

TRACE FORMULA FOR NUCLEAR PERTURBATIONS OF DISCRETE NONSELFADJOINT OPERATORS

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Abstract. It is shown that, if T is a discrete nonselfadjoint operator, and P is nuclear, than, under some condition, $T + P$ is discrete. A regularized trace formula is given. It is shown that this result is applicable to differential operators given by regular boundary conditions.

1. Introduction

In [3] it was shown that, if T is a discrete selfadjoint operator with distribution of eigenvalues $N(T, r) = O(r^p)$, $0 < p < 1 < 2$ ($N(T, r)$ denotes the number of eigenvalues of the operator T in the circle with center at 0 and radius r), and if P is a nuclear operator, then there exists an increasing sequence of natural numbers k_n such that $k_n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \text{Sp } P,$$

(Sp denotes the trace of an operator) where μ_k and λ_k denote the eigenvalues of the operators $T + P$ and T . This result is applicable to a selfadjoint differential operator T given by a differential expression of order greater than 2 and with regular boundary conditions.

In this paper the case of a nonselfadjoint operator is treated. The result obtained is applicable to nonselfadjoint differential operators of order greater than 6 given by regular boundary conditions.

2. Main result

The main result of this paper is the following theorem.

THEOREM. *Let H be a separable complex Hilbert space with inner product (\cdot, \cdot) , P a nuclear operator in H and T an operator such that*

- 1) $T^{-1} \in \sigma_q$ for some $q \geq 1$ (σ_q is defined in [1]);
- 2) all generalized eigenspaces $\mathcal{N}((\lambda_i I - T)^{m_i})$ (m_i is the ascent of the operator $\lambda_i I - T$) except finitely many contain only eigenvectors;
- 3) $|\arg(Tf, f)| < \pi/2q$, for all $f \in H$;
- 4) $N(T, r) = O(r^p)$ ($N(T, r)$ denotes the number of eigenvalues of the operator T , including multiplicities, in the circle with center at 0 and radius r), for some p , $0 < p < 1/6$;
- 5) $\|\sum_{i=1}^k P_i\| < M$, for $k = 1, 2, \dots$, where P_i denotes the projection of H onto the generalized eigenspace $\mathcal{N}((\lambda_i I - T)^{m_i})$ along $\mathcal{R}((\lambda_i I - T)^{m_i})$, where m_i is the ascent of the operator $\lambda_i I - T$.

Then, the operator $T + P$ is discrete and there exists an increasing sequence of natural numbers k_n such that $\lim_{n \rightarrow \infty} k_n = \infty$ and the following formula holds:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \text{Sp } P,$$

where μ_k and λ_k denote the eigenvalues of operators $T + P$ and T , enumerated (including multiplicities) in the non-decreasing order of absolute value.

The proof of the theorem will be exposed in several lemmas. In the following, the conditions of the theorem hold, with the notation introduced above.

LEMMA 1. *There exists an unbounded increasing sequence r_n such that:*

- 1) $\lim_{n \rightarrow \infty} r_n / d_n^{6/5} = 0$
- 2) $\sum_{n=1}^{\infty} r_n^{1+p} / d_n^{7/5} < \infty$ where $d_n = d(\sigma(T), \{z : |z| = r_n\})$.

Proof. Let $\delta > 0$ and $c > 0$ (c will be chosen later). Let

$$K_n = \{z \in \mathbf{C} : n^c < |z| < (n+1)^c\}.$$

The width of the ring K_n is a value of order n^{c-1} . In the ring K_n there exist $N(T, (n+1)^c) - N(T, n^c)$ eigenvalues of the operator T , and by condition 4) of the theorem, this is equal to $O(n^{cp})$. If we divide the ring K_n into n^{cp} rings of equal width, there will be no eigenvalue of the operator T in at least one of the new rings. Let us denote by r_n the radius of the circle which divides that ring into two rings of equal width. Then the value $r_n d_n^{-6/5}$ is of order $n^{-c(1/5-6/5p)+6/5(1+\delta)}$. The value $r_n^{1+p} d_n^{-7/5}$ is of order $n^{-c(2/5-12/5p)+7/5(1+\delta)}$. From the fact $0 < p < 1/6$ it follows that we can pick a c such that 1)–3) are fulfilled. \square

Remark. From the proof of Lemma 1 it is evident that on the circle $\{z \in \mathbf{C} : |z| = r_n\}$ there are no eigenvalues of the operator T . In the following we shall denote the distance of a point to the spectrum of the operator T by $d(\lambda)$, and $\{z \in \mathbf{C} : |z| = r_n\}$ by Γ_n .

