T_1$ (this union is balanced). It is sufficient to prove that $T'\cap F = T$ because T' is evidently a bornivorous disc in E. It is clear that $T\subset T'\cap F$. Conversely, because $T' = \{\lambda d + (1-\lambda)\omega \mid \lambda\in[0,1],\ d\in T,\ \omega\in T_1\}$ if $z\in T'\cap F$, it follows that $z = \lambda d + (1-\lambda)\omega$, where $d\in T$, $\omega\in T_1$ and $z\in F$, so that for $\lambda\neq 1$, $\omega = [1/(1-\lambda)](z-\lambda d)\in T_1$, i.e. $\omega\in T_1\cap F\subset T$, hence $z = \lambda d + (1-\lambda)\omega\in\lambda T + (1-\lambda)T\subset T$, because T is a disc. If $\lambda = 1$, then $z = d$, $d\in T\cap F$, so $z\in T$. Consequently, $T'\cap F\subset T$, that is $T'\cap F = T$, and the proof is completed.

If (E,t) is a LCG, then there exists a LCG $(E,\tilde{t})$ which has all the t-bornivorous discs as a base of neighbourhoods of zero. The topology $\tilde{t}$ is obviously the weakest topology which is finer than t and for which $(E,\tilde{t})$ is a bornological LCG. It is clear that t and $\tilde{t}$ have the same bounded subsets. $(E,\tilde{t})$ is called the associated bornological group to the group (E,t). The problem whether $\operatorname{loc}\tilde{t} = \widetilde{\operatorname{loc} t}$, where $\widetilde{\operatorname{loc} t}$ is the associated bornological locally convex topology to the topology $\operatorname{loc} t$, is open. According to the Lemma 1.1, we have the following proposition for the associated bornological group.

PROPOSITION 1.1. *If (E,t) is a LCG and if F is a finite codimensional subgroup, then $\widetilde{t|F} = \tilde{t}|F$.*

Using the associated LCG, we can prove the following theorem.

Theorem 1.2. *If (E,t) is a quasibarrelled LCG and if F is a finite codimensional subgroup, then F is quasibarrelled with respect to the induced topology.*

Proof. Let F be a hyperplane in (E,t) and $x_0 \notin F$. If $\mathrm{Cl}(F) = F$, then $T + \langle x_0 \rangle$ is an absorbing bornivorous barrel in E, because $T + \langle x_0 \rangle = \psi(T \times \langle x_0 \rangle)$, where ψ is the topological isomorphism of the spaces E and $F \times \mathcal{L}\{x_0\}$.

If $\mathrm{Cl}(F) = E$, then it is either $\mathrm{Cl}(T) \subset F$ or $\mathrm{Cl}(T) \not\subset F$. Let us prove that $T_1 = \mathrm{Cl}_{\mathrm{loc}\,t}(F) + \langle x_0 \rangle$ is an absorbing bornivorous barrel in E. T_1 is obviously absorbing and t-closed subset in E, because it is $\mathrm{loc}\,t$-closed as a sum of $\mathrm{loc}\,t$-closed and $\mathrm{loc}\,t$-compact subsets.

If $\mathrm{Cl}(T) \subset F$ and if $(F,\tilde{t})$ is a dense subgroup in $(E,\tilde{t})$, then $\mathrm{Cl}_{\mathrm{loc}\,t}(T) = \bigcap_{\lambda>1} \lambda T \subset \lambda T \subset F$, so $\mathrm{Cl}_{\mathrm{loc}\,t}(T)$ is also bornivorous in (E,t), wherefrom it follows that T_1 is a t-neighbourhood of zero. Also $T_1 \cap F \subset 2T$, which means that T is a neighbourhood of zero in F, i.e. that F is a quasibarrelled LCG.

If $\mathrm{Cl}(T) \subset F$ and if $(F,\tilde{t})$ is a closed subgroup in $(E,\tilde{t})$, then $T + \langle x_0 \rangle$ is an absorbing bornivorous barrel in $(E,\tilde{t})$, wherefrom it follows that $Cl_{\mathrm{loc}\,t}(T) + \langle x_0 \rangle$ is a t-bornivorous subset. So, T_1 is a t-neighbourhood of zero for which $T_1 \cap F \subset 2T$. Thus, T is a neighbourhood of zero in F, so in this case F is a quasibarrelled LCG, too.

