PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 50 (64), 1991, 141–151

THE REGULARIZED TRACE FOR NUCLEAR PERTURBATIONS OF DISCRETE OPERATORS

Milutin Dostanić

Abstract. The regularized trace of nuclear perturbations for semibounded (positive) discrete operators is found.

1. Introduction

In [**1**] the regularized trace was found for a perturbation of a discrete operator by an operator of finite rank. For this result the behaviour of the function $N(\lambda)$ played an important role, where $N(\lambda)$ denotes the distribution of the eigenvalues for the nonperturbed operator.

In this paper this result is generalized for perturbations of a discrete operator by a nuclear operator which satisfies some additional conditions.

2. Statement of the problem and the main result

Let L_0 be a selfadjoint discrete operator acting on a separable Hilbert space H, with domain $\mathcal{D}(L_0)$. We denote by S a nuclear operator with a Hilbert representation:

$$S = \sum_{k=1}^{\infty} s_k(\,\cdot\,, f_k)g_k, \qquad \left(\sum_{k=1}^{\infty} s_k < \infty\right).$$

Let us consider the operator $L = L_0 + SL_0$. In this case μ_ν and λ_ν are eigenvalues of the operators L and L_0 respectively, arranged in the increasing order of their real parts, each of which is repeated according to its multiplicity. The main result is the following theorem.

Theorem. *Let L_0 be a selfadjoint, discrete, semibounded (from below) operator acting on H. Suppose that the distribution function of its eigenvalues has the form $N(\lambda) = \sum_{\lambda_n \le \lambda} 1 = c\lambda + O(\lambda^p)$ where c is a constant, $0 < p < 1$, $n = 1, 2, \dots$, $0 \notin \sigma(L_0)$. Suppose that one of the following conditions is satisfied:*

AMS Subject Classification (1980): Primary 47 A 55; Secondary 47 A 70

1° $c = 0$; *for some* $q \in [0,1]$ *we have that* $g_k \in \mathcal{D}(L_0^{1-q})$, $f_k \in \mathcal{D}(L_0^q)$ *for every* k *and in this case* $\sum_{k=1}^{\infty} s_k \|L_0^{1-q} g_k\| \cdot \|L_0^q f_k\| < \infty$;

2° $c \neq 0$; *for some* $q \in [0,2]$ *we have that* $g_k \in \mathcal{D}(L_0^{2-q})$, $f_k \in \mathcal{D}(L_0^q)$ *for every* k *and in this case* $\sum_{k=1}^{\infty} s_k \|L_0^{2-q} g_k\| \cdot \|L_0^q f_k\| < \infty$.

Then the operator L *is discrete and there exists a subsequence of natural numbers* k_n $(k_n \to +\infty)$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \sum_{k=1}^{\infty} s_k (L_0^{1-q} g_k, L_0^q f_k). \tag{1}$$

Note that formula (1) gives the regularized trace of the operator L because the right-hand side of this equality can be evaluated.

3. Resolvent of the operator L

For the sake of symmetry, we shall introduce new vectors $f'_k = \sqrt{s_k} f_k$, $g'_k = \sqrt{s_k} g_k$. Then

$$S = \sum_{k=1}^{\infty} (\,\cdot\,, f'_k) g'_k, \qquad (f'_k, f'_l) = s_k \delta_{kl}, \quad (g'_k, g'_l) = s_k \delta_{kl}.$$

In order to find the resolvent of the operator L it is necessary to examine the solvability of the equation

$$L_0 u + \sum_{k=1}^{\infty} (L_0 u, f'_k) g'_k = \lambda u + f \tag{2}$$

where $u \in \mathcal{D}(L_0)$ and $f \in H$. Suppose $\lambda \notin \sigma(L_0)$. Applying the operator $R_\lambda^0 = (L_0 - \lambda I)^{-1}$ to (2) we have that

$$u + \sum_{j=1}^{\infty} (L_0 u, f'_j) R_\lambda^0 g'_j = R_\lambda^0 f.$$

Let us denote $(L_0 u, f'_j) = \xi_j$; then we obtain

$$u = -\sum_{j=1}^{\infty} R_\lambda^0 g'_j + R_\lambda^0 f.$$

Substituting the expression for u in the formula for ξ_j we shall get an infinite system of algebraic linear equations:

$$\xi_k + \sum_{j=1}^{\infty} \xi_j (L_0 R_\lambda^0 g'_j, f'_k) = (L_0 R_\lambda^0 f, f'_k), \quad k = 1, 2, 3, \dots\ . \tag{3}$$

