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# NOTE ON A PAPER BY H. L. MONTGOMERY (OMEGA THEOREMS FOR THE RIEMANN ZETA-FUNCTION)

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**Abstract.** We study Omega theorems for the expression $E = \mathrm{Re}(e^{i\theta}\log\zeta(\sigma_0 + it_0))$ where $1/2 \le \sigma_0 < 1$ and $0 \le \theta < 2\pi$ ($\sigma_0$, $\theta$ fixed) as $t_0 \to \infty$. In fact we prove $E \ge C(1-\sigma_0)^{-1}(\log t_0)^{1-\sigma_0}(\log\log t_0)^{-\sigma_0}$ for at least one $t_0$ in $[T^\varepsilon, T]$ where $C$ is a positive constant. Note that $(1-\sigma_0)^{-1} \to \infty$ as $\sigma_0 \to 1$.

**1. Introduction.** In [**8**] Montgomery developed a method for studying $\Omega$-theorems for $\mathrm{Re}(e^{-i\theta}\log\zeta(s_0))$, where $\theta$ is fixed such that $0 \le \theta < 2\pi$, the real part $\sigma_0$ of $s_0$ is fixed in $[1/2, 1)$, and the imaginary part $t_0$ tends to infinity. In fact he gets a lower bound for the maximum of this expression in $T^{(\sigma_0-1/2)/3} \le t_0 \le T$, namely

$$\frac{1}{20}(\sigma_0 - 1/2)^{1/2}(\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0}. \tag{1.1}$$

He obtains by his method that (on Riemann hypothesis)

$$|\zeta(1/2 + it)| = \Omega(\exp(1/20(\log t/\log\log t)^{1/2})) \tag{1.2}$$

and that

$$\mathrm{Arg}\,\zeta(1/2 + it) = \Omega((\log t/\log\log t)^{1/2}). \tag{1.3}$$

As a passing remark, it should be mentioned that in [**9**] Ramachandra has an alternative method for proving some of these results (sometimes in a stronger and sometimes in a weaker form); for example, for $\sigma \in [1/2, 1)$

$$\log|\zeta(\sigma + it)| = \Omega_\pm((\log t)^{1-\sigma-\varepsilon}). \tag{1.4}$$

He has also other results. (For the advantages of Ramachandra's method see Remark 1 after Main Theorem).

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The aim of this paper is (following the method of Montgomery and making an optimal use of his parameter $\alpha$) to prove the following main theorem, which is perhaps the limit of his method. To state the main theorem, it is convenient to make a hypothesis. Let $N(\mu, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfying $\beta \geq \mu$ and $|\gamma| \leq T$.

HYPOTHESIS. *For a fixed $\mu$ such that $1/2 < \mu < 1$ and all large $T$, there exists a $\delta > 0$ such that $N(\mu, T) \ll T^{1-\delta}$, where the implied constant depends on $\mu$ and $\delta$.*

*Remark.* If we assume Riemann hypothesis, it suffices to take $\delta = 1$. However the hypothesis is always satisfied by any $\delta < 1 - 3(1-\mu)/(2-\mu)$, by the well-known density result of A. E. Ingham.

MAIN THEOREM. *Let $1/2 \leq \sigma_0 < 1$, $0 \leq \theta < 2\pi$, $\varepsilon > 0$. Let $y$ be the positive solution of $e^y = 2y + 1$. Let $l$ be an integer constant satisfying $l \geq 6$, $c_2 = 2y/(2y+1)^2$, $0 < c_1 < c_2$. Then for $T \geq T_0$ depending on these constants we have*

$$\mathrm{Re}(e^{-i\theta} \log \zeta(\sigma_0 + it_0)) \geq (1-\sigma_0)^{-1} c_0 c_1 (\log t_0)^{1-\sigma_0} (\log\log t_0)^{-\sigma_0}$$

*for at least one $t_0$ in $T^\varepsilon \leq t_0 \leq T$, where $c_0 = (\cos(2\pi/l))(\delta/\log l)^{1-\sigma_0}$. Here $\delta = 1$ if we assume the Riemann hypothesis. Otherwise we have to assume $1/2 < \sigma_0 < 1$ and then we can take $\delta = 1 - 3(1-\sigma_0)/(2-\sigma_0)$.*

