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TIGHT SEMIGROUPS

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Abstract. We give a description of tight semigroups, i.e. semigroups in which for every subset M, $|M| \leq 2$ implies that $|M^2| \leq 2$.

Introduction

Freiman and Schein studied in [**3**] tight semigroups, i.e. semigroups in which, for every subset M, $|M| \leq 2$ implies $|M^2| \leq 2$. In this paper these semigroups will be completely described in two different ways.

A subset M of a semigroup S is n-closed if $M^n \subseteq M$, where n is an integer such that $n \geq 2$. A semigroup S is n-closed if every subset of S is n-closed [**4**]. Lyapin, [**4**] considered 3-closed semigroups. A semigroup S is a Rédei's band if $xy \in \{x, y\}$ for every $x, y \in S$. A semigroup S is 2-closed if and only if S is a Rédei's band [**8**]. Here will be considered semigroups in which for every subset M there exists n such that M is n-closed and semigroups in which for every subset M, $|M| \leq 2$ implies that there exists n such that M is n-closed.

A group G with the identity element e is a Boolean group if $x^2 = e$ for all $x \in G$. A semigroup S with zero 0 is a nil-semigroup if for every $x \in S$ there exists a positive integer n such that $x^n = 0$. An ideal extension S of a semigroup T is a retract extension of T if there exists a homomorphism φ of S onto T such that $\varphi(t) = t$ for all $t \in T$. Such a homomorphism we call retraction. If φ is a retraction of S onto T and if $\Phi = \ker \varphi$, then by Φ_a, $a \in T$, we denote the Φ-class containing the element $a \in T$, i.e. $\Phi_a = \{x \in S \mid \varphi(x) = a\}$.

It is clear that $S = \bigcup\{\Phi_a \mid a \in T\}$. We denote by $\mathrm{Reg}(S)$ $(E(S))$ the set of all regular (idempotent) elements of a semigroup S. If X is a subset of a semigroup S, then $C(X) = \{a \in S \mid ax = xa \text{ for all } x \in X\}$. If S is a semigroup with zero 0, then $C_0(X) = \{a \in S \mid aS = Sa = 0\}$. By $\mathbf{Z}^+$ we denote the set of all positive integers.

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In any semigroup S define the relation $\mathcal{K}$ by

$$a\,\mathcal{K}\,b \iff (\exists m,n\in \mathbf{Z}^+)\ a^m=b^n.$$

It is immediate that $\mathcal{K}$ is an equivalence relation. The $\mathcal{K}$-class containing the element a is denoted by K_a. In particular, if S is periodic, then S is the union of the classes K_e, $e\in E(S)$.

For undefined notions and notations we refer to [**1**] and [**6**].

2. Tight semigroups with exactly one idempotent

Definition 2.1 [**3**]. A semigroup S with zero 0 is a semigroup of the first type if (1) $(\forall x\in S)\ x^2=0$; (2) $(\forall x,y\in S)\ xy=0\vee yx=0\vee xy=yx$.

LEMMA 1.1. [**3**]. *A semigroup of the first type is a tight semigroup.*

Example. The three-element cyclic semigroup with zero is a tight semigroup and S is not a semigroup of the first type. It is clear that a semigroup S is a semigroup of the first type if and only if S is a tight semigroup with zero and $x^2=0$ for all $x\in S$. A construction of a semigroup of the first type is given in [**3**]. Here, a construction of a tight nil-semigroup will be given.

LEMMA 1.2. *Every subsemigroup and every homomorphic image of a tight semigroup is a tight semigroup.*

LEMMA 1.3. *Every subsemigroup and every homomorphic image of a semigroup of the first type is a semigroup of the first type.*

THEOREM 1.1. *Let A be a semigroup of the first type. Let $\{a_\alpha\mid \alpha\in Y\}\subseteq C_0(A)$ and let B_α, $\alpha\in Y$, be sets such that $B_\alpha\neq\varnothing$, $B_\alpha\cap B_\beta=\varnothing$ for $\alpha\neq\beta$ and $B\cap A=\varnothing$ for $B=\bigcup\{B_\alpha\mid\alpha\in Y\}$. Let $*$ be a multiplication on $S=A\cup B$ satisfying the following conditions*:

$$(x,y)\in A\times A \implies x*y=xy; \tag{1.1}$$
$$(x,y)\in B_\alpha\times B_\beta \implies x*y\in\{a_\alpha,a_\beta\},\quad \alpha\neq\beta; \tag{1.2}$$
$$(x,y)\in B_\alpha\times A\cup A\times B_\alpha \implies x*y\in\{a_\alpha,0\}; \tag{1.3}$$
$$(x,y)\in B_\alpha\times B_\alpha \implies x*y\in C_0(A) \tag{1.4}$$

and the following conditions hold:

(i) $x*y\neq y*x \implies x*y=a_\alpha\vee y*x=a_\alpha$;

(ii) $x*x=a_\alpha$;

(iii) $(x*y)*z=0 \implies x*(y*z)=0,\quad x,y,z\in B_\alpha$;

$$(x,y)\in B_\alpha\times A^2\cup A^2\times B_\alpha \implies x*y=0; \tag{1.5}$$
$$(x,y)\in B_\alpha\times \{a_\alpha\}\cup \{a_\alpha\}\times B_\alpha \implies x*y=0; \tag{1.6}$$

$$(x,y) \in B_\alpha \times B_\alpha^2 \cup B_\alpha^2 \times B_\alpha \implies x * y = 0, \quad \alpha \neq \beta. \tag{1.7}$$

Then $(S,*)$ is a tight nil-semigroup.

