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SOME INEQUALITIES OF ISOPERIMETRIC TYPE CONCERNING ANALYTIC AND SUBHARMONIC FUNCTIONS

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Abstract. We prove an inequality between the weighted Bergman and Hardy norm of an analytic function and generalize this to the ratio of two log-subharmonic function. These results may be regarded as a generalization of the classical isoperimetric inequality.

1. Introduction

In recent papers, Burbea [**3**] and Mateljević and Pavlović [**9**] proved the following result (Theorem B, below), which is a generalization of the classical isoperimetric inequality. Let $f \in H^p(U)$, $0 < p < \infty$, and let n be a positive integer, $n \geq 2$. Then

$$\frac{n-1}{\pi} \int_U |f(z)|^{np} (1-|z|^2)^{n-2}\, dx\, dy \leq \left[\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right]^n,$$

where U denotes the unit disc, and $H^p(U)$, $0 < p < \infty$, denote the usual Hardy classes.

In this paper we give a few extensions of this result.

In Section 3 we prove that a functional corresponding to the previous inequality is monotone and give a generalization of Theorem B (Theorem 2, below) which we need later.

In [**6**, Theorem 1] Huber proved an interesting inequality and used it to prove an isoperimetric inequality [**6**, Theorem 3] which holds on any sufficiently regular abstract surface. In Section 4, we prove a result (see Theorem 3, below) which for $n = 2$ is Huber's [**6**, Theorem 1] and which can also be regarded as a further generalization of Theorem B and Theorem 2.

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2. Isoperimetric inequalities and the theory of reproducing kernels

Let U denote the unit disc in the complex plane $\mathbf{C}$. Also, we will use the usual notations for Hardy spaces (see [**4**] and [**10**]).

For $q > 1$, let B_q denote the space of all analytic functions f in U such that $\|f\|_{B_q} < \infty$, where

$$\|f\|_{B_q} = \left\{ \frac{q-1}{\pi} \int_U |f(z)|^2 (1-|z|^2)^{q-2}\, dx\, dy \right\}^{1/2}.$$

It is known (see, for example [**2**]) that the Hilbert space B_q, $q > 1$, has the reproducing kernel $K_q(z, \zeta) = (1 - z\bar{\zeta})^{-q}$, $z, \zeta \in U$, and that the Hardy space $H^2(U^n)$, $n \geq 1$, is the Hilbert space with reproducing kernel

$$K(z, \zeta) = \prod_{i=1}^{n} (1 - z_i \bar{\zeta}_i), \qquad z = (z_1, \dots, z_n), \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{C}^n.$$

Since the diagonal restriction of $K(z, \zeta)$ is the reproducing kernel $K_n(z, \zeta) = (1 - z\bar{\zeta})^{-n}$, $z, \zeta \in U$, of B_n, $n \geq 2$, we deduce the following result from the general theory of reproducing kernels (see [**1**] and [**2**]).

Theorem A. *Let $F \in H^2(U^n)$, $n \geq 2$, and let $f(z) = F(z, \dots, z)$, $z \in U$, be the restriction of F to the diagonal of U^n. Then*

$$\frac{n-1}{\pi} \int_U |f(z)|^2 (1-|z|^2)^{n-2}\, dx\, dy \leq \|F\|_2^2,$$

where

$$\|F\|_2^2 = \lim_{r \to 1} \frac{1}{(2\pi)^n} \int_{T^n} |F(re^{i\theta_1}, re^{i\theta_2}, \dots, re^{i\theta_n})|^2\, d\theta_1\, d\theta_2 \dots d\theta_n$$

and $T = \{z : |z| = 1\}$ is the unit circle.

Theorem B. *Let $f \in H^p(U)$, $0 < p < +\infty$, and let n be a positive integer, $n \geq 2$. Then*

$$\frac{n-1}{\pi} \int_U |f(z)|^{np} (1-|z|^2)^{n-2}\, dx\, dy \leq \|f\|_p^{np},$$

where

$$\|f\|_p^p = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta.$$

The case $p = 2$ of Theorem B follows from Theorem A if we put $F(z_1, \dots, z_n) = f(z_1) \cdot \dots \cdot f(z_n)$. Now, using the well-known factorization theorem we can show that Theorem B is true for all positive p.