This system can be used to determine ξ_j. Put $a_{kj} = -(L_0 R_\lambda^0 g'_j, f'_k)$ and $c_k = (L_0 R_\lambda^0 f, f'_k)$; then the system (3) can be written in the form $\xi_k - \sum_{j=1}^{\infty} a_{kj} \xi_j = c_k$, $k = 1, 2, 3, \dots$. We can check directly

$$\sum_{k=1}^{\infty} |a_{kk}| < \infty, \qquad \sum_{j,k=1}^{\infty} |a_{kj}|^2 < \infty. \tag{4}$$

Formulas (4) imply that the infinite matrix $I-\mathcal{A}$, where $\mathcal{A}=(a_{kj})_{k,j=1}^{\infty}$ is a Koch matrix. Therefore, by deleting a fixed column and a row from the matrix $I-\mathcal{A}$ we shall get again a Koch matrix.

To the Koch matrix $I-\mathcal{A}$ we can associate its determinant:

$$\det(I-\mathcal{A})=1-\sum_{j=1}^{\infty}a_{jj}+\frac{1}{2!}\sum_{j,k=1}^{\infty}\begin{vmatrix}a_{jj} & a_{jk}\\ a_{kj} & a_{kk}\end{vmatrix}-\frac{1}{3!}\sum \quad +\cdots$$

(For more details about the Koch matrix see [**2**] or [**3**])

Since the matrix of the system $\xi_{ik}-\sum_{j=1}^{\infty}a_{kj}\xi_j=c_k$, $k=1,2,3,\dots$ is a Koch matrix, the solution of the system (3) is given by

$$\xi_k=\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}c_j/\Delta \tag{5}$$

where Δ_{jk} is the determinant of the matrix obtained from the matrix $I-\mathcal{A}$ by deleting its j-th row and k-th column; $\det(I-\mathcal{A})=\Delta$. Because of $R_\lambda f=u$ $(R_\lambda=(L-\lambda I)^{-1})$ and $u=-\sum_{j=1}^{\infty}\xi_jR_\lambda^0g_j'+R_\lambda^0f$, (5) implies that

$$R_\lambda f=R_\lambda^0f-\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}(L_0R_\lambda^0f,f_j')R_\lambda^0g_k'\right)\Delta^{-1}. \tag{6}$$

The spectrum of the operator L is discrete because of Theorem 10.1, p. 336, in [**2**]. Suppose $\lambda\notin\sigma(L_0)\cup\sigma(L)$; then

$$L-\lambda I=(I+SL_0R_\lambda^0)(L_0-\lambda I).$$

Since the operator $L-\lambda I$ vanishes only at the point $x=0$ it follows that $-1\notin\sigma(SL_0R_\lambda^0)$. The operator $SL_0R_\lambda^0$ is nuclear, so that $(I+SL_0R_\lambda^0)^{-1}=I+K$, where K is a nuclear operator. But then we have $(L-\lambda I)^{-1}=(L_0-\lambda I)^{-1}(I+K)$ so that $(L-\lambda I)^{-1}$ is a compact operator and L is a discrete operator.