Conversely, every tight nil-semigroup is isomorphic to some semigroup of the previous type.

Proof. Assume the conditions (1.1)–(1.7) hold. Let $(x,y,z) \in A \times A \times B$. Then

$$(x * y) * z = xy * z = 0 \qquad \text{(by (1.5))},$$
$$x * (y * z) = x(y * z) = 0,$$

since $y * z \in \{a_\alpha, 0\} \subseteq C_0(A)$, where $\alpha \in Y$ is such that $z \in B_\alpha$. Similar proof we have in the case $(x,y,z) \in A \times B \times A \cup B \times A \times A$

Let $(x,y,z) \in A \times B \times B$. Then $y \in B_\alpha$ for some $\alpha \in Y$ and $x * y \in \{a_\alpha, 0\}$, so by (1.6) we obtain that $(x * y) * z = 0$. Moreover, since $y * z \in C_0(A)$, then $x * (y * z) = x(y * z) = 0 = (x * y) * z$. The similar proof we have in the case $(x,y,z) \in B \times A \times B \cup B \times B \times A$.

Let $x,y,z \in B$. Assume that $(x,y,z) \in B_\alpha \times B_\beta \times B_\gamma$ for some $\alpha,\beta,\gamma \in Y$, $\alpha \neq \beta \neq \gamma \neq \alpha$. Then $x * y \in \{a_\alpha, a_\beta\}$, $y * z \in \{a_\beta, a_\gamma\}$, so by (1.6) we obtain that $x * (y * z) = 0 = (x * y) * z$. Let $(x,y,z) \in B_\alpha \times B_\beta \times B_\beta$, for some $\alpha, \beta \in Y$, $\alpha \neq \beta$. Then we have that $x * y \in \{a_\alpha, a_\beta\}$ and $y * z \in B_\beta^2$, so by (1.6) and (1.7) we have that

$$x * (y * z) = 0 = (x * y) * z.$$

The similar proof we have in the case $(x,y,z) \in B_\beta \times B_\alpha \times B_\beta \cup B_\beta \times B_\beta \times B_\alpha$, $\alpha, \beta \in Y$, $\alpha \neq \beta$.

Let $x,y,z \in B_\alpha$, $\alpha \in Y$. Then

$$(x * y) * z \in \{a_\alpha, 0\}, \qquad x * (y * z) \in \{a_\alpha, 0\},$$

since $x * y, y * z \in A$, so by (1.4) (iii) we have that

$$(x * y) * z = 0 = x * (y * z) \qquad \text{or} \qquad (x * y) * z = a_\alpha = x * (y * z),$$

Thus, $(S,*)$ is a semigroup.

Let $x \in B_\alpha$, $\alpha \in Y$. Then by (1.4) (ii) we have that $x^2 = x * x = a_\alpha$, and by (1.6) we have that $x^3 = x * a_\alpha = 0$. Therefore, $(S,*)$ is a nil-semigroup.

Let $x \in B_\alpha$, $\alpha \in Y$ and let $y \in A$. Then

$$x * y, y * x \in \{a_\alpha, 0\}, \quad y^2 = 0 \quad \text{and} \quad x^2 = a_\alpha$$

so

$$|\{x,y\}^2| = |\{x^2, x * y, y * x, y^2\}| = |\{a_\alpha, a_\beta\}| \leq 2.$$

Assume that $x,y \in B_\alpha$ for some $x \in Y$. Then $x^2 = y^2 = a_\alpha$. Assume that $x * y = y * x$. Then

$$|\{x,y\}^2| = |\{x^2, x * y, y * x, y^2\}| = |\{a_\alpha, x * y\}| \leq 2.$$

Let $x * y \neq y * x$. Then by (1.4) (i) we have that $x * y = a_\alpha$ or $y * x = a_\alpha$ so

$$|\{x,y\}^2| = |\{x^2, x*y, y*x, y^2\}| = |\{a_\alpha, x*y, y*x\}| \leq 2.$$

Finally, let $x, y \in A$. Since A is a semigroup of the first type, then $|\{x,y\}^2| \leq 2$. Therefore, S is a tight nil-semigroup.

Conversely, let S be a tight nil-semigroup with the zero element 0. By Lemma 2 [**3**] we obtain that $(xy)^2 = 0$ for all $x, y \in S$, so the relation ρ on S defined by

$$x \, \rho \, y \iff x^2 = y^2$$

is a congruence and S/ρ is a zero semigroup. Let $\varphi : S \to S/\rho$ be a natural homomorphism and let $Y = S/\rho - \{0\rho\}$, $A = \varphi^{-1}(0\rho)$, $B_\alpha = \varphi^{-1}(\alpha)$, $\alpha \in Y$, and $a_\alpha = x^2$ for $x \in B_\alpha$. Since $a_\alpha = x^4 = 0$, $x \in B_\alpha$, $\alpha \in Y$, then $\{a_\alpha \mid \alpha \in Y\} \subseteq A$.

