

PROPERTIES OF SOLUTIONS OF SOME LINEAR CLASS OF INTEGRODIFFERENTIAL EQUATIONS OF VOLTERRA TYPE

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Abstract. We present a method for calculating a fundamental matrix of the equation (3). In addition we give a formula for a particular solution of the system (1).

We shall associate the linear neutral Volterra integrodifferential equation

$$(1) \quad \begin{aligned} \frac{d}{dt} \left[x(t) - \int_{-\infty}^t C(t-s)x(s) ds - g(t) \right] \\ = A(t)x(t) + B(t) \int_{-\infty}^t F(t-s)x(s) ds + f(t) \end{aligned}$$

with

$$(2) \quad \begin{aligned} \frac{d}{dt} \left[x(t) - \int_{t_0}^t C(t-s)x(s) ds - g(t) \right] \\ = A(t)x(t) + B(t) \int_{t_0}^t F(t-s)x(s) ds + f(t) \end{aligned}$$

via the resolvent equation

$$(3) \quad \begin{aligned} \frac{d}{dt} \left[Z(t) - \int_{t_0}^t C(t-s)Z(s) ds \right] \\ = A(t)Z(t) + B(t) \int_{t_0}^t F(t-s)Z(s) ds, \quad Z(t_0) = E_n. \end{aligned}$$

Here and hereafter x is an n -vector, $A(t)$ and $B(t)$ are $n \times n$ matrices continuous on $(-\infty, \infty)$, $g, f : (-\infty, \infty) \rightarrow \mathbf{R}^n$ are continuous, E_n the $n \times n$ identity matrix, Z an $n \times n$ matrix, and $C(t)$, $F(t)$ are matrices $n \times n$ which can be represented in the form

$$C(t) = \sum_{j=1}^k \psi_j(t) \exp(\alpha_j t), \quad F(t) = \sum_{j=1}^k \varphi_j(t) \exp(\beta_j t)$$

where

$$\psi_j(t) = \sum_{s=0}^{n_j} \psi_{sj} t^s, \quad \varphi_j(t) = \sum_{s=0}^{n_j} \varphi_{sj} t^s$$

and ψ_{sj}, φ_{sj} are constant $n \times n$ matrices, $\alpha_j, \beta_j = \text{const.}$

In the case where $C(t) = g(t) = 0$ and $A(t) = A, B(t) = E_n$ T. Burton [1] proved that for any bounded solution $x(t)$ of (2) there exists an integer sequence $n_j \rightarrow \infty$ as $j \rightarrow \infty$ such that $x(t + n_j T)$ ($T > 0$) converges to a solution $x^*(t)$ of (1). A similar result can be found in [2] for the case $g(t) = 0$ under the assumptions $Z \in L^1(0, \infty)$ and $\lim_{t \rightarrow \infty} Z(t) = 0$. In [3], the discussion of the T -periodic solution of (1) (in the case $g(t) = 0$) depended heavily on the behaviour of solutions of the integral equation

$$h(t) = \int_0^t C(t-s)h(s) ds + f(t).$$

In [4] Jianhong Wu proved by using the variation of constants formula for equation (2) that if $Z, Z' \in L^1(0, \infty)$, then there exists a unique globally stable T -periodic solution

$$g(t) + \int_{-\infty}^t Z'(t-s)g(s) ds + \int_{-\infty}^t Z(t-s)f(s) ds$$

of equation (1), where $Z(t)$ is the solution of equation (3).

The present paper is an extension of [1–4]. We shall present some facts relative to the existence of periodic and almost periodic solutions of the systems (1)–(2).

Putting

$$\begin{aligned} \int_{t_0}^t \exp(\alpha_j(t-z))(t-z)^s x(z) dz &= u_{sj}(t), \\ \int_{t_0}^t \exp(\beta_j(t-z))(t-z)^s x(z) dz &= y_{sj}(t), \quad (s = 0, \dots, n_j, j = 1, \dots, k) \end{aligned}$$

the system (2) becomes equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= \left(A(t) + \sum_{j=1}^k \psi_{0j} \right) x + B(t) \sum_{j=1}^k \sum_{s=0}^{n_j} \varphi_{sj} y_{sj} + \sum_{j=1}^k \sum_{s=1}^{n_j} \psi_{sj} (\alpha_j u_{sj} + s u_{s-1j}) \\ (4) \quad &+ \sum_{j=1}^k \psi_{0j} \alpha_j u_{0j} + f(t) + g'(t) \end{aligned}$$