Now, we introduce linear functionals φ_k:

$$\varphi_k(f)=\Delta^{-1}\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}s_k^{-1/2}(L_0R_\lambda^0f,f_j'),\qquad f\in H. \tag{7}$$

LEMMA 1. *The sequence of linear functionals φ_k $(k=1,2,3,\dots)$ on the space H is uniformly bounded.*

Proof. Since

$$\varphi_k(f)=\Delta^{-1}\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}s_k^{-1/2}(L_0R_\lambda^0f,f_j')$$

it follows that

$$\varphi_k(f)=\Delta^{-1}\Delta_{kk}(L_0R_\lambda^0f,f_k)+\Delta^{-1}\sum_{j\neq k}(-1)^{j+k}\Delta_{jk}s_j^{1/2}s_k^{-1/2}(L_0R_\lambda^0f,f_j)$$

and

$$|\varphi_k(f)| \le |\Delta|^{-1}|\Delta_{kk}|\cdot\|L_0R_\lambda^0\|\cdot\|f\| + |\Delta|^{-1}\left(\sum_{j\neq k}|\Delta_{jk}|^2 s_j/s_k\cdot\sum_{j\neq k}|(L_0R_\lambda^0 f, f_j)|^2\right)^{1/2}.$$

This formulas imply, by using the Bessel inequality, that

$$|\varphi_k(f)| \le \|f\|\left(|\Delta_{kk}|\cdot|\Delta|^{-1}\cdot\|L_0R_\lambda^0\| + \|L_0R_\lambda^0\|\cdot|\Delta|^{-1}\cdot\left(\sum_{j\neq k}|\Delta_{jk}|^2 s_j/s_k\right)^{1/2}\right).$$

We show that $\sup_{j,k}|\Delta_{jk}|/s_k < \infty$ and $\sup_k|\Delta_{kk}| < \infty$. By the Hadamard inequality

$$|\Delta_{jk}|^2 \le \prod_{p\neq j,\, p\neq k}\left(\sum_{q\neq k}|\delta_{pq}-a_{pq}|^2\right)\left(\sum_{q\neq k}|\delta_{kq}-a_{kq}|^2\right)$$

and by using that $a_{kj} = -(s_ks_j)^{1/2}(L_0R_\lambda^0 g_j, f_k)$ we get

$$\sum_{q\neq k}|\delta_{kq}-a_{kq}|^2 = \sum_{q\neq k}|a_{kq}|^2 \le s_k\|L_0R_\lambda^0\|^2|S|_1$$

($|S|_1$ is the nuclear norm of operator S). The last formula implies that

$$|\Delta_{jk}|^2/s_k \le \|L_0R_\lambda^0\|^2\cdot|S|_1\cdot\prod_{p\neq j,\, p\neq k}\left(\sum_{q\neq k}|\delta_{pq}-a_{pq}|^2\right).$$

Because of

$$\prod_{p\neq j,\, p\neq k}\left(\sum_{q\neq k}|\delta_{pq}-a_{pq}|^2\right) \le \exp\left(2\sum_{p=1}^{\infty}|a_{pp}| + \sum_{p,q=1}^{\infty}|a_{pq}|^2\right) = M < \infty$$

we have $|\Delta_{jk}|^2/s_k \le M|S|_1\cdot\|L_0R_\lambda^0\|^2$.

We can show in a similar way that $|\Delta_{kk}| \le \sqrt{M}$. This completes the proof of Lemma 1.

Using (7), the equality (6) can be written in the form

$$R_\lambda - R_\lambda^0 = -\sum_{k=1}^{\infty} s_k\varphi_k(\,\cdot\,)R_\lambda^0 g_k.$$

Since Lemma 1 implies $\sum_{k=1}^{\infty} s_k\|\varphi_k\|\,\|R_\lambda^0 g_k\| < \infty$, the operator

$$\sum_{k=1}^{\infty} s_k\varphi_k(\,\cdot\,)R_\lambda^0 g_k$$

is nuclear. So we have

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = -\sum_{k=1}^{\infty} s_k\varphi_k(R_\lambda^0 g_k)$$

and

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = -\Delta^{-1} \sum_{k,j=1}^{\infty} (-1)^{j+k} \Delta_{jk} (L_0 (R_\lambda^0)^2 g_k', f_j'). \tag{8}$$

Using that $a_{jk} = -(L_0 R_\lambda^0 g_k', f_j')$, we can get $a_{jk}'(\lambda) = -(L_0 (R_\lambda^0)^2 g_k', f_j')$ and (8) can be written as