Proofs of Theorems A and B were sketched in [**9**]. Independently, Burbea [**3**, Corollaries 3.4 and 3.6, Theorem 4.2] (see also [**2**]) proved that these results are sharp and founded further generalizations.

Theorem B may be regarded as a generalization of the classical isoperimetric inequality. Namely, let D be simply-connected domain in the complex plane $\mathbf{C}$ bounded by a rectifiable curve K of length L, and let F be a conformal mapping of the unit disk U onto D. Applying Theorem B to $f = F'$, $n = 2$ and $p = 1$ (see, for example, [**7**]), we get the classical isoperimetric inequality $4\pi\,\mathrm{area}(D) \le L^2$.

3. Generalization of Theorem B

Let f be an analytic function on a neighborhood of $|z| = r$, $0 \le r < \infty$, and let p be a positive number. We define

$$I_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta.$$

THEOREM 1. *Let $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ be an analytic function on $|z| \le R$, $0 \le r < R$, $0 < p < +\infty$, and m positive integer, $m \ge 2$. Then*

$$\begin{aligned} I(R,r) = {}& \frac{m-1}{\pi}\iint_{|z|\le R} |f(z)|^{mp}(R^2-|z|^2)^{m-2}\,dx\,dy \\ &- \frac{m-1}{\pi}\iint_{|z|\le r} |f(z)|^{mp}(r^2-|z|^2)^{m-2}\,dx\,dy \\ &\le (R^{2-2/m} I_p(R,f))^m - (r^{2-2/m} I_p(r,f))^m. \end{aligned}$$

Proof. First suppose that $p = 2$. Let $F(z) = (f(z))^m = \sum_{n=0}^{\infty} c_n z^n$. Then

$$\begin{aligned} I(R,r) &= 2(m-1)\Big[\int_0^R (R^2-\rho^2)^{m-2}\sum_{n=0}^{+\infty}|c_n|^2\rho^{2n+1}\,d\rho \\ &\qquad - \int_0^r (r^2-\rho^2)^{m-2}\sum_{n=0}^{\infty}|c_n|^2\rho^{2n+1}\,d\rho\Big] \\ &= (m-1)\Big[\sum_{n=0}^{\infty}|c_n|^2 B(m-1,n+1)(R^{2(n+m-1)} - r^{2(n+m-1)})\Big]. \end{aligned}$$

Using

$$B(m-1,n+1) = \frac{\Gamma(m-1)\Gamma(n+1)}{\Gamma(n+m)} = \frac{(m-2)!n!}{(n+m-1)!},$$

we get

$$I(R,r) = \sum_{n=0}^{\infty}|c_n|^2\frac{(m-1)!n!}{(n+m-1)!}(R^{2(n+m-1)} - r^{2(n+m-1)}). \tag{1}$$

Since the number of the members of the set $\{(k_1,k_2,\dots,k_m) : k_1+k_2+\cdots+k_m = n$, $k_1,k_2,\dots,k_m$ are non-negative integers$\}$, is $\dfrac{(n+m-1)!}{(m-1)!n!}$, and

$$c_n = \sum_{k_1+k_2+\cdots+k_m=n} a_{k_1}a_{k_2}\cdot\ldots\cdot a_{k_m}$$

we have, by the Cauchy-Schwartz inequality,

$$(2)\qquad |c_n|^2 \le \frac{(n+m-1)!}{(m-1)!n!}\sum_{k_1+k_2+\cdots+k_m=n}|a_{k_1}|^2|a_{k_2}|^2\cdot\ldots\cdot|a_{k_m}|^2.$$

From (1) and (2) it follows that

$$(3)\quad I(R,r) \le \sum_{n=0}^{\infty}\Big[\Big(\sum_{k_1+\cdots+k_m=n}|a_{k_1}|^2\cdot\ldots\cdot|a_{k_m}|^2\Big)\big(R^{2(n+m-1)}-r^{2(n+m-1)}\big)\Big].$$

Next, the expression on the right hand side of (3) is equal to

$$(4)\qquad \big[R^{2-2/m}I_2(R,f)\big]^m - \big[r^{2-2/m}I_2(R,f)\big]^m.$$

Now the desired inequality follows from (3) and (4).