LEMMA 2. *The series $\sum_{i=1}^{\infty} \frac{d(\lambda)^{3/5}}{\lambda - \lambda_i} \cdot P_i$ converges uniformly in the space $B(H)$ of bounded operators, for $\lambda \in \bigcup_{n=1}^{\infty} \Gamma_n$.*

Proof. From Abel's formula it follows that

$$\begin{aligned} \sum_{k=m}^n \frac{d(\lambda)^{3/5}}{\lambda - \lambda_k} P_k &= \sum_{k=m}^n \left(\sum_{j=1}^k P_j \right) \left(\frac{d(\lambda)^{3/5}}{\lambda - \lambda_k} - \frac{d(\lambda)^{3/5}}{\lambda - \lambda_{k+1}} \right) \\ &\quad - \left(\sum_{j=1}^{m-1} P_j \right) \cdot \frac{d(\lambda)^{3/5}}{\lambda - \lambda_m} + \left(\sum_{j=1}^n P_j \right) \frac{d(\lambda)^{3/5}}{\lambda - \lambda_m}. \end{aligned}$$

Thus, by virtue of condition 5) of the theorem, it follows that it is sufficient to show that the sequence $\frac{d(\lambda)^{3/5}}{|\lambda - \lambda_j|}$ and the series $\sum_{k=1}^{\infty} \left| \frac{d(\lambda)^{3/5}}{\lambda - \lambda_k} - \frac{d(\lambda)^{3/5}}{\lambda - \lambda_{k+1}} \right|$ converges uniformly for $\lambda \in \bigcup_{n=1}^{\infty} \Gamma_n$.

Let $\varepsilon > 0$ and $r_k \leq |\lambda_i| \leq r_{k+1}$. Then,

$$\sup_{\lambda \in \bigcup \Gamma_n} \frac{d(\lambda)^{3/5}}{|\lambda - \lambda_i|} \leq \frac{1}{d_k^{2/5}} < \varepsilon,$$

for sufficiently large k , e.g. i and thus the sequence $d(\lambda)^{3/5}/|\lambda - \lambda_i|$ converges uniformly. For $r_k \leq |\lambda_{i+1}| \leq r_{k+1}$,

$$\left| \frac{d(\lambda)^{3/5}}{\lambda - \lambda_k} - \frac{d(\lambda)^{3/5}}{\lambda - \lambda_{k+1}} \right| \leq \frac{|\lambda_i| + |\lambda_{i+1}|}{|\lambda - \lambda_i| \cdot |\lambda - \lambda_{i+1}|} \cdot d(\lambda)^{3/5} \leq \frac{2r_{k+1}}{d_{k+1}^{7/5}}.$$

Since in the ring $\{r_k \leq |z| \leq r_{k+1}\}$ there are $O(r_k^p)$ eigenvalues of the operator T it follows that

$$\sum_{k=1}^{\infty} \left(\sum_{r_k \leq |\lambda_k| \leq r_{k+1}} \frac{r_{k+1}}{d_{k+1}^{7/5}} \right) \leq \text{const} \sum_{k=1}^{\infty} \frac{r_{k+1}^{1+p}}{d_{k+1}^{7/5}} < \infty,$$

by Lemma 1. \square

Let N_i denote the restriction of $\lambda_i I - T$ to $\mathcal{N}((\lambda_i I - T)^{m_i})$, $R_{\lambda}^0 = (T - \lambda I)^{-1}$, $R_{\lambda} = (T + P - \lambda I)^{-1}$.

LEMMA 3. $\sup \{(d(\lambda))^{3/5} \|R_{\lambda}^0\| : \lambda \in \bigcup_{n=1}^{\infty} \Gamma_n\} < C < \infty$.

Proof. In [2] it is shown that the resolvent of a discrete operator which satisfies condition 5) of the theorem has a representation

$$R_{\lambda}^0 = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \frac{(-N_i)^{j-1} P_i}{(\lambda - \lambda_i)^j} + \sum_{j=0}^{\infty} (\lambda_0 - \lambda)^j R_{\lambda_0} (T_{\infty})^{j+1} (I - \mathbf{P}_{\infty}) \quad \text{in } B(H),$$

where $\mathbf{P}_{\infty} = \sum_{i=1}^{\infty} P_i$ (strong convergence), and $R_{\lambda_0}(T_{\infty})$ is an operator without significance to us.

From conditions 1) and 3) of the theorem and the theorem of Lidsky [1, ch. V, §6, T 6.1] it follows that $\mathbf{P}_{\infty} = I^1$) and so

$$d(\lambda)^{3/5} R_{\lambda}^0 = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \frac{d(\lambda)^{3/5}}{|\lambda - \lambda_{ij}|} (-N_i)^{j-1} P_i + \sum_{i=m+1}^{\infty} \frac{d(\lambda)^{3/5} P_i}{\lambda - \lambda_i}$$

¹⁾ \mathbf{P}_{∞} is a projection onto $S_{\infty} = \{x \in H : x = \sum_{i=1}^{\infty} x_i, x_i \in \mathcal{N}((T - \lambda_i I)^{m_i})\} = \overline{S_{\infty}}$ [2, p. 20], and $S_{\infty} = H$.