In the second case, when $\mathrm{Cl}(T) \not\subset F$, $\mathrm{Cl}(T)$ is an absorbing one in E. If, besides, $(F,\tilde{t})$ is a dense subgroup in $(E,\tilde{t})$, then $\mathrm{Cl}(T)$ is a bornivorous absorbing barrel in $(E,\tilde{t})$, because $\mathrm{Cl}_{\tilde{t}}(T) \subset \mathrm{Cl}(T)$, so $\mathrm{Cl}(T)$ is a t-neighbourhood of zero, i.e. T_1 is a t-neighbourhood of zero. Thus, T is a neighbourhood of zero in F, so F is a quasibarrelled LCG. If $(F,\tilde{t})$ is closed in $(E,\tilde{t})$, then $\mathrm{Cl}(T) + \langle x_0 \rangle$, i.e. $\mathrm{Cl}_{\mathrm{loc}\,t}(T) + \langle x_0 \rangle = T_1$, is a t-neighbourhood of zero, so in this case T is a neighbourhood of zero in F, i.e. F is quasibarrelled.

Comparing [**6**] and the previous result with the results from [**2**] and [**9**] it is natural to ask whether a barrel, a bornivorous barrel or an absorbing barrel of a finite codimensional subgroup have extensions in a group. Before giving a partial answer to this question, let us mention that the line $\mathcal{L}\{x_0\}$ has a discrete topology if and only if $x_0 \notin L$, as well as that in every LCG (E,t) $\langle x_0 \rangle$ is t-compact if and only if $x_0 \in L$, where $\langle x_0 \rangle$ is the absolutely convex hull of x_0.

Proposition 1.2. *Let (E,t) be a LCG, F be a hyperplane inside it, and T be a barrel* (*resp. bornivorous barrel, absorbing barrel, absorbing bornivorous barrel*) *in F. Then*:

(a) *If F is a closed hyperplane in (E,t), there exists a barrel T' of the same type in (E,t), such that $T' \cap F = T$.*

(b) *If* $(\mathrm{Cl}(F) = E$ *and* $\mathrm{Cl}(T) \not\subset F)$ *or* $(\mathrm{Cl}(F) = E$ *and* $T \subset F$ *and* $x_0 \in L \setminus F)$ *and if T is an absorbing barrel* (*absorbing bornivorous barrel*) *in F, there exists the barrel T' of the same type in (E,t) such that $T' \cap F = T$.*

Proof. (a) In this case it is easy to verify that the requested extension has the form $T' = T + \langle x_0 \rangle$, $x_0 \notin F$. If, for instance, T is a barrel in F, then T' is also a barrel in E. Indeed, $T' = T + \langle x_0 \rangle = \psi(T \times \langle x_0 \rangle)$, where ψ is the topological isomorphism between E and $F \times \mathcal{L}\{x_0\}$, so T' is closed in E and $\mathcal{L}(T') = \mathcal{L}(T) + \mathcal{L}\{x_0\}$ is open in E.

(b) In the case of a dense hyperplane, when $\mathrm{Cl}(T) \subset F$ and $x_0 \in L \setminus F$, the requested extension has the same form as in the case of a closed hyperplane, i.e. $T' = T + \langle x_0 \rangle$. Indeed, T' is an absorbing disc in E. Since $x_0 \notin F$, it is obvious that T is closed in E, and according to the mentioned remark $\langle x_0 \rangle$ is compact in $\mathcal{L}\{x_0\}$ so that T' is closed, too. Like in the proof of Lemma 1.1, T' is bornivorous. If $\mathrm{Cl}(F) = E$ and $\mathrm{Cl}(T) \not\subset F$, then it is clear that $T' = \mathrm{Cl}(T)$ is the requested extension for both sorts of barrels.

Remark 1. If F is a dense hyperplane in (E, t), and T is a barrel (resp. bornivorous barrel) in F, then we do not know whether F has an extension of the same type in (E, t).

Remark 2. If F is a dense hyperplane in (E, t), $\mathrm{Cl}(T) \subset F$ and if each algebraic complement of F has a discrete topology, and if T is an absorbing (resp. absorbing bornivorous) barrel in F, then, we also do not know if there exists an extension of the same type in (E, t).