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = \Delta^{-1} \sum_{k,j=1}^{\infty} (-1)^{j+k} \Delta_{jk} a_{jk}'(\lambda). \tag{9}$$

For the Koch matrix we have

$$\sum_{j,k=1}^{\infty} (-1)^{j+k} \Delta_{jk} a_{jk}'(\lambda) = \Delta'(\lambda),$$

so (9) implies

$$\mathrm{Sp}(R_\lambda - R_\lambda^0) = \Delta'(\lambda)/\Delta(\lambda). \tag{10}$$

4. The proof of the theorem in the case when $c = 0$ in the formula for $N(\lambda)$

Let $\{\lambda_k\}$ be the eigenvalues of the operator L_0 and $\{\psi_k\}$ an orthonormal base in the space H, where ψ_k are eigenvectors (corresponding to λ_k) of the operator L_0. We denote $c_n^j = (g_j, \psi_n)$, $b_n^j = (f_j, \psi_n)$; then $g_k = \sum_{n=1}^{\infty} c_n^k \psi_n$, $f_j = \sum_{n=1}^{\infty} b_n^j \psi_n$. The spectral theorem implies

$$(L_0 R_\lambda^0 g_k, f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n c_n^k \overline{b_n^j}}{\lambda - \lambda_n}. \tag{11}$$

Since the assumptions are $g_k \in \mathcal{D}(L_0^{1-q})$, $f_j \in \mathcal{D}(L_0^q)$, the Bessel inequality implies

$$\sum_{n=1}^{\infty} \lambda_n |c_n^k b_n^j| \le \|L_0^{1-q} g_k\| \, \|L_0^q f_j\|. \tag{12}$$

For the case $c = 0$ we have $N(\lambda) = O(\lambda^p)$ $(0 < p < 1)$ and then there exists a sequence of real numbers r_n $(r_n \to \infty)$ such that $d_n \to \infty$, where $d_n = d(\Gamma_n, \sigma(L_0))$ is the distance between the circle $\Gamma_n = \{\lambda : |\lambda| = r_n\}$ and the spectrum of the operator L [**1**].

LEMMA 2. *The operators L and L_0 have the same number of eigenvalues inside the circle Γ_n, for n large enough.*

Proof. We multiply equality (10) by $1/2\pi i$ and integrate it over Γ_n. Note that the well-known Riesz theorem implies

$$\frac{1}{2\pi i} \int_{\Gamma_n} \mathrm{Sp}(R_\lambda - R_\lambda^0) \, d\lambda = N_1 - N_2,$$

where N_1 and N_2 denote the numbers of eigenvalues inside the contour Γ_n (counting their multiplicity) of the operators L and L_0, respectively. So, this implies that

$$N_1 - N_2 = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\Delta'(\lambda)}{\Delta(\lambda)} \, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{S'(\lambda)}{1 + S(\lambda)} \, d\lambda \tag{13}$$

where $S(\lambda)=\Delta(\lambda)-1$. Let us estimate $S(\lambda)$. Since $a_{jk}=-(s_js_k)^{1/2}(L_0R_\lambda^0g_k,f_j)$ by using (11) and (12) we have

$$|a_{jk}|\le (s_js_k)^{1/2}\|L_0^{1-q}g_k\|\cdot\|L_0^qf_j\|/d(\lambda) \tag{14}$$

where $d(\lambda)$ denotes the distance between a point λ and $\sigma(L_0)$. Note that for $\lambda\in\Gamma_n$ we have $d(\lambda)\ge d_n$. The definition of the function $S(\lambda)$ and (14) imply the following estimate

$$\left.\begin{aligned} |S(\lambda)| &\le \sum_{k=1}^{\infty} M(\lambda)^k \\ \text{where}\quad M(\lambda) &= \sum_{j=1}^{\infty} s_j\|L_0^qf_j\|\cdot\|L_0^{1-q}g_j\|/d(\lambda)\end{aligned}\right\} \tag{15}$$