(from the condition 2) of the Theorem it follows that $N_i \equiv 0$ for $i > m$).

The proof follows immediately from Lemma 2. \square

COROLLARY 1. *The operator $T + P$ is discrete.*

Proof. By virtue of Lemma 3 $\|PR_\lambda^0\| < 1$, for λ sufficiently distant from $\sigma(T)$, so the series

$$R_\lambda = \sum_{k=0}^{\infty} (-1)^k R_\lambda^0 (PR_\lambda^0)^k \quad (1)$$

converges, and thus R_λ is compact. \square

COROLLARY 2. *For n sufficiently large, $\Gamma_n \cap \sigma(T + P) = \emptyset$.*

Proof.

$$\|P\|Cd(\lambda)^{-3/5} < \|P\|Cd_n^{-3/5} \quad (2)$$

for $\lambda \in \Gamma_n$, and $d_n \rightarrow \infty$ (follows from Lemma 1). Thus, since for every $\lambda \in \Gamma_n$ there exists R_λ^0 (see the remark after Lemma 1) from (2) it follows that R_λ exists too. \square

COROLLARY 3. $\|R_\lambda - R_\lambda^0\|_1 \leq kd_n^{-6/5}$ if $\lambda \in \Gamma_n$, where $k > 0$ and $\|\cdot\|_1$ denotes the nuclear norm of the operator.

Proof. From (1) it follows that $\|R_\lambda - R_\lambda^0\|_1 \leq \sum_{k=1}^{\infty} \|R_\lambda^0\|^{k+1} \|P\|_1^k$. It remains only to apply Lemma 3. \square

LEMMA 4. *For n large enough, in the circle $\{z : |z| < r_n\}$ the number of eigenvalues of the operator T is equal to the number of eigenvalues of the operator $T + P$, including multiplicities.*

Proof. Let $N_1(r)$ be the number of eigenvalues of the operator $T + P$, and $N_2(r)$ the number of eigenvalues of the operator T , in the circle $\{z : |z| < r\}$. Then, by the properties of Riesz projectors:

$$N_1(r_n) - N_2(r_n) = \frac{1}{2\pi i} \int_{\Gamma_n} \text{Sp}(R_\lambda - R_\lambda^0) d\lambda. \quad (3)$$

Since $|\text{Sp}(R_\lambda - R_\lambda^0)| \leq \|R_\lambda - R_\lambda^0\|_1$, it follows that $|N_1(r_n) - N_2(r_n)| = O(r_n d_n^{-6/5})$, and that is, by virtue of Lemma 1, less than 1 for n large enough. \square

Let k_n be the number of eigenvalues of the operators T and $T + P$ in the circle $\{z : |z| < r_n\}$.

LEMMA 5. $\lim_{n \rightarrow \infty} \left(\sum_{k=1}^{k_n} (\mu_k - \lambda_k) - \text{Res}_{\lambda_k}(\text{Sp}(PR_\lambda^0)) \right) = 0$, where Res_{λ_k} is the residuum of a function at point λ_k .

Proof. By the properties of Riesz projectors:

$$\sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \frac{1}{2\pi i} \int_{\Gamma_n} \lambda \text{Sp}(R_\lambda - R_\lambda^0) d\lambda. \quad (4)$$

On the other hand, from the property of the trace $\text{Sp}(R_\lambda^0 P R_\lambda^0) = \text{Sp}(P(R_\lambda^0)^2)$ and the fact that $dR_\lambda^0/d\lambda = (R_\lambda^0)^2$, after partial integration, it follows that

$$\frac{1}{2\pi i} \int_{\Gamma_n} \lambda \text{Sp}(R_\lambda^0 P R_\lambda^0) = \frac{-1}{2\pi i} \int_{\Gamma_n} \text{Sp}(P R_\lambda^0) d\lambda$$

so

$$\frac{1}{2\pi i} \int_{\Gamma_n} \lambda \text{Sp}(R_\lambda^0 P R_\lambda^0) = - \sum_{k=1}^{k_n} \text{Res}_{\lambda_k}(\text{Sp}(P R_\lambda^0)). \quad (5)$$