2. Strongly-quasibarrelled LCG. Considering the definitions of the mentioned classes of LCGs from [**6**] and [**7**], it is natural to ask whether the LCGs for which every closed bornivorous disc is a neighbourhood of zero, are different from bornological ones, i.e. quasibarrelled LCGs. If such LCGs, are LCSs, then there is no difference between them and the quasibarrelled ones. Such LCGs will further be called strongly-quasibarrelled. Like in bornological LCG, the connected component of zero in strongly-quasibarrelled LCG is open, too. It is easy to see that the mentioned LCS $(E, \mathrm{loc}\, t)$ is quasibarrelled for every strongly-quasibarrelled LCG (E, t). The example from [**7**] shows that the converse does not necessarily hold. But, if L is finite codimensional in E, then (E, t) is strongly-quasibarrelled if and only if $(E, \mathrm{loc}\, t)$ is a quasibarrelled space.

From the definition it follows that every bornological LCG is strongly-quasibarrelled, as well as that every strongly-quasibarrelled LCG is quasibarrelled. The following propositions give characterizations of the strongly-quasibarrelled LCGs.

PROPOSITION 2.1. *For every LCG (E, t) the following statements are equivalent*:

(a) *(E, t) is strongly-quasibarrelled*;

(b) *Every open subgroup is strongly-quasibarrelled*;

(c) *There exists an open subgroup which is strongly-quasibarrelled.*

Proof. (a) $\Rightarrow$ (b). If D is a closed bornivorous disc in an open subgroup F of a strongly-quasibarrelled LCG (E, t), then D is closed in E, because F is also closed in E. Since $L \subset F$, which is open-closed, D is also bornivorous in E, because every bounded subset of E is contained in L. Since (E,t) is strongly-quasibarrelled, it follows that D is a t-neighbourhood of zero in E; so $D \cap F = D$ is a t-neighbourhood of zero in F. Then F is strongly-quasibarrelled.

The implication (b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a). Let F be an open strongly-quasibarrelled subgroup of E and let D be a closed bornivorous disc in E. Then $D \cap F$ is a closed bornivorous disc in F, and the neighbourhood of zero is in F. Since F is an open subgroup in E, $D \cap F$ is a neighbourhood of zero in E, from which it follows that D is also a neighbourhood of zero in E. Thus, E is strongly-quasibarrelled.

Comparing this proposition to the corresponding ones in [**6**] and [**7**], which characterize barrelled and quasibarrelled LCGs, let us mention that the condition:

(d) *Every absorbing closed bornivorous disc is a neighbourhood of zero*;

is not equivalent to (a), (b) or (c), though (d) follows from each of them. Let us prove, for instance, that (c) $\Rightarrow$ (d). Let D be an absorbing closed bornivorous disc in a strongly-quasibarrelled LCG (E,t). Then $D \cap F$, where F is an open subgroup of (E,t), is a closed bornivorous disc in F, which is a neighbourhood of zero in F, and therefore in E. So, D is a neighbourhood of zero in E.

We give later the example which shows that (d) $\not\Rightarrow$ (c), i.e. that there exists a quasibarrelled LCG which is not strongly-quasibarrelled. That example is also a justification for introducing the class of strongly-quasibarrelled LCG.

Proposition 2.2. *A LCG (E,t) is strongly-quasibarrelled if and only if the connected component of zero L is open in E and L is a quasibarrelled LCS.*

Proof. Let (E,t) be a strongly-quasibarrelled LCG. Then the connected component of zero L contains all the bounded subsets from E, which means that L is a closed bornivorous disc in E, so that L is a neighbourhood of zero in E, i.e. $\operatorname{Int} L \neq \varnothing$, that is, L is open. If D is a bornivorous barrel in L, then D is a closed bornivorous disc in E, so that D is a neighbourhood of zero in L. Since L is open in E, D is a neighbourhood of zero in L, so L is a quasibarrelled LCS.

Conversely, let D be a closed bornivorous disc in E. Then $D \cap L$ is closed bornivorous disc in LCS L, which is absorbing in L, so $\mathcal{L}(D \cap L) = \mathcal{L}(D) \cap L = L$, i.e. $D \cap L$ is a barrel in L. Since L is quasibarrelled LCS, $D \cap L$ is a neighbourhood of zero in L, so that D is a neighbourhood of zero in E, because L is open in E. Thus E is a strongly-quasibarrelled LCG.

Proposition 2.3. *A LCG (E,t) is strongly-quasibarrelled if and only if L is open in E and (E,t) is a quasibarrelled LCG.*

Proof. If (E,t) is a strongly-quasibarrelled then it is clear that L is open in E. Now, if B is a bornivorous barrel in E, then B is closed bornivorous disc in E, so that B is a neighbourhood of zero in E. Thus, E is quasibarrelled.