If $B = \varnothing$, then $S = A$ is a semigroup of the first type. Let $B \neq \varnothing$. Let $x \in B$. Then

$$|\{x, x^2\}^2| = |\{x^2, x^3, x^4\}| \leq 2,$$

so $x^3 = 0$. Let $\alpha \in Y$, $x \in B_\alpha$ and $y \in A$. Then by

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{a_\alpha, 0, xy, yx\}| \leq 2$$

it follows that $xy, yx \in \{a_\alpha, a_\beta\}$, since $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. Therefore (1.3) holds.

Let $x \in B_\alpha$, $y \in B_\beta$, $\alpha, \beta \in Y$ and $\alpha \neq \beta$. Then by

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{a_\alpha, a_\beta, xy, yx\}| \leq 2$$

it follows that $xy, yx \in \{a_\alpha, a_\beta\}$, since $a_\alpha \neq a_\beta$ for $\alpha \neq \beta$. Thus, (1.2) holds.

Let $x, y \in B_\alpha$ for some $\alpha \in Y$. Then by

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{a_\alpha, xy, yx\}| \leq 2$$

it follows that $xy = a_\alpha$ or $yx = a_\alpha$ or $xy = yx$.

Let $z \in A$. Since (1.3) holds, we have that $xz = a_\alpha$ or $xz = 0$, so $a_\alpha z = xz^2 = x0 = 0$, if $xz = a_\alpha$, and $a_\alpha z = x^2 z = 0$ if $xz = 0$. Therefore, $a_\alpha z = 0$. In a similar way we prove that $za_\alpha = 0$, so $\{a_\alpha \mid \alpha \in Y\} \subseteq C_0(A)$. Moreover, $yz = a_\alpha$ or $yz = 0$, whence it follows that

$$(xy)z = x(yz) = xa_\alpha = x^3 = 0, \quad \text{or} \quad (xy)z = x(yz) = x0 = 0.$$

Therefore $xy \in C_0(A)$, so (1.4) holds.

Let $x \in B$, $y \in A^2$, i.e. $y = uv$, $u, v \in A$. By (1.3) we have that

$$xu = a_\alpha = x^2 \quad \text{or} \quad xu = 0,$$

and

$$xv = a_\alpha = x^2 \quad \text{or} \quad xv = 0.$$

Assume that $xu = x^2$. Then

$$xy = xuv = x^2 v = x^3 = 0,$$

if $xv = x^2$, and

$$xy = xuv = x^2v = 0,$$

if $xv = 0$. If $xu = 0$, then $xy = xuv = 0$. Therefore $xy = 0$ in any case. In a similar way we prove that $yx = 0$, so (1.5) holds.

Let $x \in B_\alpha$, $\alpha \in Y$. Then $xa_\alpha = xx^2 = x^3 = 0 = x^2x = a_\alpha x$, so (1.6) holds.

Finally, let $x \in B_\alpha$, $y \in B_\alpha^2$, $\alpha \neq \beta$. Then $y = uv$, where $u, v \in B_\alpha$ and $xu, xv \in \{a_\alpha, a_\beta\}$. Assume that $xu = a_\beta$. Then $xy = xuv = a_\beta v = v^2v = v^3 = 0$. Let $xu = a_\alpha$. Then

$$xy = xuv = a_\alpha v = x^2v = xa_\beta = xv^2 = a_\beta v = v^3 = 0,$$

if $xv = a_\beta$ and

$$xy = xuv = a_\alpha v = x^2v = xa_\alpha = xx^2 = x^3 = 0,$$

if $xv = a_\alpha$. Therefore $xy = 0$. In a similar way we prove that $yx = 0$, so (1.7) holds. □

A group G is a tight group if G is a tight semigroup.

Lemma 1.4. *G is a tight group if and only if G is a Boolean group.*

Proof. This follows by Lemma 1 [**3**] and by the commutativity of Boolean groups. □

Theorem 1.2. *S is a tight semigroup with exactly one idempotent if and only if one of the following conditions holds*:

1. *S is a tight nil-semigroup*;

2. *S is a retract extension of a Boolean group G by a semigroup of the first type with the retraction φ such that*:

$$\Phi_a \subseteq C(\Phi_b) \qquad \textit{for all } a, b \in G \textit{ such that } a \neq b, \tag{1.8}$$

where $\Phi = \ker\varphi$.

Proof. Let S be a tight semigroup with exactly one idempotent e. Since S is periodic, then S is an ideal extension of a group G by a nil-semigroup $Q = S/G$. If $|G| = 1$, then $S = Q$ is a tight nil-semigroup. Assume that $|G| \geq 2$. By Proposition III 4.5. [**6**], we have that the mapping $\varphi : S \to G$ given by $\varphi(x) = ex$, $x \in S$, is a retraction of S onto G. By Lemma 1.2 and Lemma 1.4, we have that G is a Boolean group.