If $0 < p < +\infty$, then using a Blaschke product, one can find an analytic function g on $|z| \le R$ such that

$$|g(z)|^2 \ge |f(z)|^p \text{ for } |z| \le R \quad\text{and}\quad |g(z)|^2 = |f(z)|^p \text{ on } |z| = R,$$

and consequently,

$$(5)\qquad I_p(R,f) = I_2(R,g) \quad\text{and}\quad I_p(\rho,f) \le I_2(\rho,g) \qquad \text{for } \rho \le R.$$

Since, by (5)

$$\begin{aligned} I(R,f) = \frac{m-1}{\pi}\Big\{&\iint_{r\le|z|\le R}|f(z)|^{mp}(R^2-|z|^2)^{m-2}\,dx\,dy\\ &+\iint_{|z|\le r}|f(z)|^{mp}\big[(R^2-|z|^2)^{m-2}-(r^2-|z|^2)^{m-2}\big]\,dx\,dy\Big\}\\ \le \frac{m-1}{\pi}\Big\{&\iint_{r\le|z|\le R}|g(z)|^{2m}(R^2-|z|^2)^{m-2}\,dx\,dy\\ &+\iint_{|z|\le r}|g(z)|^{2m}\big[(R^2-|z|^2)^{m-2}-(r^2-|z|^2)^{m-2}\big]\,dx\,dy\Big\}\end{aligned}$$

the proof follows from the case $p = 2$ and (5).

Theorem B is an easy corollary of Theorem 1. Namely, if $f \in H^1$ then $I_p(R,f) \to \|f\|_p^p$ when $R \to 1_-$. Combining this with Theorem 1 and the fact that for $m \ge 2$

$$r^{2-2/m}I_p(r,f) \to 0 \quad\text{when } r \to 0_+,$$

one can deduce Theorem B.

THEOREM 2. *Let $f \in H^p(U)$, $0 < p < +\infty$, let n be a positive integer, $n \geq 2$, and let α be a non-negative number such that $n\alpha < 2$. Then*

$$I = \int_U |f(z)|^{np}|z|^{-n\alpha}(1-|z|^2)^{n-2}\,dx\,dy \leq \pi B(1-b, n-1)\|f\|_p^{np}, \tag{6}$$

where $b = n\alpha/2$ and B denotes the beta function, $B(1-\alpha, n-1) = \int_0^1 t^{-a} \times (1-t)^{n-2}\,dt$. Equality holds if and only if f is of the form $f = K(\phi')^{1/p}$ for some Möbius transformation ϕ of U onto U and some constant K.

Proof. Using polar coordinates, we have

$$I = \int_0^1 \rho^{-2b} I(\rho)(1-\rho^2)^{n-2}\rho\,d\rho,$$

where $I(\rho) = \int_0^{2\pi} |f(\rho e^{i\theta})|^{np}\,d\theta$ and $2b = n\alpha$.

An application of Čebišev's inequality with respect to the probability measure $d\mu(\rho) = 2(n-1)(1-\rho^2)^{n-2}\rho\,d\rho$ on $[0,1]$, gives

$$\int_0^1 \rho^{-2b} I(\rho)\,d\mu(\rho) \leq \left[\int_0^1 \rho^{-2b}\,d\mu(\rho)\right]\left[\int_0^1 I(\rho)\,d\mu(\rho)\right].$$

Hence

$$I \leq (n-1)B(1-b, n-1)\int_U |f(z)|^{np}(1-|z|^2)^{n-2}\,dx\,dy.$$

Now the inequality (7) follows from Theorem B.