Conversely, let L be open in E, (E,t) be a quasibarrelled LCG and let D be a closed bornivorous disc in E. Then $D \cap L$ is a closed bornivorous disc in L which is LCS, and $D \cap L$ is absorbing in L, so that $\mathcal{L}(D \cap L) = \mathcal{L}(D) \cap L = L$, i.e. $D \cap L$ is a bornivorous barrel in L, and in E as well. Since E is quasibarrelled, $D \cap L$ is a neighbourhood of zero in E, thus D is a neighbourhood of zero in E, so that E is a strongly-quasibarrelled LCG.

The following example shows that the openness of the component of zero is important in previous propositions, i.e. there exists a barrelled LCG, whose connected component of zero is not open. This example gives a negative answer to the question from [**7**] whether the following two statements are equivalent

a) (E, t) is a quasibarrelled LCG;

b) Every $b(E', E)$-bounded subset in E' is t-equicontinous.

However, it is easy to check that b) is equivalent to the statement that (E, t) is a strongly-quasibarrelled LCG.

Example. According to the Proposition 2.3 candidates for $L \subset G$ whose component of zero is not open, are, among others, all so-called topologized linear spaces, which are not with a discrete topology (see [**5**; §10]) because $L = \{0\}$ in them, and, according to the definition, they are Hausdorff spaces. Some of these spaces are quasibarrelled LCGs, but not strongly-quasibarreled.

Let, then, X be an arbitrary vector space, and let X^* be its algebraic dual equipped with a product topology of the field R, where R has the discrete topology (see [**5**; §10.10]. X^*, with the introduced topology is not a discrete topological space, it is a barrelled LCG [**6**] where $L = \{0\}$ and, obviously, L is not open. So, X^* is not strongly-quasibarrelled LCG (for details see [**6**, Example 5.3]).

From the example, it can also be seen that the property of being "bornological", that is "strongly-quasibarrelled" is not stable with respect to the products.

In the similar way as in strongly-quasibarrelled LCGs, the following proposition can be proved in bornological LCGs:

PROPOSITION 2.4. *A LCG (E, t) is bornological if and only if L is open in E and L is a bornological LCS.*

Finally, let us prove that the class of strongly-quasibarrelled LCGs is stable with respect to the finite codimensional subgroups.

THEOREM 2.1. *If (E, t) is a strongly-quasibarrelled LCG and F is a finite codimensional subgroup, then F is strongly-quasibarrelled with respect to the induced topology.*

We shall prove even more: *if F is a hyperplane in E and if D is a closed bornivorous disc in F, then there exists a closed bornivorous disc D' in E such that $D' \cap E = D$.*

Proof. If L is the connected component of zero in E, then L contains all bounded subsets from E. Two cases are possible: $L \subset F$ and $L \not\subset F$.

In the first case, if D is a closed bornivorous disc in F, then D is a bornivorous disc also in E, so that $\mathrm{Cl}(D)$ is a closed bornivorous disc in E for which it still holds that $\mathrm{Cl}(D) \cap F = \mathrm{Cl}_F(D) = D$. So $\mathrm{Cl}(D) = D'$.

If $L \not\subset F$, then $L \cap F$ is a hyperplane in LCS L, and $D \cap (L \cap F) = D \cap L$ is a closed bornivorous disc in $L \cap F$. According to [**9**], in LCS L there exists a closed bornivorous disc W, such that $W \cap (L \cap F) = D \cap (L \cap F)$. It is known from [**9**] that W has the form $\mathrm{Cl}_L(D \cap L)$, if $\mathrm{Cl}(D \cap L) \not\subset L \cap F$; that is $D \cap L + \langle x_0 \rangle$, if $\mathrm{Cl}(D \cap L) \subset L \cap F$. In the first case $W = \mathrm{Cl}_L(D \cap L) \subset \mathrm{Cl}(D) \cap L \subset \mathrm{Cl}(D)$ which means that $\mathrm{Cl}(D)$ is a bornivorous disc in E, for which $\mathrm{Cl}(D) \cap F = \mathrm{Cl}_F(D) = D$ holds. In the second case $W = D \cap L + \langle x_0 \rangle \subset D + \langle x_0 \rangle \subset \mathrm{Cl}(D) + \langle x_0 \rangle$ so $D' = \mathrm{Cl}(D) + \langle x_0 \rangle$ is a closed bornivorous disc in E, because $x_0 \in L$, for which $D' \cap F = D$. The proof of the Theorem is completed.

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