Let $a \in G$ and let $x \in \Phi_a$. Let $b \in G$ and $b \neq a$. Since G is a Boolean group, then $ab = ba \neq e$, so

$$xb = \varphi(x)b = ab = ba = b\varphi(x) = bx \neq e.$$

Since

$$|\{x, b\}^2| = |\{x^2, xb, bx, b^2\}| = |\{x^2, ab, e\}| \leq 2$$

and $x^2 \in \Phi_a \cdot \Phi_a \subseteq \Phi_e$, it follows that $x^2 = e$ for all $x \in S$. Therefore, by this and by Lemma 1.2, we have that Q is a semigroup of the first type.

Let $a, b \in G$ be such that $a \neq b$, and let $x \in \Phi_a$, $y \in \Phi_b$. Then $ab = ba \neq e$ and $xy, yx \in \Phi_{ab} = \Phi_{ba} \neq \Phi_e$, so by

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{e, xy, yx\}| \leq 2$$

it follows that $xy = yx$, so (1.8) holds.

Conversely, let S be a retract extension of a Boolean group G by a semigroup of the first type Q with the retraction φ for which (1.8) holds. Let $x, y \in S$. Then $x \in \Phi_a$, $y \in \Phi_b$ for some $a, b \in G$. Since Q is a semigroup of the first type, then we have that $x^2, y^2 \in G$, so

$$x^2 = \varphi(x^2) = \varphi(x)\varphi(x) = a^2 = e \quad \text{and} \quad y^2 = \varphi(y^2) = \varphi(y)\varphi(y) = b^2 = e.$$

If $a \neq b$, then by (1.8) it follows that $xy = yx$, so

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{e, xy\}| \leq 2.$$

Assume that $a = b$. Then $xy, yx \in \Phi_a \cdot \Phi_a \subseteq \Phi_e$. If $x \in G$ or $y \in G$, then $xy, yx \in G \cap \Phi_e = \{e\}$, so $xy = yx = e$, whence

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{e\}| \leq 2.$$

Let $x, y \in S - G = Q - \{0\}$. Since Q is a semigroup of the first type, then

$$xy = 0 \quad \text{or} \quad yx = 0 \quad \text{or} \quad xy = yx \neq 0 \qquad \text{in } Q,$$

so

$$xy = e \quad \text{or} \quad yx = e \quad \text{or} \quad xy = yx \notin G \qquad \text{in } S,$$

whence

$$|\{x,y\}^2| = |\{x^2, xy, yx, y^2\}| = |\{e, xy, yx\}| \leq 2.$$

Therefore, S is a tight semigroup. □

2. Regular tight semigroups

The following theorem gives a characterization of regular tight semigroups.

THEOREM 3.1. *The following conditions on a semigroup S are equivalent*:

(i) *S is a regular tight semigroup*;

(ii) *for every subset M of S, $|M| \leq 2$ implies that there exists n such that M is n-closed*;

(iii) *for S one of the following conditions holds*:

  (a) *S is a Rédei's band*;

  (b) *$S = P \cup Q$, where P is a Boolean group, Q is a Rédei's band, $P \cap Q = \varnothing$ and*

$$(\forall x \in P)(\forall y \in Q) \;\; xy = yx = y; \tag{2.1}$$

  (c) *S is a Boolean group.*

Proof. (i) $\Longrightarrow$ (iii). Let S be a regular tight semigroup. Then by Lemmas 1 and 5 [**3**] we have that S is periodic and that $E(S)$ is a Rédei's band. Let $a \in S$. Then $a = axa$ for some $x \in S$, so

$$|\{a,x\}^2| = |\{a^2, ax, xa, x^2\}| \le 2.$$

If $ax = xa$, then a is completely regular. Let $ax \neq xa$. Then $a^2 = ax$ or $a^2 = xa$, whence it follows that $a = axa = a^3$, so a is completely regular. Therefore, S is a completely regular semigroup.

If $|E(S)| = 1$, then S is a group and by Lemma 1.4 it follows that S is a Boolean group. Let $|E(S)| \ge 2$.

Assume that $E(S)$ has the identity e. Let $y \in G_f$, $f \in E(S)$ and $f \neq e$. Since G_f is a Boolean group (by Lemmas 1.2 and 1.4) we have that $y^2 = f$ and

$$ey = efy = fy = y = yf = yfe = ye.$$

Now, by

$$|\{e,y\}^2| = |\{e, ey, ye, f\}| = |\{e, y, f\}| \le 2$$

it follows that $y = f$, i.e. $G_f = \{f\}$ for all $f \in E(S) - \{e\}$.

Let $G_e = P$, $Q = S - P = E(S) - \{e\}$. Then P is a Boolean group, Q is a Rédei's band and $P \cap Q = \varnothing$. Let $x \in P$ and $y \in Q$. By

$$|\{x,y\}^2| = |\{e, y, xy, yx\}| \le 2$$

it follows that $xy = e$ or $xy = y$. Assume that $xy = e$. Then we have that $e = xy = xyy = ey = y$. Thus, $xy = y$. In a similar way we prove that $yx = y$. Therefore, (2.1) holds.