From $\sum_{j=1}^\infty s_j\|L_0^qf_j\|\,\|L_0^{1-q}g_j\|<\infty$, and (15) it follows that $S(\lambda)\to 0$ when $n\to\infty$ and $\lambda\in\Gamma_n$; the convergence is uniform with respect to λ. Then (13) implies

$$N_1-N_2=\sum_{\nu=0}^{\infty}(-1)^\nu\frac{1}{2\pi i}\int_{\Gamma_n}S'(\lambda)S^\nu(\lambda)\,d\lambda. \tag{16}$$

From (15) we get

$$|S^\nu(\lambda)|\le K_1^\nu/d(\lambda)^\nu \tag{17}$$

for $\lambda\in\Gamma_n$ and n is large enough (K_1 does not depend on λ and ν).

By differentiating (11) with respect to λ and using (12) we have

$$|a'_{jk}(\lambda)|\le (s_js_k)^{1/2}\|L_0^{1-q}g_k\|\cdot\|L_0^qf_j\|/d^2(\lambda). \tag{18}$$

From (14) and (18) and the definition of the function $S(\lambda)$ we can get the following estimate

$$|S'(\lambda)|\le\frac{1}{d(\lambda)}\cdot\sum_{n=1}^{\infty}nM^n(\lambda)\le\frac{K_2}{d^2(\lambda)} \tag{19}$$

where the constant K_2 does not depend on λ, $\lambda\in\Gamma_n$ and n is large enough. Then (17) and (19) imply

$$\left|\int_{\Gamma_n}S'(\lambda)S^\nu(\lambda)\,d\lambda\right|\le K_2K_1^\nu\int_{\Gamma_n}\frac{|d\lambda|}{d^{2+\nu}(\lambda)}.$$

Next we can easily check that

$$\int_{\Gamma_n}\frac{|d\lambda|}{d^2(\lambda)}\le\frac{C_0}{d_n}$$

(the constant C_0 does not depend on n). Because of $d(\lambda)\ge d_n$ on Γ_n the inequality

$$\int_{\Gamma_n}\frac{|d\lambda|}{d^{2+\nu}(\lambda)}\le\frac{1}{d_n^\nu}\int_{\Gamma_n}\frac{|d\lambda|}{d^2(\lambda)}$$

holds and consequently

$$\left|\int_{\Gamma_n}S'(\lambda)S^\nu(\lambda)\,d\lambda\right|\le C_0K_2K_1^\nu/d_n^{\nu+1} \tag{20}$$

for n large enough. Formula (16) implies $|N_1 - N_2| \le \sum_{\nu=0}^{\infty} C_0 K_2 K_1^\nu / d_n^{\nu+1}$ (n large enough) and $|N_1 - N_2| = O(1/d_n)$. Since $|N_1 - N_2|$ is a natural number or 0, this is possible if and only if $N_1 = N_2$. So the lemma is proved.

Let us now prove the Theorem in the case $c = 0$. If we multiply (10) by $\lambda/2\pi i$ and integrate it over Γ_n we have

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\,\mathrm{Sp}(R_\lambda - R_\lambda^0)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_n} \lambda \frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda. \tag{21}$$

The properties of Riesz projectors imply that

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\,\mathrm{Sp}(R_\lambda - R_\lambda^0)\,d\lambda = \sum_{\nu=1}^{k_n} (\mu_\nu - \lambda_\nu) \tag{22}$$

where k_n denotes the number of eigenvalues for operators L and L_0 inside the contour Γ_n (this number is the same for both operators by Lemma 2). From (21) and (22) it follows that

$$\sum_{\nu=1}^{k_n} (\mu_\nu - \lambda_\nu) = \frac{1}{2\pi i}\int_{\Gamma_n} \lambda \frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda. \tag{23}$$

Since

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda \frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{1}{2\pi i}\int_{\Gamma_n} \lambda S'(\lambda) S^\nu(\lambda)\,d\lambda \tag{24}$$

for n large enough (because of $S(\lambda) \rightrightarrows 0$, as $n \to \infty$, $\lambda \in \Gamma_n$), it is enough to estimate each of the integrals $\int_{\Gamma_n} \lambda S'(\lambda) S^\nu(\lambda)\,d\lambda$.