*Remark* 1. An advantage of Montgomery's method is that it gives a better $\Omega$-result for $|\zeta(\sigma + it)|$ than the result of Levinson and Ramachandra, namely

$$\log|\zeta(\sigma + it)| = \Omega_\pm\big((\log t)^{1-\sigma}/\log\log t\big) \qquad \text{for } 1/2 < \sigma < 1.$$

A disadvantage of Montgomery's method and that of Levinson [**7**] is that it does not work for short intervals. The method of Ramachandra does not assume Riemann hypothesis, works for short intervals, yields a slightly weaker result for $1/2 < \sigma_0 < 1$ than the result of Montgomery. Also Ramachandra's method, as developed by Ramachandra and Balasubramanian [**2**], works for $L$-series and so on. For example, it gives (without the assumption of Riemann hypothesis, etc.)

$$\log|L(1/2 + it, \chi)| = \Omega_+\big((\log t/\log\log t)^{1/2}\big)$$

and other results (see also [**10**]). Continuing [**2**] Balasubramanian [**1**] has shown that the constant can be taken to be $3/4$ for $\zeta(s)$.

*Remark* 2. The positive root of the equation $e^y = 2y + 1$ is $y = 1.2564312086\ldots$; so $c_2 = 1/4.910814964\ldots$. For example if we take $l = 10$, then $\cos(2\pi/l) \geq 0.809$ and $(\log l)^{1-\sigma_0} \geq 1/1.5175$. So we get, on the assumption of Riemann hypothesis,

$$|\zeta(1/2 + it)| = \Omega\big(\exp(c_3(\log t/\log\log t)^{1/2})\big) \qquad \text{as } t \to \infty,$$

where $c_3 = 1/4.60578\ldots$. This improves Theorem 2 of [**8**].

*Remark* 3. If $\sigma_0$ is very close to 1, then the constant we get, is boosted up very much because of the factor $(1-\sigma_0)^{-1}$.

*Remark* 4. By the method of this paper, it follows that for $1/2 < \sigma < 1$,

$$|\log\zeta(\sigma_0 + it_0)| \geq 2(1-\sigma_0)^{-1} c_0 c_1 (\log t_0)^{1-\sigma_0} (\log\log t_0)^{-\sigma_0}$$

for some $t_0$ satisfying $T^{\varepsilon} \leq t_0 \leq T$, where $c_0$ and $c_1$ are as in the main theorem.

*Remark* 5. The method of Montogomery is to start with

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \left(e^{-i\theta}\log\zeta(s+s_0)\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 (2 + x^s e^{i\theta} + x^{-s}e^{-i\theta})\,ds$$
$$= \sum_{|\log(p/x)|\leq 2\alpha} p^{-s_0}\left(2\alpha - |\log(p/x)|\right) + \text{terms of smaller order},$$

break off $|\operatorname{Im} s| \geq \tau = (\log t_0)^2$ and then, move the line of integration in the rest of this integral to $\operatorname{Re} s = 0$. Notice now that $ds/i = dt$ and the quantity multiplying $e^{-i\theta}\log\zeta(s+s_0)$ is non-negative and hence we can take the real part of both sides and take out the maximum of the real part of $e^{-i\theta}\log\zeta(s+s_0)$ in $|t| \leq \tau$. This leads to a lower bound provided the real part of right-hand side is big. This is provided by the Dirichlet box principle. We work out the details in Sections 3 and 4. To get the result of Remark 4, we have to consider

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \left(\log\zeta(s+s_0)\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 x^s\,ds$$

and treat it similarly.

*Remark* 6. The extension of the results to zeta functions of algebraic number fields presents no difficulties. It is verbatim the same as that for $\zeta(s)$. Some remarks will be made in Section 5 and the appendix.

**2. Notation.** Let

(1) $A, B, C, \ldots, C_0, C_1, C_2, \ldots$ denote positive constants;

(2) $f \ll g$ denote that $|f| \leq Ag$, where $A$ is some positive constant;

(3) $\|\theta\|$ denote $\min_n |\theta - n|$, where $n$ runs through all integers;

(4) $[x]$ denote the integral part of $x$.