$$\begin{aligned} \frac{du_{0j}}{dt} &= x + \alpha_j u_{0j}, \quad \frac{dy_{0j}}{dt} = x + \beta_j y_{0j}, \quad j = 1, \dots, k \\ \frac{du_{sj}}{dt} &= \alpha_j u_{sj} + s u_{s-1j}, \quad \frac{dy_{sj}}{dt} = s y_{s-1j} + \beta_j y_{sj}, \quad \begin{matrix} s = 1, \dots, n_j, \\ j = 1, \dots, k, \end{matrix} \end{aligned}$$

with initial conditions

$$(5) \quad u_{sj}(t_0) = 0, \quad y_{sj}(t_0) = 0, \quad s = 0, \dots, n_j, \quad j = 1, \dots, k$$

as only $g'(t)$ exists and is continuous for $t \geq t_0$.

Of course the derivative $x'(t)$ and $\int_{t_0}^t C(t-s)x(s)ds$ must exist for the representation (4).

THEOREM 1. *Let $W(t, t_0)$, $(W(t_0, t_0) = E_p)$ be the $p \times p$ fundamental matrix of the system (4) with $f(t) = g(t) = 0$; then the fundamental $n \times n$ matrix $X(t, t_0)$, $(X(t_0, t_0) = E_n)$ of system (3) is the upper left minor of degree n of $W(t, t_0)$, i.e.*

$$W(t, t_0) = \begin{bmatrix} X & P \\ Q & R \end{bmatrix}.$$

Proof. Let $W(t, t_0)$ denote the fundamental $p \times p$ matrix of system (4) with $f(t) = g(t) = 0$, where $p = n(2k+1) + 2 \sum_{j=1}^k n_j$. Since the general solution of system (4) with $f(t) = g(t) = 0$ has the form $W(t, t_0)C$, where C is a p -dimensional vector, then in order to obtain a general solution of problem (4)–(5) it is necessary to equate all components of $W(t, t_0)C$ except of the first n to zero and express some of the $p - n$ arbitrary constants of the vector C in terms of any constants of the vector C . Since $W(t_0, t_0) = E_p$, then $c_{n+1} = \dots = c_p = 0$, where c_i ($i = 1, \dots, p$) are components of the vector C . Hence the fundamental $n \times n$ matrix $X(t, t_0)$, $(X(t_0, t_0) = E_n)$ of system (3) is equal to the upper left minor of degree n of $W(t, t_0)$.

Let $W(t)$ ($W(t_0) = E_p$) denote the fundamental matrix of (4) with $f(t) = g(t) = 0$ and let $A(t)$, $B(t)$ be periodic of period ω , then by the Floquet Theorem

$$(6) \quad W(t) = Q(t) \exp(\Lambda(t - t_0))$$

where $Q(t)$ is periodic of period ω , $Q(t_0) = E_p$, and Λ is a constant matrix.

THEOREM 2. *Let (6) be the fundamental matrix solution of (4) such that $W(t_0) = E_p$, then*

$$X(t) = Q^*(t) \exp(\Lambda(t - t_0))M$$

will be a fundamental matrix solution of (3) such that $X(t_0) = E_n$, where $Q^(t)$ is an $n \times p$ periodic matrix of period ω obtained by deleting the last $p - n$ rows in the matrix $Q(t)$, $M = (m_{ij})$ is a $p \times n$ matrix with the property that $m_{ii} = 1$ for $i = 1, \dots, n$ and $m_{ij} = 0$ for $i \neq j$.*

Proof. From (6) the general solution of system (4) with $f(t) = g(t) = 0$ has the form $Q(t) \exp(\Lambda(t - t_0))C$, where C is a p -dimensional vector. Let $x(t_0) = c^*$, where c^* is an n -dimensional vector. By Theorem 1, in order to obtain a general solution of (2) with $f(t) = g(t) = 0$ it is necessary to equate the components c_{n+1}, \dots, c_p to zero.