Assume that $E(S)$ is without the identity. Let $e \in E(S)$ and $x \in G_e$. Since e is not the identity of $E(S)$, there exists $f \in E(S)$ such that $ef \neq f$ or $fe \neq f$. Since $E(S)$ is a Rédei's band we obtain that $ef = e$ or $fe = e$. Clearly, $e \neq f$. Now we have

$$\begin{aligned} xf &= xef = xe = x, && \text{if} \quad ef = e, \qquad \text{or} \\ fx &= fex = ex = x, && \text{if} \quad fe = e. \end{aligned}$$

Therefore,

$$|\{x,f\}^2| = |\{x^2, xf, fx, f\}| = |\{e, f, xf, fx\}| \le 2,$$

whence $x \in \{xf, fx\} \subseteq \{e, f\}$, so $x = e$. Thus $G_e = \{e\}$ for all $e \in E(S)$, so $S = E(S)$ is a Rédei's band.

(iii) $\Longrightarrow$ (i). This follows immediately.

(ii) $\Longrightarrow$ (iii). Let for every subset M of S for which $|M| \le 2$ there exists n such that M is n-closed. Let $x \in S$. Then there exists n such that $\{x\}$ is n-closed, i.e. $x^n = x$ $(n \ge 2)$, so S is periodic and completely regular. Thus S is a union of periodic groups G_e, $e \in E(S)$. Let $e \in E(S)$ and $x \in G_e$. Then there exists n such that $\{x, e\}$ is n-closed. Then

$$x^2 = x^2 e^{n-2} \in \{x,e\}^n \subseteq \{x,e\} \qquad (\text{if } n = 2, \text{ then } x^2 \in \{x,e\})$$

whence $x^2 = e$ for all $x \in G_e$, so G_e is a Boolean group for all $e \in E(S)$.

If $|E(S)| = 1$, then S is a Boolean group. Assume that $|E(S)| \geq 2$. Let $e, f \in E(S)$. Then $\{e, f\}$ is n-closed for some n, whence

$$ef = ef^{n-1} \in \{e, f\}^n \subseteq \{e, f\},$$

so $E(S)$ is a Rédei's band. Assume that $E(S)$ does not contain an identity. Let $e \in E(S)$ and let $x \in G_e$. Then there exists $f \in E(S)$ such that $f \neq e$ and $ef = e$ or $fe = e$. Let $ef = e$ (in a similar way we consider the case $fe = e$). Then $\{x, f\}$ is n-closed for some n. If $n = 2$, then

$$e = x^2 \in \{x, f\}^2 \subseteq \{x, f\}.$$

In $n \geq 3$, then

$$e = ef = x^2 f^{n-2} \in \{x, f\}^n \subseteq \{x, f\}.$$

Therefore, $e \in \{x, f\}$, whence we have that $x = e$, so $G_e = \{e\}$ for all $e \in E(S)$. Thus $S = E(S)$, i.e. S is a Rédei's band.

Assume that $E(S)$ contains the identity e. Let $f \in E(S)$, $f \neq e$ and let $y \in G_f$. Then $\{e, y\}$ is n-closed for some n. If $n = 2$, then

$$f = y^2 \in \{e, y\}^2 \subseteq \{e, y\}.$$

Let $n \geq 3$. Then

$$f = ef = e^{n-2}y^2 \in \{e, y\}^2 \subseteq \{e, y\}.$$

Therefore, $f \in \{e, y\}$, whence $y = f$, so $G_f = \{f\}$ for every $f \in E(S) - \{e\}$. Let $G_e = P$, $Q = S - P = E(S) - \{e\}$. Then we have that P is a Boolean group, Q is a Rédei's band and $P \cap Q = \varnothing$, Let $x \in P$, $y \in Q$. Then $\{x, y\}$ is n-closed for some n, so

$$xy = xy^{n-1} \in \{x, y\}^n \subseteq \{x, y\}.$$

If $xy = x$, then we have that $e = x^2 = x^2 y = ey = y$ (since e is the identity of $E(S)$), which is not possible. Thus, $xy = y$. In a similar way we prove that $yx = y$. Therefore, (2.1) holds.

(iii) $\Longrightarrow$ (ii). If S is a Rédei's band, then we have that $M^2 \subseteq M$ for every subset M of S. If S is a Boolean group, then $M^3 = M$ for every subset M of S such that $|M| \leq 2$. By this and by (2.1) we obtain that (ii) holds. □

By Theorem 2.1 the following corollaries follow:

Corollary 2.1. *S is a semigroup in which for every subset M of S there exists n such that M is n-closed if and only if one of the following conditions holds*:

1. *S is a Rédei's band*;

2. *S is a group of order* 2;

3. *$S = P \cup Q$, where P is a group of order* 2, *Q is a Rédei'S band, $P \cap Q = \varnothing$ and* (2.1) *holds.*

Proof. Let S be a semigroup in which for every subset M there exists n such that M is n-closed. Then by Theorem 2.1 it follows that S satisfies one of the conditions (a), (b) or (c) of this theorem. Let G be a Boolean subgroup of S with the identity e and let $x, y \in G$. Then $\{x, y, e\}$ is n-closed for some n, so

$$xy = xye^{n-2} \in \{x, y, e\}^n \subseteq \{x, y, e\}, \qquad \text{if } n \geq 2,$$

and

$$xy \in \{x, y, e\}^2 \subseteq \{x, y, e\}, \qquad \text{if } n = 2.$$

If $xy = x$, then it follows that $y = e$, and, if $xy = y$, then $x = e$. If $xy = e$, then $y = x^{-1} = x$. Therefore, G is a group of order 2, so one of the conditions 1, 2 or 3 holds.