By the definition of the function S (because of the structure of a_{jk}) we know that S is a holomorphic function on $\mathbf{C}$ except at the points $\lambda_1, \lambda_2, \ldots$. Inside the contour Γ_n, the function S has singularities at the points $\lambda_1, \lambda_2, \ldots, \lambda_{k_n}$. We can easily check that

$$\int_{\Gamma_n} \lambda S'(\lambda) S^\nu(\lambda)\,d\lambda = -\frac{1}{\nu+1}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda. \tag{25}$$

Formulas (24) and (25) imply

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda \frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_n} S(\lambda)\,d\lambda + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu+1}\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda. \tag{26}$$

From (17), for n large enough, it follows that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda\right| \le \frac{C_0 K_1}{2\pi}\left(\frac{K_1}{d_n}\right)^\nu,$$

therefore

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu+1}\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda = O(d_n^{-1}). \tag{27}$$

Next, (26) and (27) imply

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=-\frac{1}{2\pi i}\int_{\Gamma_n}S(\lambda)\,d\lambda+O(d_n^{-1}). \tag{28}$$

Let us consider the function R given by

$$R(\lambda)=S(\lambda)+\sum_{j=1}^{\infty}a_{jj}(\lambda). \tag{29}$$

In a way similar to the proof of inequality (15), we can prove that $|R(\lambda)|\leq P\cdot(d(\lambda))^{-2}$ for $\lambda\in\Gamma_n$, for n large enough and a constant P which does not depend on λ. This inequality implies

$$-\frac{1}{2\pi i}\int_{\Gamma_n}R(\lambda)\,d\lambda=O(d_n^{-1}). \tag{30}$$

Then, from (28), (29) and (30) it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_n}\left(\sum_{j=1}^{\infty}a_{jj}(\lambda)\right)d\lambda+O(d_n^{-1}). \tag{31}$$

Since the series $\sum_{j=1}^{\infty}a_{jj}(\lambda)$ converges uniformly on Γ_n, we have

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int_{\Gamma_n}a_{jj}(\lambda)\,d\lambda+O(d_n^{-1}). \tag{32}$$

Because of $a_{jj}=-s_j\sum_{\nu=1}^{\infty}\lambda_\nu a_\nu^j\overline{b_\nu^j}/(\lambda_\nu-\lambda)$, (32) gives

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\sum_{j=1}^{\infty}\left(s_j\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}\right)+O(d_n^{-1}). \tag{33}$$

From (23) and (24) we can get

$$\sum_{\nu=1}^{k_n}(\mu_\nu-\lambda_\nu)=\sum_{j=1}^{\infty}\left(s_j\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}\right)+O(d_n^{-1}). \tag{34}$$

Since $\sum_{\nu=1}^{\infty}\lambda_\nu|a_\nu^jb_\nu^j|\leq\|L_0^{1-q}g_j\|\cdot\|L_0^qf_j\|$ the series $\sum_{j=1}^{\infty}s_j\|L_0^{1-q}g_j\|\cdot\|L_0^qf_j\|$ is convergent and $\lim_{n\to\infty}\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}=(L_0^{1-q}g_j,L_0^qf_j)$, then we have from (34), when $n\to\infty$,

$$\lim_{n\to\infty}\sum_{\nu=1}^{k_n}(\mu_\nu-\lambda_\nu)=\sum_{j=1}^{\infty}s_j(L_0^{1-q}g_j,L_0^qf_j).$$

So, the theorem is proved in the case $c=0$.