Hence the general solution of (2) with $f(t) = g(t) = 0$ and the fundamental matrix of (3) can be represented by

$$x(t) = Q^*(t) \exp(\Lambda(t - t_0))M c^* \quad \text{and} \quad X(t) = Q^*(t) \exp(\Lambda(t - t_0))M.$$

Let $W(t)$ ($W(t_0) = E_p$) denote the fundamental matrix of (4). Equation (4) with initial values $x(t_0) = x_0$, $y_{sj}(t_0) = 0$, $u_{sj}(t_0) = 0$, $s = 0, \dots, n_j$, $j = 1, \dots, k$, is equivalent to the equation

$$(7) \quad z(t) = W(t)z_0 + \int_{t_0}^t W(t)W^{-1}(r)[\Phi_1(r) + \Phi_2(r)] dr$$

where

$$\begin{aligned} \Phi_1(t) &= \text{col}(f(t), 0, 0, 0, 0), & \Phi_2(t) &= \text{col}(g'(t), 0, 0, 0, 0), \\ z(t) &= \text{col}(x(t), y_{0j}(t), y_{sj}(t), u_{0j}(t), u_{sj}(t)) \\ z_0 &= \text{col}(x_0, 0, 0, 0, 0), & (s = 0, \dots, n_j, j = 1, \dots, k). \end{aligned}$$

The first n components of (7) give the solution of (2) with initial conditions $x(t_0) = x_0$, i.e.

$$(8) \quad x(t) = X(t, t_0)(x_0 - g_0) + g(t) + \int_{t_0}^t X(t, r)f(r) dr + \int_{t_0}^t X'_r(t, r)g(r) dr,$$

where $g_0 = g(t_0)$ and $X(t, s)$ is the fundamental $n \times n$ matrix of (3) defined in Theorem 1.

For further consideration we assume that $t_0 = 0$.

LEMMA. If $1^\circ A(t), B(t), f(t), g(t)$ are periodic of period ω ; $2^\circ x(t)$ is the solution of (2), then $x(t + \omega)$ is the solution of (2) if and only if

$$(9) \quad \int_0^\omega [C_t(t + \omega - r) + B(t)F(t + \omega - r)]x(r) dr = 0.$$

Proof. From the identity

$$\begin{aligned} & \frac{d}{dt} \left[x(t + \omega) - \int_0^{t+\omega} C(t + \omega - r)x(r) dr - g(t) \right] \\ &= A(t)x(t + \omega) + B(t) \int_0^{t+\omega} F(t + \omega - r)x(r) dr + f(t) \end{aligned}$$

and

$$\begin{aligned} & \frac{d}{dt} \left[x(t + \omega) - \int_0^t C(t - r)x(r + \omega) dr - g(t) \right] \\ &= A(t)x(t + \omega) + B(t) \int_0^t F(t - r)x(r + \omega) dr + f(t) \end{aligned}$$

we see that

$$\frac{d}{dt} \int_0^\omega C(t + \omega - r)x(r) dr + B(t) \int_0^\omega F(t + \omega - r)x(r) dr = 0.$$

Hence

$$\int_0^\omega (C_t(t + \omega - r) + B(t)F(t + \omega - r))x(r) dr = 0.$$

The sufficiency is obvious.

THEOREM 3. *If $A(t), B(t), f(t)$ and $g(t)$ are periodic of period ω , then the solution $x(t)$ of (2) is periodic if and only if (9) and $x(0) = x(\omega)$ hold.*

Proof. If (9) and $x(0) = x(\omega)$ hold, then the solutions $x(t + \omega)$ and $x(t)$ coincide for $t = 0$. Hence, according to the uniqueness theorem, they coincide for any t and thus $x(t)$ is periodic of period ω . If the solution $x(t)$ of (2) is periodic of period ω , then the above conditions are obviously verified.

Example. Consider the scalar equation

$$\begin{aligned} \frac{d}{dt} \left[x(t) - \int_0^t \exp(-4(t-s))x(s) ds - \frac{3}{2} \cos t \right] \\ = -x(t) + 4 \int_0^t \exp(-4(t-s))x(s) ds + 2 \cos t + \sin t. \end{aligned}$$

It is not difficult to show that this equation has a periodic solution

$$x^*(t) = 2 \sin t + (1/2) \cos t$$

THEOREM 4. *If 1° $A(t), B(t), f(t), g(t)$ are periodic of period ω ;
2° $\det(E_n - X(\omega, 0)) \neq 0$,*

then the system (2) admits a periodic solution of period ω if and only if

$$\begin{aligned} (10) \quad x(t) = X(t, 0)[E_n - X(\omega, 0)]^{-1} \left\{ \int_0^\omega X(\omega, r)f(r) dr + \int_0^\omega X'_r(\omega, r)g(r) dr \right\} \\ + g(t) + \int_0^t X(t, r)f(r) dr + \int_0^t X'_r(t, r)g(r) dr \end{aligned}$$

satisfies (9). This solution is (10).