The converse follows by [**4**]. □

Corollary 2.2 [**7**, **8**]. *For every $m \in \mathbf{Z}^+$ the following conditions on a semigroup S are equivalent*:

(i) *S is $2m$-closed*;

(ii) *S is 2-closed*;

(iii) *S is a Rédei's band.* □

Corollary 2.3 [**4**, **5**, **7**]. *For every $m \in \mathbf{Z}^+$ the following conditions on a semigroup S are equivalent*:

(i) *S is $(2m+1)$-closed*;

(ii) *S is 3-closed*;

(iii) *for every subset M of S there exists n such that M is n-closed*;

(iv) *for every subset M of S, $|M| \leq 3$ implies that there exists n such that M is n-closed*;

(v) $(\forall x, y, z \in S)\ xyz \in \{x, y, z\}$. □

3. The general case

In this part, in two different ways, the general case of tight semigroups will be considered. First,

Theorem 3.1. *A semigroup S is a tight semigroup if and only if one of the following conditions holds*:

(A1) *S is a Rédei's band Y of semigroups of the first type S_α with the zero e_α, $\alpha \in Y$, and*

$$S_\alpha \cdot S_\beta = \{e_{\alpha\beta}\} \tag{3.1}$$

for all $\alpha, \beta \in Y$ such that $\alpha \neq \beta$.

(A2) *$S = P \cup Q$, where P is a tight semigroup with exactly one idempotent, Q is a semigroup from* (A1), *$P \cap Q = \varnothing$ and*

$$(\forall x \in P)(\forall y \in Q)\ \ xy = yx = y^2; \tag{3.2}$$

(A3) *S is a tight semigroup with exactly one idempotent.*

Proof. Let S be a tight semigroup. Then by Lemma 1 of [**3**] we have that S is periodic. By Lemma 4 of [**3**], $E(S)$ is a Rédei's band and by Proposition 1 of [**2**] $\mathrm{Reg}(S)$ is a subsemigroup of S. Also, by Lemma 4 of [**3**] it follows that the function $\psi : S \to E(S)$ defined by $\psi(x) = x^4$, $x \in S$, is a retraction. It is clear that $\mathcal{K} = \ker \psi$, so K_e is a subsemigroup for every $e \in E(S)$.

If $|E(S)| = 1$, then (A3) holds. Assume that $|E(S)| \geq 2$ and that $E(S)$ does not contain the identity. Then by Theorem 2.1 we have that $\mathrm{Reg}(S) = E(S)$. Let $e \in E(S)$ and let $x \in K_e$. Then there exists $f \in E(S) - \{e\}$ such that $ef = e$ or $fe = e$. Let $ef = e$ (the similar proof we have in the case $fe = e$). Then

$$|\{x, f\}^2| = |\{x^2, xf, fx, f\}| \leq 2.$$

If $xf = f$, then $x^n f = f$ for every $n \in \mathbf{Z}^+$, so $ef = f$, which is not possible. Thus, $xf = x^2$. Now we have that

$$x^2 = xf = xff = x^2 f = xxf = xx^2 = x^3,$$

whence $x^2 = e$. By this and by Lemma 1.2 it follows that K_e is a semigroup of the first type for every $e \in E(S)$.

Let $x \in K_e$, $y \in K_f$, $e, f \in E(S)$, $e \neq f$. Since

$$\psi(xy) = \psi(x)\psi(y) = ef,$$

then $xy \in K_{ef}$, so S is a Rédei's band $E(S)$ of semigroups of the first type K_e, $e \in E(S)$. Moreover, by

$$|\{x, y\}^2| = |\{e, f, xy, yx\}| \leq 2,$$

it follows that $xy \in \{e, f\}$, so $K_e K_f = \{ef\}$. Therefore, (A1) holds.

Let $E(S)$ contain the identity e. Then by Theorem 2.1 we have that $\mathrm{Reg}(S) = G \cup H$, where G is a Boolean group with the identity e and $H = \mathrm{Reg}(S) - G = E(S) - \{e\}$ is a Rédei's band. Let $Q = \bigcup\{K_f \mid f \in H\}$, $P = S - Q = K_e$. Then P is a tight semigroup with exactly one idempotent. Let $f \in H$ and let $x \in K_f$. Then

$$|\{x, e\}^2| = |\{x^2, xe, ex, e\}| \leq 2.$$

Since $x^2 \neq e$, then $xe, ex \in \{x^2, e\}$. If $xe = e$, then $x^4 e = e$, so $e = x^4 e = fe = f$, which is not possible. Therefore, $xe = x^2$, so

