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A PROPERTY OF CANONICAL GRAPHS

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Abstract. A finite connected graph is called canonical if no two of its vertices have the same neighbours. In this paper we prove that in all but a sequence of exceptional cases, deleting of a suitable chosen vertex in a canonical graph also gives a connected canonical graph. This property can have applications in various hereditary problems in the spectral Theory of Graphs.

In this paper we consider only finite connected graphs without loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and the number of its vertices by $|G|$. Relation $H \subseteq G$ will always mean that H is an induced subgraph of a graph G. For any two vertices u, v of G, $uv = 1$ will mean that u is adjacent to v, while $uv = 0$ will mean that u is nonadjacent to v.

The graph obtained by deleting a vertex $x \in V(G)$ from G is denoted by $G - x$. It can be connected or disconnected. But, as is well known, there is at least one vertex $x \in V(G)$ such that the corresponding graph $G - x$ is also connected.

Next, we say that two vertices $u, v \in V(G)$ are equivalent in G and we denote it by $u \sim v$ if we have

$$uv = 0 \qquad \text{and} \quad ru = rv \tag{1}$$

for any vertex $r \in V(G)\backslash\{u, v\}$, thus if and only if u and v have the same neighbours in G. Relation $\sim$ is obviously an equivalence relation on the vertex set $V(G)$. The corresponding quotient graph is denoted by g and called the *canonical graph* of G. This graph is also connected.

For instance, if $G = K_{m_1 \ldots m_p}$ $(p \geq 2)$ is a complete p-partite graph, then its canonical graph is the complete graph K_p. The canonical graph of the complete graph K_n is the same graph K_n.

We say that a graph G is *canonical* if $|G| = |g|$, i.e. if G has no two equivalent vertices.

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If g is the canonical graph of G, $|g| = k$ and $N_1, \dots, N_k$ are the corresponding sets of equivalent vertices in G, we denote

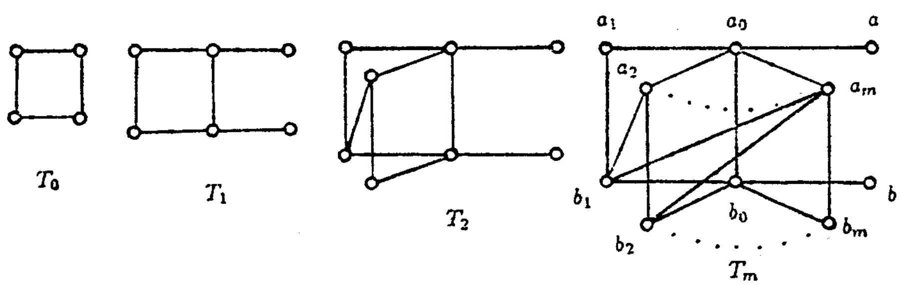
$$G = g(N_1, \dots, N_k).$$

We call $N_1, \dots, N_k$ the characteristic sets of G. Obviously, each set $N_i \subseteq V(G)$ $(i = 1, \dots, k)$ consists only of isolated vertices, and if at least one edge between the sets N_i and N_j $(i \neq j)$ is present, then all possible edges between these sets are also present. Therefore, it is very convenient to display the sets N_i $(i = 1, \dots, k)$ by white (that is, empty) circles, and all possible edges between the sets N_i and N_j $(i \neq j)$ by only one edge between the corresponding circles. If, for example, G is the complete bipartite graph with characteristic sets N_1, N_2, we can simply draw

$$G = \underset{N_1}{\bigcirc}\!\!\longrightarrow\!\!\!\!\!-\!\!\underset{N_2}{\bigcirc}$$

We note that many hereditary problems in the Spectral Theory of Graphs can be reduced to finding firstly the corresponding sets of canonical graphs. Compare for instance the papers [**3**], [**4**], [**5**], [**6**], [**7**], [**8**], [**9**], [**10**], [**11**], or the monograph [**2**], where many results from these papers are presented. Therefore, the importance of the following hereditary property of canonical graphs is clear.

THEOREM. *In all but a sequence of exceptional cases, each canonical graph G with n vertices $(n \geq 2)$ contains, as an induced subgraph, a connected and canonical subgraph H on $n - 1$ vertices. The exceptional graphs are*



where $a_j b_i = 1$ $(i \leq j;\ i, j = 1, \dots, m)$. We obviously have that $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$.