**3. Some lemmas.** LEMMA 3.1. [**8**] *Let $\theta_1, \theta_2, \ldots, \theta_M$ be distinct positive real numbers and suppose that $l \geq 6$, where $l$ is an integer constant. Then there exist at least $R$ integers $r'_k$ such that $1 \leq r'_k \leq J = l^M R$ and $\|r'_k\theta_m\| < 1/l$ for $1 \leq m \leq M$.*

*Proof*. For the sake of completeness, we give the proof of this lemma. Consider the $M$-dimensional cube ($0 \leq x_i < 1$, $i = 1, 2, \ldots, M$) in the $M$-dimensional

Euclidean space. Let $l \geq 6$ be an integer constant. Divide each side of the cube into $l$ equal parts:

$$[0, 1/l), \quad [1/l, 2/l), \quad \ldots, \quad [(l-1)/l, 1).$$

to get $l^M$ small cubes. Put $J_0 = l^M R + 1$. Let $x_{j,r} = r\theta_j - m_{j,r}$ where $m_{j,r} = [r\theta_j]$, $1 \leq j \leq M$ and $1 \leq r \leq J_0$. Then clearly $0 \leq x_{j,r} < 1$. Consider the points $x_r = (x_{1,r}, x_{2,r}, \ldots, x_{M,r})$ in the unit cube. At least one small cube contains $R+1$ such points. Otherwise, since there are $l^M$ small cubes and if each of them contained at most $R$ points, then the total number of points would be $\leq l^M R$, which is a contradiction because of the fact that $1 \leq r \leq J_0$. Choose that particular small cube which contains $R+1$ points.

The set of these points is a subset of $[1, J_0]$. Let the corresponding values of $r$ be $r_1, \ldots, r_{R+1}$ and $r_1 < r_2 < \ldots < r_{R+1}$. We see that there exists an $A_j$ (independent of $k$) such that $0 \leq A_j \leq l-1$ and $A_j/l \leq x_{j,r_k} < (A_j+1)/l$ for every $k = 1, 2, \ldots, R+1$, $j = 1, 2, \ldots, M$. This implies that for suitable integers $m_j^*$ we have $-1/l < (r_k - r_1)\theta_j - m_j^* < 1/l$ $\forall k = 2, 3, \ldots, R+1$, i.e. $\|(r_k - r_1)\theta_j\| < 1/l$, $\forall k = 2, 3, \ldots, R+1$.

Also we have $1 \leq r_k' \leq l^M R$ where $r_k' = r_k - r_1$, $\forall k = 2, 3, \ldots, R+1$. This proves the lemma.

LEMMA 3.2. *Let $\theta_1, \theta_2, \ldots, \theta_M$ of Lemma 3.1 be $P \log p$ where $P$ is a fixed positive integer and $p$ runs through a finite set of primes containing those $p$ satisfying $|\log(p/x)| \leq 2\alpha$, where $x \geq 10$ is fixed and $\alpha$ is a positive constant which will be fixed later. Let $l_{k-1} = r_k' P$ for every $k = 2, 3, \ldots, R+1$. Then we have*

(i) $\cos(2\pi l_k \log p) \geq \cos(2\pi/l)$ $\forall k = 1, 2, \ldots, R$.

(ii) $P \leq l_k \leq l^M RP$, $\forall k = 1, 2, \ldots, R$.

*Proof*. (i) From Lemma 3.1 we have $\|l_k \log p\| < 1/l$, $\forall k = 1, 2, 3, \ldots, R$. This proves (i).

(ii) From Lemma 3.1 we have $1 \leq r_k' \leq l^M R$, $\forall k = 2, 3, \ldots, R+1$. This implies that $P \leq l_k \leq l^M RP$, $\forall k = 1, 2, \ldots, R$.

LEMMA 3.3. *For $1/2 \leq \sigma_0 < 1$, we have*

$$\sum_{|\log(p/x)| \leq 2\alpha} p^{-\sigma_0} \left(2\alpha - \left|\log\left(\frac{p}{x}\right)\right|\right) \sim \left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2 \frac{x^{1-\sigma_0}}{\log x}.$$

*Proof*. The proof uses the fact that $\pi(x) \sim x/\log x$ and also Stieltjes integral and integration by parts.

LEMMA 3.4. *If $x > 0$, $c > 0$, then we have*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2}\, ds = \begin{cases} \log x, & \text{if } x \geq 1, \\ 0, & \text{if } 0 < x \leq 1. \end{cases}$$

*Proof*. The proof follows by moving the line of integration to the extreme left and the extreme right respectively.