Suppose that equality holds in (6). It is clear that equality holds in Čebišev's inequality (7). Hence,

$$[I(x) - I(y)][\psi(x) - \psi(y)] = 0$$

for all $x, y \in [0,1]$, where $\psi(x) = x^{-2b}$.

So, we can conclude that either $I(x)$ or $\psi(x)$ is a constant function on $[0,1]$. In other words $f(z) = c$ for some constant c or $b = 0$. If $b = 0$, we have, by [**3**, Corollary 3.6], that $f = K(\phi')^{1/p}$ for some Möbius transformation ϕ of U onto itself and some constant K.

4. Inequalities for subharmonic functions

In preparation for an extension of Huber's result mentioned above, we will sketch an extension of the classical isoperimetric inequality to log-subharmonic functions.

Let D be a region in $\mathbf{C}$. We say that a non-negative function u is log-subharmonic (l.s.h.) on D if the function $\log u$ is subharmonic on D. Suppose that

u is l.s.h. on $\overline{U} = \{z : |z| \le 1\}$, i.e. on a domain D containing U. If, in addition u is continuous on $\overline{U}$ then the analytic function

$$F(z) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log u(e^{it})\,dt\right), \qquad z \in U,$$

can be extended to be continuous on $\overline{U}$ and $|F(e^{i\theta})| = u(e^{i\theta})$ on T. Since $\log|F(z)|$ is harmonic on $\overline{U}$, by the maximum-principle, we get $u(z) \le |F(z)|$, $z \in U$. Hence, by Theorem B,

$$\frac{1}{\pi}\int_U [u(z)]^2\,dx\,dy \le \left[\frac{1}{2\pi}\int_U u(e^{i\theta})\,d\theta\right]^2.$$

If u is only upper semi-continuous on U then, by [**5**, Theorem 1.4] there exists a decreasing sequence $u_n(z)$ of functions continuous on U such that $u_n(z) \to u(z)$ as $n \to \infty$. Then the proof can be reduced to the case in which $u(z)$ is continuous on $\overline{U}$.

In order to obtain further generalizations we need the Riesz decomposition theorem, one of the most fundamental results in the theory of subharmonic functions (see [**5**, Theorem 3.14).

THEOREM R. *Let $S(z)$ be s.h. and not identically $-\infty$ on $\overline{U}$. Then we have for $z = re^{i\theta} \in U$*

$$S(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t)S(e^{it})\,dt - \int_U \log\left|\frac{1-z\bar\zeta}{z-\zeta}\right| d\mu(\zeta),$$

where $P(r,t)$ is the Poisson kernel and μ is a positive, finite Borel measure on $\overline{U}$.

THEOREM 3. *Let u_j $(j = 1, 2)$ be log-subharmonic functions on $\overline{U}$ and $u = u_1/u_2$. Let $\mu = \mu_2$ be the measure in the Riesz decomposition of $\log u_2$ and $\alpha = \mu(U) < 2/n$, $n = 2, 3, \dots$. Then*

$$\int_U u(z)^n(1-|z|^2)^{n-2}\,dx\,dy \le \pi B(1 - n\alpha/2, n-1)\left[\frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\,dt\right]^n, \tag{8}$$

where B denotes the beta function.

Equality holds if and only if $\log u_1$ and $\log u_2$ are harmonic functions on U and $u(z) = K|\phi'(z)|$, $z \in U$, where K is a positive constant and ϕ is a Möbius transformation of the unit disk U onto itself.

Proof. By the Riesz decomposition,

$$\log u_i(z) = h_i(z) - \int_U \log\left|\frac{1-z\bar\zeta}{z-\zeta}\right| d\mu_i(\zeta), \qquad (i = 1, 2), \tag{9}$$

where $h_i(z) = (2\pi)^{-1}\int_0^{2\pi} P(r,\theta - t)\log u_i(e^{it})\,dt$, $z = re^{it} \in U$, and μ_i positive finite Borel measures on $\overline{U}$.