Proof. First, it is trivial to check that all graphs T_m $(m \geq 0)$ are exceptional graphs. Graphs $G - a_0$ and $G - b_0$ are disconnected, while for any $m \geq 1$ and $i = 0, \dots, m-1$ we have

$$\begin{aligned} a_0 &\sim b_1 \quad (\text{in } G - a), & b_0 &\sim a_m \quad (\text{in } G - b),\\ a &\sim a_1 \quad (\text{in } G - b_1), & b &\sim b_m \quad (\text{in } G - a_m),\\ a_i &\sim a_{i+1} \quad (\text{in } G - b_{i+1}), & b_i &\sim b_{i+1} \quad (\text{in } G - a_i). \end{aligned}$$

Next, we consider any connected canonical graph $G \neq T_m$ $(m = 0, 1, 2, \dots)$, which is "bad" in the sense that deleting of any its vertex gives a disconnected or a noncanonical graph. We shall prove that this assumption in all possible cases gives a contradiction, thus that such graphs really do not exist.

Firstly choose any vertex x of G such that the subgraph $G-x$ is connected. Then $G-x$ is a noncanonical graph. Let $G-x=f(M_1,\dots,M_k)$ $(k\geq 1)$, where $M_1,\dots,M_k$ are the characteristic sets of $G-x$, f is the corresponding canonical graph of $G-x$ (also connected), and at least one of the sets $M_1,\dots,M_k$ is not a singleton. Obviously $k\geq 2$. Rearranging the vertices of f, we can assume that $|M_1|\geq 2$.

If $|M_i|\geq 2$ for some $i\in\{1,\dots,k\}$ and if $u,v\in M_i$ $(u\neq v)$, the fact that G is a canonical graph implies that exactly one of u, v is adjacent to x. Therefore, we immediately get

$$(2)\qquad |M_i|\leq 2 \qquad (i=1,\dots,k),$$

and consequently $|M_1|=2$. We assume that $M_1=\{y,z\}$, where $xy=1$ and $xz=0$.

Now assume that some $|M_j|=2$ $(j>1)$, for instance that $M_2=\{c,d\}$, where $xc=1$ and $xd=0$. Deleting then the vertex d from G we obviously get a connected graph. It is also easily seen that $G-d$ is a canonical graph, what is a contradiction. Hence we must have

$$(3)\qquad |M_j|=1 \qquad (j=2,\dots,k).$$

Thus, except y and z, there is no other equivalent pair of vertices in $G-x$.

Denoting $P=V(G)\setminus\{x,y,z\}$, we obviously have that $P\neq\varnothing$.

Deleting the vertex z from G we evidently get a connected subgraph. Since G is bad, we see that $G-z$ is a noncanonical graph. Since for any two vertices $r,s\in P$ we also have

$$r\not\sim s,\quad r\not\sim y \qquad (\text{in } G-z),$$

we conclude that there is a vertex $t\in P$ such that

$$(4)\qquad x\sim t \qquad (\text{in } G-z).$$

Hence $xt=0$, and we easily conclude that $yt=zt=1$. Moreover, $rx=rt$ and $ry=rz$ for any other vertex $r\in P\setminus\{t\}$.

Since $G\neq xytz=T_0$, we have that $|P|\geq 2$. If $|P|=2$, thus if $P=\{t,r\}$, we necessarily have that $rt=rx=ry=rz=1$ since $f=try$ is a canonical graph, whence we obviously get that $G-r$ is a connected and canonical graph (a contradiction). Hence, we can assume that $|P|\geq 3$. Now delete the vertex y of G.

1° Assume first that $G-y$ is a disconnected graph.

Since f is a connected graph, it is easy to see that $rx=0$ and consequently $rt=0$ for any vertex $r\neq x,y,z,t$.

Denote next the vertex t by p_1 and the vertex z by q_1.

Deleting now the vertex p_1 from G, we obviously get a connected graph $G-p_1$. Since G is bad, $G-p_1$ is a noncanonical graph. Discussing all the possible pairs of vertices in $G - p_1$ as candidates for equivalent vertices, we conclude there is a vertex $q_2 \neq x, y, p_1, q_1$ such that

$$q_1 \sim q_2 \qquad (\text{in } G - p_1). \tag{5}$$

Then $q_2 x = q_2 y = q_2 p_1 = q_2 q_1 = 0$, and $rx = rp_1 = 0$, $ry = rq_1 = rq_2$ for any vertex $r \neq x, y, p_1, q_1, q_2$. Making use of the last relation, we easily conclude that $G - q_2$ is a connected, thus a noncanonical graph. Discussing now all the possible pairs of vertices as candidates for two equivalent vertices in $G - q_2$, we conclude that there is a new vertex $p_2 \neq x, y, p_1, q_1, q_2$ such that

$$p_1 \sim p_2 \qquad (\text{in } G - q_2). \tag{6}$$

Hence $p_2 x = p_2 p_1 = 0$, $p_2 y = p_2 q_1 = p_2 q_2 = 1$, and $rp_2 = 0$ for any other vertex $r \neq x, y, p_1, q_1, p_2, q_2$.