By adding (5) and (4), we obtain

$$\sum_{k=1}^{k_n} \left(\mu_k - \lambda_k - \text{Res}_{\lambda_k}(\text{Sp}(P R_\lambda^0)) \right) = \frac{1}{2\pi i} \int_{\Gamma_n} \lambda \text{Sp}(R_\lambda - R_\lambda^0 + R_\lambda^0 P R_\lambda^0) d\lambda. \quad (6)$$

From (2) it follows that

$$R_\lambda - R_\lambda^0 + R_\lambda^0 P R_\lambda^0 = \sum_{k=2}^{\infty} (-1)^k R_\lambda^0 (P R_\lambda^0)^k \quad (7)$$

and since $\text{Sp}(d(P R_\lambda^0)^k/d\lambda) = \text{Sp}(k R_\lambda^0 (P R_\lambda^0)^k)$, after substituting (7) into (6) and applying partial integration to each member of the uniformly convergent series (7) we obtain

$$\sum_{k=1}^{k_n} \left(\mu_k - \lambda_k - \text{Res}_{\lambda_k}(\text{Sp}(P R_\lambda^0)) \right) = - \frac{1}{2\pi i} \int_{\Gamma_n} \text{Sp} \left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k} (P R_\lambda^0)^k \right) d\lambda. \quad (8)$$

By applying the inequality $|\text{Sp}(P R_\lambda^0)| \leq \|R_\lambda^0\|^k \cdot \|P\|_1^k$ and Lemma 3 to each member of the series (8) we obtain

$$\begin{aligned} \left| \sum_{k=1}^{k_n} \left(\mu_k - \lambda_k - \text{Res}_{\lambda_k}(\text{Sp}(P R_\lambda^0)) \right) \right| &\leq r_n \sum_{k=2}^{\infty} \frac{\|P\|_1^k}{k} \cdot \left(\frac{c}{d_n^{3/5}} \right)^k \\ &= \frac{r_n}{d_n^{6/5}} \sum_{k=2}^{\infty} \left(\frac{\|P\|_1 C}{d_n^{3/5}} \right)^{k-2} \cdot (\|P\|_1 C)^2 = O \left(\frac{r_n}{d_n^{6/5}} \right) \end{aligned}$$

since for n large enough, $\|P\|_1 C d_n^{-3/5} < 1$. The last expression converges to zero, by virtue of Lemma 1. \square

Proof of the Theorem. From [2] and the fact that $\mathbf{P}_\infty = I$ which has already been proved, it follows that

$$P R_\lambda^0 = \sum_{i=1}^{\infty} \sum_{j=1}^{m_i} \frac{1}{(\lambda - \lambda_i)^j} P(-N_i)^{j-1} P_i$$

and thus $\text{Res}_{\lambda_k} P R_\lambda^0 = P R_k$. This proves the theorem, because

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} \text{Res}_{\lambda_k}(\text{Sp}(P R_\lambda^0)) = \lim_{n \rightarrow \infty} \text{Sp} P \sum_{k=1}^{k_n} P_k = \text{Sp} P.$$

3. Applications

The theorem proved above can be applied to differential operators of order not less than 7, as in the following example:

Let $H = L^2[0, 1]$ be a complex-valued continuous function defined on $[0, 1]$ with values in the angle $\{z : |\operatorname{Arg} z| < \pi/4\}$ and T a differential operator in H given by the differential expression $l(y) = y^{(\text{VIII})} + q(x)y$ and boundary conditions of Sturm type

$$y^{(j)}(0) = 0, \quad y^{(j)}(1) = 0, \quad j = 0, 1, 2, 3,$$

which are regular [4, II, §4.8].

Let K be a nuclear integral operator in the space H :

$$(\mathbf{K}f)(x) = \int_0^1 K(x, t)f(t) dt.$$

The conditions 1) and 4) of the theorem are fulfilled by virtue of the theorem on the asymptotic behaviour of the eigenvalues of regular boundary problem [4, II, §4.9].

The condition 5) of the theorem is fulfilled since the union of bases of the spaces $\mathcal{N}((\lambda_i I - T)^{m_i})$ form a Riesz basis in H [4, II, §5.3] and [1, VI, §5].

The condition 1) is fulfilled because the Green function of the operator T is continuous and thus belongs to $L^2(0, 1)^2$, so $T^{-1} \in \sigma_2$.

Since

$$(Ty, y) = \int_0^1 (y^{(\text{VIII})} + q(x)y)\bar{y} dy = \int_0^1 |y^{(\text{IV})}|^2 dx + \int_0^1 q(x)|y|^2 dx$$

(after 4 partial integrations), the condition 1) of the theorem is fulfilled, by virtue of the assumption $|\arg q(x)| < \pi/4$ and thus, with the notation introduced before

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \int_0^1 K(x, x) dx,$$

for some increasing unbounded sequence of natural numbers k_n .

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