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AN IDENTITY FOR THE INDEPENDENCE POLYNOMIALS OF TREES

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Abstract. The independence polynomial $\omega(G)$ of a graph G is a polynomial whose k-th coefficient is the number of selections of k independent vertices in G. The main result of the paper is the identity:

$$\omega(T-u)\omega(T-v)-\omega(T)\omega(T-u-v)=-(-x)^{d(u,v)}\omega(T-P)\omega(T-[P])$$

where u and v are distinct vertices of a tree T, $d(u,v)$ is the distance between them and P is the path connecting them; the subgraphs $T-P$ and $T-[P]$ are obtained by deleting from T the vertices of P and the vertices of P together with their first neighbors. A conjecture of Merrifield and Simmons is proved with the help of this identity, which is also compared to some previously known analogous results.

The independence polynomial of a graph G is defined by:

$$\omega(G)=\omega(G,x)=\sum_{k=0}^{|G|}n(G,k)x^k \tag{1}$$

where $n(G,0)=1$, $n(G,1)=|G|$ is the number of vertices of G, whereas, for $k\geq 2$, $n(G,k)$ is equal to the number of ways in which k independent vertices can be selected in G. The basic properties of the independence polynomial were determined in [**3**]. Here we need the following two properties:

1° If v is a vertex of G, then

$$\omega(G)=\omega(G-v)+x\omega(G-[v]) \tag{2}$$

where $G-v$ is the subgraph obtained by deleting v from G, whereas $G-[v]$ is the subgraph obtained by deleting from G both v and the vertices adjacent to it.

2° If $G_1\cup G_2$ denotes a graph composed of disjoint graphs G_1 and G_2, then

$$\omega(G_1\cup G_2)=\omega(G_1)\omega(G_2). \tag{3}$$

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Let $G - H$ and $G - [H]$ be the graphs obtained from G by deleting respectively the vertices of a subgraph H and the vertices of H together with their first neighbors.

The main result of this paper is the following theorem:

THEOREM 1. *Let T be a tree and u and v distinct vertices of it. Let P be the (unique) path connecting u and v. Then the following identity holds*:

$$\omega(T-u)\omega(T-v) - \omega(T)\omega(T-u-v) = -(-x)^{d(u,v)}\omega(T-P)\omega(T-[P]) \qquad (4)$$

where $d(u,v)$ is the distance between u and v.

Instead of Theorem 1, we prove a somewhat more general theorem; namely, Theorem 2. To do this, we need some preparations.

An auxiliary class of graphs

Denote by P_n the path with n vertices, $n \geq 2$. Label the vertices of P_n by $v_1, v_2, \ldots, v_n$ so that v_i and v_{i+1} are adjacent, $i = 1, \ldots, n-1$. Let $R_1, R_2, \ldots, R_n$ be n distinct rooted graphs with mutually disjoint vertex sets. Then the compound graph $P_n(1,n)$ is obtained by identifying the root r_i of R_i with the vertex v_i of P_n; we do this simultaneously for $i = 1, 2, \ldots, n$ (see Fig. 1).

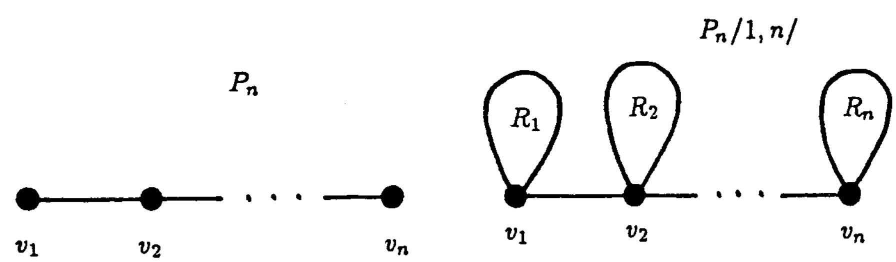

Fig. 1

The vertices v_1 and v_n of $P_n(1,n)$ are connected by a unique path, namely, P_n. As a matter of fact, every tree with a diameter not smaller than $n-1$ can be viewed as a special case of the graph $P_n(1,n)$. Then any two vertices of a tree, whose distance is $n-1$, can be considered as the vertices v_1 and v_n of an appropriately chosen graph $P_n(1,n)$. Bearing this in mind, it is evident that Theorem 1 is a special case of the following theorem:

THEOREM 2. *It $n \geq 2$, then*:

$$\omega(P_n(1,n) - v_1)\omega(P_n(1,n) - v_n) - \omega(P_n(1,n))\omega(P_n(1,n) - v_1 - v_n)$$
$$= (-x)^n \prod_{i=1}^{n} \omega(R_i - r_i)\omega(R_i - [r_i]). \qquad (5)$$

Note that because of (3)

$$\prod_{i=1}^{n} \omega(R_i - r_i) = \omega(P_n(1,n) - P_n), \qquad \prod_{i=1}^{n} \omega(R_i - [r_i]) = \omega(P_n(1,n) - [P_n]).$$

In accordance with the notation just introduced, we have:

$$P_n(1,n) - v_1 = P_{n-1}(2,n) \cup R_1 - r_1 \tag{6}$$
$$P_n(1,n) - v_n = P_{n-1}(1,n-1) \cup R_n - r_n \tag{7}$$
$$P_n(1,n) - v_1 - v_n = P_{n-2}(2,n-1) \cup R_1 - r_1 \cup R_n - r_n. \tag{8}$$

Before proceeding with the proof of formula (5), we consider the special case when all the rooted graphs R_i, $i = 1, 2, \ldots, n$, are mutually isomorphic. Then, by applying (2) to the vertex v_n of $P_n(1,n)$, one arrives at the recurrence relation:

$$\omega(P_n(1,n)) = \omega(R-r)\omega(P_{n-1}(1,n-1)) + x\omega(R-r)\omega(R-[r])\omega(P_{n-2}(1,n-2)). \tag{9}$$

The solution of (9) reads:

$$\begin{aligned}\omega(P_n(1,n),x) &= (2B)^{-1}[(A+B)^{n+1} - (A-B)^{n+1}] \\ &\quad + x\omega(R-[r])(2B)^{-1}[(A+B)^n - (A-B)^n]\end{aligned} \tag{10}$$

where

$$A = \frac{1}{2}\omega(R-r), \qquad B = \left[x\omega(R-r)\omega(R-[r]) + \frac{1}{4}\omega(R-[r])^2\right]^{1/2}.$$

A special case of formula (10) for $x = 1$ and $R = P_2$ was reported previously in [**5**].

Proof of Theorem 2

We prove Theorem 2 by induction on the number of vertices of the path P_n.

For $n = 2$ the validity of formula (5) is checked by direct application of the relations (2) and (3) to the vertices v_1 and v_2 of $P_2(1,2)$, $P_2(1,2) - v_1$ and $P_2(1,2) - v_2$ and by noting that $P_2(1,2) - v_1 - v_2 = R_1 - r_1 \cup R_2 - r_2$.

Suppose now that the identity (5) holds for $n = m$. By using this assumption, we have to deduce that formula (5) is satisfied also for $n = m+1$. Applying (2) and (3) to the vertex v_{m+1}, and having (6)–(8) in mind, we obtain:

$$\begin{aligned}\omega(P_{m+1}(1,m+1)) &= \omega(R_{m+1} - r_{m+1})\omega(P_m(1,m)) \\ &\quad + x\omega(R_m - r_m)\omega(R_{m+1} - [r_{m+1}])\omega(P_{m-1}(1,m-1)), \\ \omega(P_{m+1}(1,m+1) - v_1) &= \omega(R_{m+1} - r_{m+1})\omega(P_m(1,m) - v_1) \\ &\quad + x\omega(R_m - r_m)\omega(R_{m+1} - [r_{m+1}])\omega(P_{m-1}(1,m-1) - v_1)\end{aligned}$$

which together with:

$$\omega(P_{m+1}(1,m+1)-v_{m+1}) = \omega(R_{m+1}-r_{m+1})\omega(P_m(1,m)),$$
$$\omega(P_{m+1}(1,m+1)-v_1-v_{m+1}) = \omega(R_{m+1}-r_{m+1})\omega(P_m(1,m)-v_1)$$

yields:

$$\begin{aligned}
&\omega(P_{m+1}(1,m+1)-v_1)\omega(P_{m+1}(1,m+1)-v_{m+1})\\
&-\omega(P_{m+1}(1,m+1))\omega(P_{m+1}(1,m+1)-v_1-v_{m+1})\\
&= -x\omega(R_m-r_m)\omega(R_{m+1}-r_{m+1})\omega(R_{m+1}-[r_{m+1}])\times\\
&\big\{\omega(P_m(1,m)-v_1)\omega(P_{m-1}(1,m-1))-\omega(P_m(1,m))\omega(P_{m-1}(1,m-1)-v_1)\big\}.
\end{aligned} \tag{11}$$

Because of (6)–(8), the right-hand side of (11) is equal to:

$$\begin{aligned}
&-x\omega(R_{m+1}-r_{m+1})\omega(R_{m+1}-[r_{m+1}])\times\\
&\big\{\omega(P_m(1,m)-v_1)\omega(P_m(1,m)-v_m)-\omega(P_m(1,m))\omega(P_m(1,m)-v_1-v_m)\big\}
\end{aligned}$$

which by the induction hypothesis becomes:

$$-x\omega(R_{m+1}-r_{m+1})\omega(R_{m+1}-[r_{m+1}])\left[(-x)^m\prod_{i=1}^{m}\omega(R_i-r_i)\omega(R_i-[r_i])\right].$$

Thence, (11) is transformed into the form:

$$\begin{aligned}
&\omega(P_{m+1}(1,m+1)-v_1)\omega(P_{m+1}(1,m+1)-v_{m+1})\\
&-\omega(P_{m+1}(1,m+1))\omega(P_{m+1}(1,m+1)-v_1-v_{m+1})\\
&= (-x)^{m+1}\prod_{i=1}^{m+1}\omega(R_i-r_i)\omega(R_i-[r_i])
\end{aligned}$$

which is sufficient for the proof of Theorem 2. ■

Discussion

Identities having forms similar to (4) are known for some other graph polynomials ([**1**], [**2**]). It is especially worth mentioning the following two from [**1**]:

$$\phi(G-u)\phi(G-v)-\phi(G)\phi(G-u-v) = \left[\sum_P \phi(G-P)\right]^2$$
$$\alpha(G-u)\alpha(G-v)-\alpha(G)\alpha(G-u-v) = \sum_P \big[\alpha(G-P)\big]^2$$

where ϕ and α stand respectively for the characteristic and the matching polynomial. In the expressions above, G denotes an arbitrary graph and the summations go over all paths P connecting the vertices u and v. These formulas lead to an obvious generalization of (4), namely:

$$\omega(G-u)\omega(G-v)-\omega(G)\omega(G-u-v) = \sum_P (-x)^{|P|}\omega(G-P)\omega(G-[P]). \tag{12}$$

Unfortunately, (12) turns out to be false already for unicyclic graphs. At this time, we are unable to propose an extension of identity (4) for cyclic graphs, even as a conjecture. So, we leave this problem for the future.

For $x = 1$, the independence polynomial (1) becomes equal to the number of independent-vertex sets of G. This quantity, denoted by $\sigma(G)$, was extensively studied in connection with certain topological problems of chemistry [**4**]. On page 144 of [**4**], a property of $\sigma(G)$ is stated without proof, which for nonadjacent vertices u and v can be formulated as follows:

$$\sigma(G-u)\sigma(G-v) - \sigma(G)\sigma(G-u-v)\begin{cases} > 0, & \text{if } d(u,v) \text{ is odd,} \\ < 0, & \text{if } d(u,v) \text{ is even.} \end{cases}$$

Our Theorem 1 shows that this assertion is true at least for all trees.

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