Now, delete the vertex p_2 from G. If $G - p_2$ is disconnected, we conclude that $G = xyp_1q_1p_2q_2 = T_1$, which contradicts the assumption $G \neq T_m$ $(m = 0, 1, 2, \dots)$. Thus, $G - p_2$ must be a connected (and hence a noncanonical) graph. Therefore, we conclude that there is a new vertex q_3 such that

$$q_2 \sim q_3 \qquad (\text{in } G - p_2). \tag{7}$$

Then $q_3 x = q_3 y = q_3 p_1 = q_3 q_1 = q_3 p_2 = q_3 q_2 = 0$, and $rx = rp_1 = rp_2 = 0$, $ry = rq_1 = rq_2 = rq_3$ for any other vertex $r \neq x, y, p_1, q_1, p_2, q_2, q_3$. Deleting now the vertex q_3 from G, we conclude that $G - q_3$ is a connected (thus a noncanonical) graph. Discussing all the possible pairs of vertices in $G - q_3$ as candidates for a pair of equivalent vertices, we conclude that there is a new vertex p_3 such that

$$p_2 \sim p_3 \qquad (\text{in } G - q_3). \tag{8}$$

Then $p_3 x = p_3 p_1 = p_3 p_2 = 0$, $p_3 y = p_3 q_1 = p_3 q_2 = p_3 q_3 = 1$ and $rp_3 = 0$ for any other vertex $r \neq x, y, p_1, q_1, p_2, q_2, p_3, q_3$. Assuming that $G - p_3$ is a disconnected graph, we get the contradiction $G = xyp_1q_1p_2q_2p_3q_3 = T_2$. Hence, $G - p_3$ is a connected (thus a noncanonical) graph. Continuing this procedure, after finitely many steps, we conclude that there is a positive integer m such that

$$G = xyp_1q_1 \dots p_{m+1}q_{m+1} = T_m,$$

which is a contradiction again.

Hence, the case when $G - y$ is a disconnected graph is contradictory.

2° Now assume that $G - y$ is a connected (noncanonical) graph.

Since $G \neq T_0$, there is at least one vertex $r \neq x, y, z, t$ and we have $rx = rt$, $ry = rz$ for any such a vertex r. Since $G - y$ is a connected noncanonical graph, we may conclude that there is a new vertex $x_1 \neq x, y, z, t$ such that

$$x \sim x_1 \qquad (\text{in } G - y). \tag{9}$$

Then $x_1 x = x_1 y = x_1 z = x_1 t = 0$ and $rx = rt = rx_1$, $ry = rz$ for any vertex $r \neq x, y, z, t, x_1$.

Since G is a connected graph, we see that there is at least one vertex $r \neq x, y, z, t, x_1$. By $rx = rt = rx_1$ for any such r, we may conclude that $G - x_1$ is a connected (thus a noncanonical) graph. Hence, one may conclude that there is a new vertex $y_1 \neq x, y, z, t, x_1$ such that

$$y_1 \sim y \qquad (\text{in } G - x_1). \tag{10}$$

Therefore, we easily get $y_1x = y_1t = y_1x_1 = 1$, $y_1y = y_1z = 0$, and $rx = rt = rx_1$, $ry = rz = ry_1$ for any other vertex $r \neq x, y, z, t, x_1, y_1$.

Now we delete the vertex y_1 from G. If $G - y_1$ is a disconnected graph, we can put $x_1 = x'$, $y_1 = y'$, $y = z'$, $x = t'$, to get a contradiction, exactly as it has been done in the case 1°. Hence, we can assume that $G - y_1$ is a connected (thus a noncanonical) graph. Now, one can see that there must exist a new vertex $x_2 \neq x, y, z, t, x_1, y_1$ of G such that

$$x_2 \sim x_1 \qquad (\text{in } G - y_1). \tag{11}$$

Then $x_2x = x_2y = x_2z = x_2t = x_2x_1 = x_2y_1 = 0$, and $rx = rt = rx_1 = rx_2$, $ry = rz = ry_1$ for any vertex $r \neq x, y, z, t, x_1, y_1, x_2$. Therefore, we easily conclude that $G - x_2$ is a connected (hence a noncanonical) graph. Moreover, one can conclude that, as the only possible case, there is a new vertex $y_2 \neq x, y, z, t, x_1, y_1, x_2$ such that

$$y_2 \sim y_1 \qquad (\text{in } G - x_2). \tag{12}$$

Then $y_2x = y_2t = y_2x_1 = y_2x_2 = 1$, $y_2y = y_2z = y_2y_1 = 0$, and $rx = rt = rx_1 = rx_2$, $ry = rz = ry_1 = ry_2$ for any vertex $r \neq x, y, z, t, x_1, y_1, x_2, y_2$.

Continuing this procedure, we conclude that there is a positive integer m such that

$$G = xytzx_1y_1x_2y_2 \dots x_my_m = T_m,$$

which is again a contradiction.

This proves that case 2° is also impossible, hence our theorem is completely proved.

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