Proof. Let $x(t)$ be a periodic solution of period ω of the system (2), then from (8)

$$x(0) = X(\omega, 0)(x_0 - g_0) + g(\omega) + \int_0^\omega X(\omega, r)f(r) dr + \int_0^\omega X'_r(\omega, r)g(r) dr.$$

Hence, since $\det(E_n - X(\omega, 0)) \neq 0$ we have that

$$x_0 = g_0 + (E_n - X(\omega, 0))^{-1} \left\{ \int_0^\omega X(\omega, r)f(r) dr + \int_0^\omega X'_r(\omega, r)g(r) dr \right\}.$$

The solution $x(t)$ can therefore be written as (10).

The rest of this proof is very similar and therefore is omitted.

Following an argument similar to those of [4, Theorem 2; 3, Theorem 2] we get

THEOREM 5. *Let $C, F \in L^1(0, \infty)$ and let $A(t), B(t), f(t), g(t)$ be periodic of period ω . If $x(t) = x(t, 0, x_0)$ is a bounded solution of (2) on $(0, \infty)$, then there is a sequence of positive integers $\{n_j\}$, $n_j \rightarrow \infty$ as $j \rightarrow \infty$, such that $\{x(t + n_j\omega)\}$ converges uniformly on compact subsets of $(-\infty, \infty)$ to a function $x^*(t)$ which is a solution of (1).*

Proof. Let $C, F \in L^1\langle 0, \infty \rangle$ and let $x(t)$ be a bounded solution of (2) on $\langle 0, \infty \rangle$. We want to show that $\{x(t + n\omega) : n = 1, \dots\}$ is equicontinuous and uniformly bounded on any fixed interval $\langle -k, k \rangle$.

For $t_2 \geq t_1 \geq -n\omega$, we integrate (2) from $t_1 + n\omega$ to $t_2 + n\omega$ and get

$$\begin{aligned} x(t_2 + n\omega) - x(t_1 + n\omega) &= \int_0^{t_2 + n\omega} C(t_2 + n\omega - s)x(s) ds - \int_0^{t_1 + n\omega} C(t_1 + n\omega - s)x(s) ds \\ &\quad + \int_{t_1 + n\omega}^{t_2 + n\omega} \left(A(t)x(t) + B(t) \int_0^t F(t - s)x(s) ds + f(t) \right) dt \\ &\quad + g(t_2 + n\omega) - g(t_1 + n\omega). \end{aligned}$$

The functions $x(t), f(t), A(t), B(t)$ are bounded and $F \in L^1\langle 0, \infty \rangle$, hence there exist M and N such that $|x(t)| \leq M$, $|f(t)| \leq M$, $|A(t)| \leq M$, $|B(t)| \leq M$ for $t \in \langle 0, \infty \rangle$ and $\int_0^\infty |F(t)| dt = N < \infty$. Thus

$$\int_{t_1 + n\omega}^{t_2 + n\omega} \left| A(t)x(t) + B(t) \int_0^t F(t - s)x(s) ds + f(t) \right| dt \leq M_1 |t_2 - t_1|$$

where $M_1 = M(1 + MN + M)$. Since $C \in L^1\langle 0, \infty \rangle$, then for any $\varepsilon > 0$, there exists a $T > 0$ such that

$$\int_t^\infty |C(s)| ds < \frac{\varepsilon}{8M} \quad \text{for } t \geq T \quad \text{and so} \quad \int_T^\infty |C(t_2 - t_1 + v) - C(v)| dv < \frac{\varepsilon}{4M}.$$

By the continuity of C , there exists a $\delta_1 > 0$ such that for $v \in \langle 0, T \rangle$ and $0 \leq t_2 - t_1 \leq \delta_1$ we have

$$|C(t_2 - t_1 + v) - C(v)| < \frac{\varepsilon}{4TM} \quad \text{and} \quad \int_0^{t_2 - t_1} |C(v)| dv < \frac{\varepsilon}{4M}.$$

Thus

$$\begin{aligned} &\left| \int_0^{t_2 + n\omega} C(t_2 + n\omega - s)x(s) ds - \int_0^{t_1 + n\omega} C(t_1 + n\omega - s)x(s) ds \right| \\ &\leq M \int_0^{t_1 + n\omega} |C(t_2 - t_1 + v) - C(v)| dv + M \int_0^{t_2 - t_1} |C(v)| dv \\ &\leq M \int_0^T |C(t_2 - t_1 + v) - C(v)| dv + M \int_T^\infty |C(t_2 - t_1 + v) - C(v)| dv \\ &\quad + M \int_0^{t_2 - t_1} |C(v)| dv \leq \frac{3}{4} \end{aligned}$$

if $0 \leq t_2 - t_1 \leq \delta_1$.