5. The proof of the theorem in the case when $c \neq 0$ in formula for $N(\lambda)$

If $g_k \in \mathcal{D}(L_0^{2-q})$ the spectral theorem implies $g_k \in \mathcal{D}(L_0^{1-q})$ and

$$\|L_0^{1-q} g_k\| \leq \lambda_1^{-1} \|L_0^{2-q} g_k\|. \tag{35}$$

Because of the assumption $\sum_{k=1}^{\infty} s_k \|L_0^{2-q} g_k\| \cdot \|L_0^q f_k\| < \infty$ (which holds in the case $c \neq 0$) (35) implies

$$\sum_{k=1}^{\infty} s_k \|L_0^{1-q} g_k\| \cdot \|L_0^q f_k\| < \infty.$$

If $g_k \in \mathcal{D}(L_0^{2-q})$, $f_j \in \mathcal{D}(L_0^q)$ using the Bessel inequality, we have

$$\sum_{n=1}^{\infty} \lambda_n^2 |c_n^k \overline{b_n^j}| \leq \|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|. \tag{36}$$

In the case $c \neq 0$, the formula for $N(\lambda)$ can be written in the form

$$\lambda_n = n(k + o(1)) \tag{37}$$

where $k = c^{-1}$.

The following lemma was proved in [**1**].

Lemma 3. *Suppose that $\alpha > 0$, N_α is so large that $N_\alpha > 8/\alpha$ and for $o(1)$ in (37) the following holds: $|o(1)| \leq \alpha k / 8(1+\alpha)$, for every $n > N_\alpha$. Then between λ_n and $\lambda_{n'}$, where $n > N_\alpha$ and $n' = [n(1+\alpha)] > N_\alpha$, there are eigenvalues λ_{n_ν} and $\lambda_{n_\nu+1}$ of the operator L_0 such that*

$$\lambda_{n_\nu+1} - \lambda_{n_\nu} \geq k/2. \tag{38}$$

Let us define $\Gamma_{n_\nu} = \{\lambda : |\lambda| = r_{n_\nu}\}$, where $r_{n_\nu} = (\lambda_{n_\nu} + \lambda_{n_\nu+1})/2$. Since

$$(L_0 R_\lambda^0 g_k, f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n c_n^k \overline{b_n^j}}{\lambda_n - \lambda},$$

we have

$$(L_0 R_\lambda^0 g_k, f_j) + \frac{1}{\lambda}(L_0^{1-q} g_k, L_0^q f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n^2 c_n^k \overline{b_n^j}}{\lambda(\lambda_n - \lambda)}. \tag{39}$$

From (35), (36), and (39) it follows that

$$|(L_0 R_\lambda^0 g_k, f_j)| \leq \frac{\|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|}{|\lambda| d(\lambda)} + \frac{1}{\lambda_1} \frac{\|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|}{|\lambda|}$$

and then, $d(\lambda) \geq k/4$ on Γ_{n_ν}, implies

$$|a_{jk}(\lambda)| \leq K(s_j s_k)^{1/2} \cdot \|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\| \cdot |\lambda|^{-1} \tag{40}$$

on Γ_{n_ν} $(K = 4 \cdot k^{-1} + \lambda_1^{-1})$. Using (40), we can get an estimate for the function

$$|S(\lambda)| \le \sum_{n=1}^{\infty} M_1^n(\lambda) \tag{41}$$

where $M_1(\lambda) = |\lambda|^{-1} K \sum_{j=1}^{\infty} s_j \|L_0^{2-q} g_j\| \cdot \|L_0^q f_j\|$.

Clearly, for $\lambda \in \Gamma_{n_\nu}$ and ν large enough, we have $|S(\lambda)| \le Q/|\lambda|$ where the constant Q does not depend on λ $(\lambda \in \Gamma_{n_\nu})$. This implies that $S(\lambda) \to 0$, for $\lambda \in \Gamma_{n_\nu}$, $\nu \to \infty$, therefore the function $\Delta(\lambda)$ does not have zeros in region $\{\lambda : |\lambda| > r_{n_\nu}\}$ for ν large enough.

In a similar way, as it was done in Lemma 2, we can show that the operators L and L_0 have the same number of eigenvalues inside the contour Γ_{n_ν}, for ν large enough. This number is equal to n_ν.