Suppose first that $\alpha > 0$. Let $h(z) = h_1(z) - h_2(z)$. By (9),

$$u(z)^n \leq \exp\left[nh(z) + \int_U \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|^{n\alpha} \frac{d\mu(\zeta)}{\alpha}\right]. \tag{10}$$

Let h^* be the conjugate harmonic function of h in U and let $f = \exp(h + ih^*)$. Since $|f| = e^h$ by Jensen's inequality and (10), we find that

$$u(z)^n \leq |f(z)|^n \int_U \left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|^{n\alpha} d\nu(\zeta), \tag{11}$$

where $d\nu(\zeta) = d\mu(\zeta)/\alpha$. From (11) and Fubini's theorem, it follows that

$$\int_U u(z)^n (1-|z|^2)^{n-2}\, dx\, dy \leq \int_U I(\zeta)\, d\nu(\zeta),$$

where

$$I(\zeta) = \int_U |f(z)|^n (1-|z|^2)^{n-2} \left|\frac{1-z\bar{\zeta}}{z-\zeta}\right|^{n\alpha} dx\, dy.$$

Let $w = w(z) = (z-\zeta)/(1-z\bar{\zeta})$. Since the hyperbolic metric is invariant under a conformal map of the unit disc onto itself, we have $1-|z|^2 = (1-|w|^2)|z'(w)|$. Thus

$$I(\zeta) = \int_U \frac{|f(z(w))|^n}{|w|^{n\alpha}} (1-|w|^2)^{n-2} |z'(w)|^n\, du\, dv.$$

Let $F(w) = F_\zeta(w) = f(z(w))z'(w)$. By Theorem 2,

$$I(\zeta) \leq \pi B(1-b, n-1)\|F\|_1^n, \tag{12}$$

where $n\alpha = 2b$. Since

$$\int_T |F(w)|\,|dw| = \int_T |f(z(w))|\,|z'(w)|\,|dw| = \int_T |f(z)|\,|dz|$$

and $f(z) = u_1(z)/u_2(z)$ for $z \in T$, the desired inequality follows from (12).

If $\alpha = 0$, then $u(z)^n \leq |f(z)|^n$ and the inequality (8) follows from Theorem B.

We proceed to prove that if equality holds in (8), then $\alpha = 0$. Suppose on the contrary, that $\alpha > 0$ and that equality holds in (8). Then an inspection of the proof of inequality (8) shows that we have

$$\int_U I(\zeta)\, d\nu(\zeta) = K,$$

where K denotes the right-hand side of (8). (From the previous proof it is clear that K is also the right hand side of (12). Hence $I(\zeta) = K$ for at least one $\zeta \in U$ and consequently, by Theorem 2, $\alpha = 0$. Thus we have a contradiction and therefore $\alpha = 0$. Next, equality holds in (10) and (11) for almost all $z \in U$ with respect to the area measure. In particular, we have

$$\int_U \log\left|\frac{1-z\bar{\zeta}}{z-\zeta}\right| d\mu(\zeta) = 0$$

for some $z \in U$ and consequently $\mu_1(U) = 0$, i.e. $\log u_1(z) = h_1(z)$. Now, by Theorem 2, $u(z)$ has the form stated in Theorem 3.

Since $B(1-\alpha, n-1) = \int_0^1 t^{-\alpha}(1-t)^{n-2}\,dt$ we have $B(1-\alpha, 1) = (1-\alpha)^{-1}$. Hence, in the case $n = 2$, we get Huber's Theorem 1 [**6**]. We refer to Huber's paper [**6**], where the reader can find some interesting corollaries of this result.

Finally, note that although our motivation is Huber's paper mentioned above, our proof is different. Only the starting point, Riesz's decomposition theorem, is the same. Huber's proof is based on the change of variables $w = F(z) = \int_0^z \exp\{h(\zeta) + ih^*(\zeta)\}\,d\zeta$ and the fact that the area surrounded by the level line of the Green's function of a domain is not greater than the corresponding area of the corresponding disk. The main ingredients of our proof are Jensen's inequality and Theorem 2.

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