By the continuity of g , there exists a $\delta_2 > 0$ such that

$$0 \leq (t_2 + n\omega) - (t_1 + n\omega) \leq \delta_2 \quad \text{imply} \quad |g(t_2 + n\omega) - g(t_1 + n\omega)| \leq \varepsilon/8.$$

Let $\delta = \min(\delta_1, \varepsilon/8M_1, \delta_2)$. Then we have

$$|x(t_2 + n\omega) - x(t_1 + n\omega)| \leq \varepsilon \quad \text{if} \quad 0 \leq t_2 - t_1 \leq \delta.$$

This implies that $\{x(t + n\omega)\}$ is equicontinuous and uniformly bounded on any fixed interval $\langle -k, k \rangle$, $k = 1, 2, \dots$. By Ascoli's theorem there is a subsequence $\{x(t + n_1\omega)\}$ of the $x(t + n\omega)$'s converging uniformly on $\langle -1, 1 \rangle$, which contains a subsequence $\{x(t + n_2\omega)\}$ on $\langle -2, 2 \rangle$. Proceeding inductively we obtain a subsequence, say $\{x(t + n_j\omega)\}$, converging uniformly on any fixed interval $\langle -k, k \rangle$ to a continuous function $x^*(t)$.

Now, we show that $x^*(t)$ is a solution of (1). Integrating (2) from $n_j\omega$ to $t + n_j\omega$, we have

$$\begin{aligned} x(t + n_j\omega) - x(n_j\omega) &= \int_0^{t+n_j\omega} C(t + n_j\omega - s)x(s) ds - \int_0^{n_j\omega} C(n_j\omega - s)x(s) ds \\ &\quad + \int_{n_j\omega}^{t+n_j\omega} \left(A(s)x(s) + B(s) \int_0^s F(s-v)x(v) dv + f(s) \right) ds \\ &\quad + g(t + n_j\omega) - g(n_j\omega) \\ &= \int_{-n_j\omega}^t C(t-v)x(v + n_j\omega) dv - \int_{-n_j\omega}^0 C(-v)x(v + n_j\omega) dv \\ &\quad + \int_0^t \left(A(v + n_j\omega)x(v + n_j\omega) \right. \\ &\quad \left. + B(v + n_j\omega) \int_{-n_j\omega}^v F(v-u)x(u + n_j\omega) du + f(v) \right) dv + g(t) - g(n_j\omega) \\ &= \int_{-n_j\omega}^t C(t-v)x(v + n_j\omega) dv - \int_{-n_j\omega}^0 C(-v)x(v + n_j\omega) dv \\ &\quad + \int_0^t \left(A(v)x(v + n_j\omega) + B(v) \int_{-n_j\omega}^v F(v-u)x(u + n_j\omega) du + f(v) \right) dv \\ &\quad + g(t) - g(n_j\omega). \end{aligned}$$

Since $C, F \in L^1\langle 0, \infty \rangle$, by Lebesgue's dominated convergence theorem by letting $j \rightarrow \infty$, we have

$$\begin{aligned} x^*(t) - x^*(0) &= \int_{-\infty}^t C(t-v)x^*(v) dv - \int_{-\infty}^0 C(-v)x^*(v) dv \\ &\quad + \int_0^t \left(A(v)x^*(v) + B(v) \int_{-\infty}^v F(v-u)x^*(u) du + f(v) \right) dv + g(t) - g(\infty). \end{aligned}$$

Hence, by differentiation, we have

$$\begin{aligned} \frac{d}{dt} \left[x^*(t) - \int_{-\infty}^t C(t-v)x^*(v) dv - g(t) \right] \\ = A(t)x^*(t) + B(t) \int_{-\infty}^t F(t-v)x^*(v) dv + f(t) \end{aligned}$$

and so the limit function $x^*(t)$ is a solution of (1).