As in the previous case, we have

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = -\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)\, d\lambda + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r+1} \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)^{r+1}\, d\lambda. \tag{42}$$

Using the estimate $|S(\lambda)| \le Q/|\lambda|$, which holds for $\lambda \in \Gamma_{n_\nu}$ and ν large enough, we can get easily

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r+1} \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)^{r+1}\, d\lambda = O(r_{n_\nu}^{-1}), \qquad \nu \to \infty$$

and consequently (42) implies

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = -\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)\, d\lambda + O(r_{n_\lambda}^{-1})\,. \tag{43}$$

Since $\left| S(\lambda) + \sum_{j=1}^{\infty} a_{jj}(\lambda) \right| \le \text{const}/|\lambda|^2$ on Γ_{n_ν} (ν large enough) we have

$$\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} \left(S(\lambda) + \sum_{j=1}^{\infty} a_{jj}(\lambda) \right) d\lambda = O(r_{n_\nu}^{-1}). \tag{44}$$

From (43) and (44) it follows that

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} \sum_{j=1}^{\infty} a_{jj}(\lambda)\, d\lambda + O(r_{n_\nu}^{-1}). \tag{45}$$

Because of

$$\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} a_{jj}(\lambda)\, d\lambda = s_j \sum_{j=1}^{n_\nu} \lambda_s a_s^j \overline{b_s^j}$$

and the uniform convergence of the series $\sum a_{jj}(\lambda)$ on Γ_{n_ν}, (45) becomes

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = \sum_{j=1}^{\infty} \left(s_j \sum_{s=1}^{n_\nu} \lambda_s a_s^j \overline{b_s^j} \right) + O(r_{n_\nu}^{-1}). \tag{46}$$

Hence

$$\sum_{j=1}^{\infty} s_j \sum_{s=1}^{\infty} \lambda_s |a_s^j b_s^j| \leq \sum_{j=1}^{\infty} s_j \|L_0^{1-q} g_j\| \cdot \|L_0^q f_j\| < \infty$$

(46) and the fact that $\lim_{\nu\to\infty} \sum_{s=1}^{n_\nu} (\mu_s - \lambda_s) = (L_0^{1-q} g_j, L_0^q f_j)$ imply

$$\lim_{\nu\to\infty} \sum_{s=1}^{n_\nu} (\mu_s - \lambda_s) = \sum_{j=1}^{\infty} s_j (L_0^{1-q} g_j, L_0^q f_j).$$

So, the proof of the theorem in case $c \neq 0$ is completed.

Remark. Under the condition stated in the theorem can get directly that

$$\sum_{j=1}^{\infty} s_j (L_0^{1-q} g_j, L_0^q f_j) = \sum_{j=1}^{\infty} \lambda_j (S\varphi_j, \varphi_j) \tag{47}$$

so the statement of the theorem can be formulated in terms of the operator S by formula (47).

Example. Let L_0 be a semibounded ($L_0 > 0$), discrete, selfadjoint operator which acts on H. Suppose that $S > 0$ is a nuclear operator which has properties $R(S) \subset \mathcal{D}(L_0)$ and $\sum \lambda_n (S\varphi_n, \varphi_n) < \infty$ (λ_n denotes the eigenvalues and φ_n the corresponding eigenvectors of the operator L_0). Then

$$\lim_{n\to\infty} \sum_{\nu=1}^{k_n} (\mu_\nu - \lambda_\nu) = \sum_{\nu=1}^{\infty} \lambda_\nu (S\varphi_\nu, \varphi_\nu).$$

REFERENCES

[**1**] В. А. Садовничий, В. А. Любишкин, *Конечномерные возмущениj дискретных операторов и формулы следов*, Функц. анализ и его прилож. **20** (1986), 55–65.

[**2**] И. Ц. Гохберг, М. Г. Крейн, *Введение в теорию линейных несамосопряженных операторов в гильбертовом пространстве*, Наука, Москва, 1965.

[**3**] В. Б. Лидский, *О суммируемости рjдов по главным векторам несамосопряженных операторов*, Труды Москов. мат. общ. **11** (1962), 3–35.

Institut za primenjenu matematiku i elektroniku
Kneza Miloša 37
11000 Beograd, Jugoslavija

(Received 21 11 1989)