LEMMA 3.5. *If $\alpha > 0$, $x > 0$ and $c > 0$, then we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{2}\right)^2 x^s\,ds = \begin{cases} 2\alpha - |\log x| & \text{if } |\log x| \le 2\alpha, \\ 0 & \text{if } |\log x| \ge 2\alpha. \end{cases}$$

*Proof*. Follows from Lemma 3.4.

LEMMA 3.6. *Let $0 \le \theta < 2\pi$, $\alpha > 0$ and $1/2 \le \sigma_0 < 1$ be constants and let $s = \sigma + it$, $s_0 = \sigma_0 + it_0$. Then for all $x$, with $10 \le x \ll (\log t_0)\log\log t_0$ we have*

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(e^{-i\theta}\log\zeta(s+s_0)\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds$$
$$= \sum_{|\log(p/x)|\le 2\alpha} p^{-s_0}\left(2\alpha - |\log(p/x)|\right) + O((\log x)^2).$$

*Proof*. Follows from Lemma 3.5.

LEMMA 3.7. *Let $\tau = (\log t_0)^2$. If $\{\sigma > 0, |t| \le 2\tau\}$ is zero free for $\zeta(s+s_0)$, then in $0 < \sigma \le 1$ we have $\log\zeta(s+s_0) = O((\log t_0)(\log(2/\sigma)))$.*

*Proof*. See Theorem 9.6 (B) of [**12**].

LEMMA 3.8. *Let $\theta$, $\alpha$, $\sigma_0$ and $t_0$ be as in Lemma 3.6. The contribution of $|t| \ge (\log t_0)^2$ to the integral in Lemma 3.6 is $O((\log x)^2)$. Also the contributions from the integrals over $[i\tau, 1+i\tau]$ and $[-i\tau, 1-i\tau]$ are $O((\log x)^2)$.*

*Proof*. Follows from Lemma 3.7.

LEMMA 3.9. *With $\tau = (\log t_0)^2$, we have,*

$$\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{-i\tau}^{i\tau}\left(e^{-i\theta}\log\zeta(s+s_0)\right)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds\right\}$$
$$= \sum_{|\log(p/x)|\le 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|) + O((\log x)^2)$$

*Proof*. Follows from lemmas 3.6, 3.7 and 3.8.

LEMMA 3.10. *We have*

$$\max_{|t|\le\tau}\left(\mathrm{Re}\, e^{-i\theta}\log\zeta(s+s_0)\right)\frac{1}{2\pi i}\int_{|t|\le\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds$$

$$\geq \sum_{|\log(p/x)|\leq 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|) + O((\log x)^2).$$

*Proof*. Follows from Lemma 3.9.

LEMMA 3.11. *For $\tau = (\log t_0)^2$ and $2\alpha \leq |\log x|$ we have*

$$\frac{1}{2\pi i}\int_{|t|<\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds = 4\alpha + O(1/\tau).$$

*Proof*. Since the integrand is $O(t^{-2})$ for $|t|\geq\tau$, we have

$$\frac{1}{2\pi i}\int_{|t|\leq\tau,\,\sigma=0}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds$$
$$= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2(2+x^s e^{i\theta}+x^{-s}e^{-i\theta})\,ds + O\left(\frac{1}{\tau}\right) = 4\alpha + O\left(\frac{1}{\tau}\right)$$

by using Lemma 3.5 and continuity.

LEMMA 3.12. *Let $c$ be a positive constant to be chosen later. Let $p$ be the set of primes satisfying*

$$ce^{-2\alpha}(\log P\log\log P) \leq p \leq ce^{2\alpha}(\log T\log\log T)$$

*where we now refer to Lemma 3.2 and put $T = l^M RP$. $M$ will be greater than or equal to the number of primes satisfying the inequalities just stated. We put $M = [(ce^{2\alpha}+\varepsilon)\log T]$ where $\varepsilon > 0$ is an arbitrary but fixed constant. Let $x = c\log t_0\log\log t_0$ where $c$ is a small positive constant and $t_0 = 2\pi l_k$ $(k = 1, 2, \ldots, R)$ for any $k$. Then for all primes $p$ satisfying $|\log(p/x)| \leq 2\alpha$, we have $\cos(t_0\log p) \geq \cos(2\pi/l)$. Thus*

$$\sum_{|\log(p/x)|\leq 2\alpha} p^{-\sigma_0}\cos(t_0\log p)(2\alpha - |\log(p/x)|)$$
$$\geq \cos\left(\frac{2\pi}{l}\right)c^{1-\sigma_0}\left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}.$$

*Proof*. The lemma follows from Lemma 3.3.