Let $A(t)$ and $B(t)$ be periodic and f, g — almost periodic and $\operatorname{Re} \alpha_j < 0$, $\operatorname{Re} \beta_j < 0$. Let $x(t)$ be an almost periodic solution of (2). Then

$$\begin{aligned} & \int_0^t e^{\beta_j(t-z)} (t-z)^s x(z) dz \\ &= - \int_{-\infty}^0 e^{\beta_j(t-z)} (t-z)^s x(z) dz + \int_{-\infty}^t e^{\beta_j(t-z)} (t-z)^s x(z) dz, \end{aligned}$$

where $\int_{-\infty}^t e^{\beta_j(t-z)} (t-z)^s x(z) dz$ is an almost periodic function. Since $x(t)$ is an almost periodic solution of (2), then

$$(11) \quad \sum_{s=0}^{n_j} B(t) \varphi_{sj} \int_{-\infty}^0 e^{\beta_j(t-z)} (t-z)^s x(z) dz \equiv 0 \quad (j = 1, \dots, k).$$

It is easy to see that

$$\sum_{s=0}^{n_j} B(t) \varphi_{sj} \int_{-\infty}^0 e^{\beta_j(t-z)} (t-z)^s x(z) dz \quad (j = 1, \dots, k)$$

is an almost periodic function. In this case, if

$$(11') \quad \sum_{s=0}^{n_j} \varphi_{sj} \int_{-\infty}^0 e^{\alpha_j(t-z)} (t-z)^s x(z) dz \equiv 0 \quad (j = 1, \dots, k).$$

then

$$\sum_{s=0}^{n_j} \varphi_{sj} \int_0^t e^{\alpha_j(t-z)} (t-z)^s x(z) dz$$

is an almost periodic function.

On the ground of conditions (11), (11'), the solution $x(t)$ of the system (2) is the solution of the system (1).

Note that (1) is equivalent to the system

$$\begin{aligned} \frac{dx}{dt} &= \left[A(t) + \sum_{j=1}^k \psi_{0j} \right] + B(t) \sum_{j=1}^k \sum_{s=0}^{n_j} \varphi_{sj} w_{sj} \\ (12) \quad &+ \sum_{j=1}^k \sum_{s=1}^{n_j} \psi_{sj} (\alpha_j v_{sj} + s v_{s-1j}) + \sum_{j=1}^k \psi_{0j} \alpha_j v_{0j} + f(t) + g'(t) \end{aligned}$$

$$\begin{aligned} dv_{0j}/dt &= x + \alpha_j v_{0j}, & dw_{0j}/dt &= x + \beta_j w_{0j} & s &= 1, \dots, n_j, \\ dv_{sj}/dt &= \alpha_j v_{sj} + s v_{s-1j}, & dw_{sj}/dt &= s w_{s-1j} + \beta_j w_{sj} & j &= 1, \dots, k \end{aligned}$$

where

$$w_{sj}(t) = \int_{-\infty}^t e^{\beta_j(t-z)} (t-z)^s x(z) dz, \quad v_{sj}(t) = \int_{-\infty}^t e^{\alpha_j(t-z)} (t-z)^s x(z) dz.$$

According to the assumption, $x(t)$ is an almost periodic function, so $w_{sj}(t)$ and $v_{sj}(t)$ are almost periodic functions.

THEOREM 6. *If $A(t)$ and $B(t)$ are periodic, f, g are almost periodic and $\operatorname{Re} \alpha_j < 0$, $\operatorname{Re} \beta_j < 0$ ($j = 1, \dots, k$), then $x(t)$ is an almost periodic solution of the system (2) if and only if $(x(t), w_{sj}(t), v_{sj}(t))$ is an almost periodic solution of the system (12) and conditions (11)–(11') are satisfied.*

Proof. If the system (2) admits an almost periodic solution $x(t)$, then the conditions (11)–(11') are obviously verified and (12) has an almost periodic solution.

Conversely, if (12) has an almost periodic solution and conditions (11)–(11') are satisfied, then the solution $x(t)$ of (2) is almost periodic.

Remark. If $A(t) = A = \text{const}$, $B(t) = B = \text{const}$, $t_0 = 0$, then the system (2) has a general solution of the form

$$x(t) = X(t)(x_0 - g_0) + g(t) + \int_0^t X(t-s)f(s) ds + \int_0^t X'(t-s)g(s) ds.$$

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