$$x^2 = xe = xee = x^2 e = xxe = xx^2 = x^3.$$

Hence, $x^2 = f$, so K_f is a semigroup of the first type for every $f \in H$. As in the previous case we prove that Q is a Rédei's band H of semigroups of the first type K_f, $f \in H$ and that $K_f K_g = \{fg\}$ for all $f, g \in H$, $f \neq g$. Therefore, Q is a semigroup from (A1). Let $x \in P$, $y \in Q$. Then $y \in K_f$ for some $f \in H$, so

$$|\{x, y\}^2| = |\{x^2, f, xy, yx\}| \leq 2.$$

If $xy = x^2$, then

$$xf = xy^2 = xyy = x^2 y = xxy = xx^2 = x^3,$$

so $f = ef = x^4 f = x^6$, which is not possible. Therefore, $xy = f = y^2$. In a similar way we prove that $yx = f = y^2$, so (3.2) holds. The converse follows immediately. □

By Theorem 3.1 we obtain the following corollary:

COROLLARY 3.1. *Let S be a tight semigroup. Then* $\mathrm{Reg}(S)$ *is an ideal of S and the mapping $\varphi : S \to \mathrm{Reg}(S)$ defined by*

$$\varphi(x) = ex \quad \textit{if} \quad x \in K_e, \qquad e \in E(S), \tag{3.3}$$

is a retraction.

Proof. Let S be a Rédei's band Y of semigroups S_α of the first type with the zero e_α, $\alpha \in Y$, and let (3.1) hold. Then we have $\mathrm{Reg}(S) = E(S) = \{e_\alpha \mid \alpha \in Y\}$. Let $x \in S_\alpha$ for some $\alpha \in Y$. Then

$$xe_\alpha = e_\alpha x = e_\alpha, \tag{3.4}$$

since S_α is a semigroup of the first type. Moreover, if $\beta \in Y$, then by (3.1) it follows that $xe_\beta = e_{\alpha\beta}$ and $e_\beta x = e_{\beta\alpha}$, so $\mathrm{Reg}(S)$ is an ideal of S. It is clear that $\mathcal{K}$-classes of S are semigroups S_α, $\alpha \in Y$, so by (3.4) we obtain that (3.3) has the form

$$\varphi(x) = e_\alpha x = e_\alpha \quad \text{if} \quad x \in S_\alpha, \quad \alpha \in Y. \tag{3.5}$$

So φ is a retraction.

Let $S = P \cup Q$, where P is a tight semigroup with exactly one idempotent, Q is a semigroup from (A1), $P \cap Q = \varnothing$ and (3.2) holds. Then $\mathrm{Reg}(S) = G \cup H$, where G is a Boolean group, $\mathrm{Reg}(P) = G$, H is a Rédei's band and $\mathrm{Reg}(Q) = H$. Let $x \in S$ and let $a \in \mathrm{Reg}(S)$. Since

$$GP \cup PG \subseteq G \subseteq \mathrm{Reg}(S) \quad \text{and} \quad QH \cup HQ \cup PQ \cup QP \subseteq H \subseteq \mathrm{Reg}(S),$$

(by the previous case and by (3.2)), then we have that $\mathrm{Reg}(S)$ is an ideal of S. Let φ be a mapping defined by (3.3) and let $x, y \in S$. If $x, y \in Q$ or $x, y \in P$, then by the previous case and by proof of Theorem 1.2 we have that $\varphi(xy) = \varphi(x)\varphi(y)$. Assume that $x \in P$ and $y \in Q$. Then $x \in K_e$, $y \in K_f$ where e is an identity of G, $f \in H$ and by (3.2) we have that $xy = yx = y^2 = f$, so

$$\varphi(x)\varphi(y) = exyf = f = fxy = \varphi(xy),$$

and

$$\varphi(y)\varphi(x) = fyex = f = fyx = \varphi(yx).$$

Therefore, φ is a retraction.

If S is a tight semigroup with exactly one idempotent, then the proof follows by the proof of Theorem 1.2. □

Secondly, we give a description of tight semigroups by retract extensions.

THEOREM 3.2. *A semigroup S is a tight semigroup if and only if one of the following conditions holds*:

(B1) *S is a retract extension of a Rédei's band E by a semigroup of the first type with the retraction φ satisfying the condition*

$$\Phi_a \cdot \Phi_b = \{ab\} \qquad \text{for all } a, b \in E,\ a \neq b, \tag{3.6}$$

where $\Phi = \ker \varphi$.

(B2) *S is a retract extension of a Rédei's band E with the identity e by a tight nil-semigroup with the retraction φ satisfying conditions*

$$x^2 = a \qquad \text{for all } a \in E,\ a \neq e \text{ and all } x \in \Phi_a; \tag{3.7}$$

$$\Phi_a \cdot \Phi_b = \{ab\} \qquad \text{for all } a, b \in E,\ a \neq b; \tag{3.8}$$

where $\Phi = \ker \varphi$.