**4. Proof of the theorem.** Consider the rectangles $\sigma \geq \sigma_0$, $|t_j - t| \leq 2(\log t_0)^2$, $(j = 1, 2, \ldots, R)$. These rectangles are disjoint and their number is $R$. If $R > DT^{1-\delta}+2$, where $D$ is the constant coming from the hypothesis, then at least two of these rectangles are zero-free. We select the rectangle for which $t_0 + \tau \leq T$ ($T$ to be defined) and we fix $P = T^{\varepsilon_1}$, $R = T^{1-\delta+\varepsilon_2}$, where $\varepsilon_1$, $\varepsilon_2$ are small positive constants. Then we put $M = [(ce^{2\alpha}+\varepsilon)\log T]$ and $l^M RP = T$. If we choose $c = \dfrac{\delta}{e^{2\alpha}\log l} - \dfrac{\varepsilon_3}{e^{2\alpha}\log l}$ for a small positive constant $\varepsilon_3$, then from lemmas 3.10, 3.11 and 3.12 we get (by putting $2\alpha(1-\sigma_0) = \beta$)

$$\max_{|t|\le\tau} \mathrm{Re}(e^{-i\theta}\log\zeta(s_0+it))$$
$$\ge \frac{1}{4\alpha}\cos\left(\frac{2\pi}{l}\right)(\log l)^{-(1-\sigma_0)}\frac{\delta^{1-\sigma_0}}{e^{2\alpha(1-\sigma_0)}}\left(\frac{2\sinh(\alpha(1-\sigma_0))}{1-\sigma_0}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}$$
$$= \frac{1}{2}\frac{\cos(2\pi/l)\delta^{1-\sigma_0}}{(\log l)^{1-\sigma_0}(1-\sigma_0)}\left(\frac{1-e^{-\beta}}{\sqrt{\beta}}\right)^2\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}.$$

By choosing $\beta > 0$ such that $(1-e^{-\beta})/\sqrt{\beta}$ is maximum we see that this expression becomes

$$\frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\cdot\frac{\delta^{1-\sigma_0}}{1-\sigma_0}\cdot c_1\cdot\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}$$

where $c_1$ is a positive constant independent of $\delta$, $l$ and $\sigma_0$, and $c_2 = 2y/(2y+1)^2 > c_1$, where $y$ is the root of the equation $e^y = 2y+1$. This proves the theorem.

**5. Generalizations.** Let $K$ be an algebraic number field. The Dedekind zeta-function of $K$ is defined for $\mathrm{Re}\, s > 1$ by $\zeta_K(s) = \sum_{\mathfrak{A}\neq 0}(N\mathfrak{A})^{-s}$, where $N\mathfrak{A}$ denotes the norm of the ideal $\mathfrak{A}$ and the sum is extended over all non-zero integral ideals of the ring of integers of the field $K$. We know that $\zeta_K(s)$ can be continued analytically in $\sigma \ge 0$ and there we have $|(s-1)\zeta_K(s)| \ll (|t|+4)^{D_0}$, where $D_0$ is some positive constant.

Let $\log\zeta_K(s) = \sum_{n=1}^{\infty} e_n n^{-s}$ for $\sigma \ge 2$. We notice that $e_n \ge 0$, $\forall n$. Also it is well-known that

$$\sum_{n\le x} e_n \sim \sum_{p\le x} e_p \sim \sum_{N\mathfrak{P}\le x} 1 \sim \frac{x}{\log x} \tag{5.1}$$

holds, where the third sum from the left of (5.1) is extended over prime ideals. Let $N_K(\mu, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ where $\beta \ge \mu$ and $|\gamma| \le T$. We make the following hypothesis.

HYPOTHESIS $(*)$. *For a fixed $\mu$ such that $1 > \mu > 1/2$, there exists a $\delta > 0$ satisfying $N_K(\mu, T) \ll T^{1-\delta}$, where the constant depends on $K$, $\mu$ and $\delta$.*

Assuming $(*)$ we can prove

THEOREM. *Let $\mu < \sigma_0 < 1$, $T > T(\sigma_0)$. For any real $\theta$, there is a $t_0$ such that $T^{\varepsilon} \le t_0 \le T$ and*

$$\mathrm{Re}\big(e^{-i\theta}\log\zeta_K(s_0)\big) \ge \frac{\cos(2\pi/l)}{(\log l)^{1-\sigma_0}}\cdot\frac{\delta^{1-\sigma_0}}{1-\sigma_0}\cdot c_1\cdot\frac{(\log t_0)^{1-\sigma_0}}{(\log\log t_0)^{\sigma_0}}$$

*where $c_1$ is a positive constant independent of $\delta$, $l$ and $\sigma_0$ and $c_2 = 2y/(2y+1)^2 > c_1$, where $y$ is the positive root of the equation $e^y = 2y+1$.*

*Remark.* For the special cases see the appendix. The proof of this theorem is verbatim the same as that for $\zeta(s)$.

**Appendix.** Let $K$ be an algebraic number field abelian over $K'$. Let the degrees of $K$ and $K'$ be $n$ and $k$ respectively. Then $\zeta_K(s)$ splits in the following way into abelian $L$-functions of $K$

$$\zeta_K(s) = L_1(s) \cdot \ldots \cdot L_j(s), \tag{A1}$$

where $j = n/k$. Let $N(L_i, \sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $L_i(s)$ in (A1) such that $\beta \geq \sigma$ and $|\gamma| \leq T$. We distinguish the following two cases.

*Case* i. Suppose for every $L_i(s)$ in (A1) we have

$$|L_i(1/2 + it)| \ll t^{k/6+\varepsilon}, \qquad t \geq 1.$$

(This has been proved by Peter Sohne following the method of [**5**]. We are thankful to Professors D. R. Heath-Brown and W. Schaal for this information). In this case, it is not hard to prove by the standard methods that $N(L_i, \sigma, T) \ll T^{\lambda(1-\sigma)+\varepsilon}$ where $\lambda = (2k+6)/3$. So, we get $N(L_i, \sigma, T) \ll T^{1-\delta}$ if $\sigma > 1 - 3/(2k+6)$. Since the number of zeros is additive, we get

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{if} \qquad \sigma > 1 - 3/(2k+6).$$

In our hypothesis $(*)$, we can take any $\mu > 1 - 3/(2k+6)$.

*Case* ii. Let $\mu'$ be the smallest real number such that

$$\int_0^T |L_i(\mu' + it)|^2 \, dt \ll_\varepsilon T^{1+\varepsilon}.$$

(a) $\mu' = 1/2$ happens when $K' = \mathbf{Q}$ or $\mathbf{Q}(\sqrt{d})$, where $d$ is an integer, not a perfect square. Then by the standard methods (see [**4**] or [**6**] or [**11**]), it follows that

$$N(L_i, \sigma, T) \ll T^{\frac{4(1-\sigma)}{3-2\sigma}+\varepsilon}.$$

In fact this is known uniformly for ordinary $L$-functions to the modulus $q \leq T$ (see [**4**]). Since the number of zeros is additive, we have

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{for} \qquad \sigma > 1/2.$$

In our hypothesis $(*)$, we can take any $\mu > 1/2$.

(b) If $\mu' > 1/2$ then by standard methods (see [**4**] or [**6**] or [**11**]), we get

$$N(L_i, \sigma, T) \ll T^{1-\delta} \qquad \text{for} \quad \sigma > \mu'.$$

Since the number of zeros is additive, it follows that

$$N_K(\sigma, T) \ll T^{1-\delta} \qquad \text{for} \quad \sigma > \mu'.$$

In our hypothesis $(*)$, we can take any $\mu > \mu'$.

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**Added in proof.** Independently of us J.-P. Kahane has some results in his paper: *Produits de Riesz et séries de Dirichlet*, in: *Analysis and Partial Differential Equations*, Marcel Dekker, New York and Basel, 1990, p.p. 231–238. However, his point of view is different and his methods are also different. We proved our result earlier.

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