(B3) *S is a retract extension of a regular tight semigroup T with a nontrivial maximal subgroup G by a semigroup of the first type with the retraction φ satisfying the conditions*

$$\Phi_a \subseteq C(\Phi_b) \qquad \text{for all } a, b \in G \text{ such that } a \neq b; \tag{3.9}$$

$$\Phi_a \cdot \Phi_b = \{ab\} \qquad \text{for all } a, b \in T,\ a \neq b \text{ and } a \notin G \text{ or } b \notin G; \tag{3.10}$$

where $\Phi = \ker \varphi$.

(B4) *S is a tight semigroup with exactly one idempotent.*

Proof. Let S be a tight semigroup. Then the conditions of Theorem 2.1 hold. By Corollary 3.1 it follows that $T = \mathrm{Reg}(S)$ is an ideal of S and the mapping φ of S in T defined by (3.3) is a retraction. Let $\Phi = \ker \varphi$. If $|E(S)| = 1$, then we obtain (B4). Let $|E(S)| \geq 2$.

Assume that S is a Rédei's band Y of semigroups of the first type S_α with the zero e_α, $\alpha \in Y$, and that (3.1) holds. Then $T = E(S) = \{e_\alpha \mid \alpha \in Y\}$. Since $x^2 \in T$ for all $x \in S$, we have that S/T is a semigroup of the first type. It is clear that S_α, $\alpha \in Y$, are all Φ-classes, so by (3.1) we obtain (3.6).

Let $S = P \cup Q$, where P is a tight semigroup with exactly one idempotent, Q is a semigroup from (A1), $P \cap Q = \varnothing$ and (3.2) holds. Then $T = G \cup H$, where G is a Boolean group, $\mathrm{Reg}(P) = G$, H is a Rédei's band and $\mathrm{Reg}(Q) = H$. Assume that $|G| = 1$, i.e. $G = \{e\}$. Then T is a Rédei's band with the identity e. Let $a \in T$, $a \neq e$ and let $x \in \Phi_a$. Since $\Phi_a \subseteq Q$ and Q is a semigroup from (A1), we have that $x^2 \in H \subseteq T$, so (3.7) holds. If $a, b \in H$, $a \neq b$, then by (3.6) (i.e. by (3.1)) we obtain (3.8). Let $a \in H$. Since $\Phi_e = P$ and $\Phi_a \subseteq Q$, then by (3.2) and by (3.7) we get

$$\Phi_a \cdot \Phi_e = \Phi_e \cdot \Phi_a = \{a\} = \{ea\} = \{ae\}.$$

Therefore, (3.8) holds. Let $|G| \geq 2$. Then by the proof of Theorem 1.2 we have that P/G is a semigroup of the first type, so $x^2 \in G \subseteq T$ for all $x \in P$. Since Q/H is a semigroup of the first type, then $x^2 \in H \subseteq T$ for all $x \in Q$, so $x^2 \in T$ for all $x \in S$, whence it follows that S/T is a semigroup of the first type. By Theorem 1.2

it follows that (3.9) holds. Let $a, b \in T$, $a \neq b$. If $a, b \in H$, $b \in G$. Since $\Phi_b = P$ and $\Phi_a \subseteq Q$, then by (3.2) and by (3.7) we have

$$\Phi_a \cdot \Phi_b = \Phi_b \cdot \Phi_a = \{a\} = \{ba\} = \{ab\}.$$

Therefore, (3.10) holds.

Conversely, let S be a semigroup from (B1), i.e. let S be a retract extension of a Rédei's band E by a semigroup of the first type with the retraction φ for which (3.6) holds. Then, for every $a \in E$, Φ_a is a semigroup isomorphic to some subsemigroup of S/E, so Φ_a is a semigroup of the first type for every $a \in E$. By this and by (3.6) it follows that S is a tight semigroup.

Let S be a semigroup from (B2), i.e. S is a retract extension of a Rédei's band E with the identity e by a tight nil-semigroup with the retraction φ for which (3.7) and (3.8) hold. Then Φ_a is a subsemigroup of S and is isomorphic to some subsemigroup of S/E, so Φ_a is a tight semigroup for all $a \in E$. By this and by (3.7) and (3.8) we obtain that S is a tight semigroup.

Let S be a semigroup from (B3), i.e. S is a retract extension of a regular tight semigroup T with a nontrivial maximal subgroup G by a semigroup of the first type with the retraction φ for which (3.9) and (3.10) hold. Then by Theorem 2.1 it follows that G is a Boolean group, $T = G \cup H$, where H is a Rédei's band and (2.1) holds. Let $P = \bigcup\{\Phi_a \mid a \in G\}$. Then P is a ret ract extension of G. Since P/G is isomorphic to some subsemigroup of S/T, then P/G is a semigroup of the first type. By this, by (3.9) and by Theorem 2.2 we obtain that P is a tight semigroup. Moreover, for every $a \in H$, Φ_a is a semigroup isomorphic to some subsemigroup of S/T, so Φ_a is a semigroup of the first type. By this and by (3.10) and (2.1) it follows that

$$|\{x, y\}^2| = |\{y, x\}^2| = |\{x^2, xy, yx, y^2\}| = |\{x^2, a\}| \leq 2,$$

for all $x \in P$, $y \in \Phi_a$, $a \in H$. Therefore, S is a tight semigroup.

If S is a semigroup from (B4), then the proof follows immediately. □

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