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THE NEG.-PROPOSITIONAL CALCULUS

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Abstract. Consistency and completeness are proved for an axiomatic system intended to be a formalization of propositional contradictions.

By changing in the truth tables the values of the logical operations of conjunction, disjunction and implication (but not of negation $\sim$), so as to obtain new functions designated by ε, υ, and $\rightarrow$, respectively:

ε	$\top$	$\perp$
$\top$	$\perp$	$\top$
$\perp$	$\top$	$\top$

υ	$\top$	$\perp$
$\top$	$\perp$	$\perp$
$\perp$	$\perp$	$\top$

$\rightarrow$	$\top$	$\perp$
$\top$	$\perp$	$\top$
$\perp$	$\perp$	$\perp$

and by taking an interest in formulas with the truth value $\perp$, we may construct an *algebra of contradictions* (identically *false* formulas) $\mathfrak{A} = \{\langle \top, \perp \rangle; \varepsilon, \upsilon, \rightarrow, \sim\}$, which could serve as the principal model of a propositional calculus called *neg.-propositional calculus*.

This formal system may be axiomatized with the following *axiom-schemata*:

1. $(A \rightarrow B) \rightarrow A$
2. $((A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow B)) \rightarrow (C \rightarrow B)$
3. $A \rightarrow \sim(A \;\varepsilon\; A)$
4. $((A \;\varepsilon\; B) \rightarrow \sim B) \rightarrow \sim A$

5*a*. $\sim A \rightarrow A \;\varepsilon\; B$

5*b*. $\sim B \rightarrow A \;\varepsilon\; B$

6. $((\sim C \rightarrow A \;\upsilon\; B) \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)$

7*a*. $A \;\upsilon\; B \rightarrow \sim A$

7*b*. $A \;\upsilon\; B \rightarrow \sim B$

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$$8. \quad (\sim B \to (A \to B)) \to (\sim A \to B)$$

and *the rule of inference*

$$\langle I \rangle \quad \frac{A \to B,\ B}{A}$$

We use $A, B, C, \dots, A_1, \dots,$ as schematic letters for formulas, and Γ, Γ_1, as letters for lists of formulas.

We can easily prove the following:

LEMMA 1. (i) $A \vdash A$; (ii) *if* $\Gamma \vdash A$, *then* $B, \Gamma \vdash A$; (iii) *if* $C, C, \Gamma \vdash A$, *then* $C, \Gamma \vdash A$; (iv) *if* $\Gamma_1, C, D, \Gamma_2 \vdash A$, *then* $\Gamma_1, D, C, \Gamma_2 \vdash A$; (v) *if* $\Gamma \vdash A_1, \dots, \Gamma \vdash A_n$ *and* $A_1, \dots, A_n \vdash B$, *then* $\Gamma \vdash B$, *or in particular*: *if* $\vdash A_1, \dots, \vdash A_n$, *and* $A_1, \dots, A_n \vdash B$, *then* $\vdash B$.

THEOREM 1. *If* $B_2, \dots, B_m \vdash A \to B_1$, *then* $B_1, B_2, \dots, B_m \vdash A$.

Proof. By adding, according to Lemma 1, the formula B_1 to the sequence $B_2, \dots, B_m$, by the rule of inference $\langle I \rangle$ we obtain A.

COROLLARY 1. *If* $\vdash A \to B$, *then* $B \vdash A$.

COROLLARY 2. *If* $\vdash (\dots(A \to B_1) \to \dots) \to B_m$, *then* $B_1, \dots, B_m \vdash A$.

THEOREM 2 (Deduction theorem). *If* $\Gamma, B \vdash A$, *then* $\Gamma \vdash A \to B$.

Proof. By induction on the length k of the given deduction.

1° When $k = 1$, the formula A is either an axiom or one of formulas Γ or B. In the first case, such a deduction is

1.	A	axiom-hypothesis
2.	$(A \to B) \to A$	axiom-schema 1
3.	$A \to B$	$\langle I \rangle$ 1, 2

and analogously in the remaining two cases.

2° Assume the theorem holds for all lengths $\leq k$, and let the length of the given deduction be $k + 1$.

Besides the three cases already discussed, an additional case arises here: A is an immediate consequence, by the rule $\langle I \rangle$, of two preceding formulas of the sequence.

Let it be the formulas P (at the r-th, $r \leq k$), and $A \to P$ (at the s-th, $s \leq k$) step. In both cases, by the hypothesis, it is possible to construct two deductions of $P \to B$ and $(A \to P) \to B$. Let us add to them the following instance of axiom-schema 2:

$$((A \to B) \to ((A \to P) \to B)) \to (P \to B)$$

Then a double application of the rule of inference yields $A \to B$.

COROLLARY 1. *If* $B \vdash A$*, then* $\vdash A \to B$*.*

COROLLARY 2. *If* $B_1, \dots, B_{m-1}, B_m \vdash A$*, then* $\vdash (\dots(A \to B_1) \to \dots \to B_{m-1}) \to B_m$*.*

We derive the introduction and the elimination rules for connectives in the theorem after the following lemma:

LEMMA 1. (a) *If* $B \vdash A$ *and* $B \vdash {\sim}A$*, then* $\vdash {\sim}B$; (b) *if* $B \vdash A$*, then* ${\sim}A \vdash {\sim}B$; (c1) ${\sim}{\sim}A \vdash A$; (c2) $A \vdash {\sim}{\sim}A$.

THEOREM 3.

	Introduction	*Elimination*
$\to$	*If* $\Gamma, B \vdash A$, *then* $\Gamma \vdash A \to B$	$A \to B,\ B \vdash A$
ε	${\sim}A,\ {\sim}B \vdash A \,\varepsilon\, B$	$A \,\varepsilon\, B \vdash {\sim}A$ $A \,\varepsilon\, B \vdash {\sim}B$
υ	${\sim}A \vdash A \,\upsilon\, B$ ${\sim}B \vdash A \,\upsilon\, B$	*If* $\Gamma, C \vdash A$ *and* $\Gamma, C \vdash B$, *then* $\Gamma, A \,\upsilon\, B \vdash {\sim}C$
$\sim$	*If* $\Gamma, B \vdash A$ *and* $\Gamma, B \to {\sim}A$, *then* $\Gamma \vdash {\sim}B$	${\sim}{\sim}A \vdash A$

Proof. The rule of $\to$-introduction is just the deduction theorem, and $\to$-elimination is our rule of inference $\langle I \rangle$. For the remaining rules we use the axiom-schema 4 for ε introduction, 5a and 5b for ε-elimination, 7a and 7b for υ-introduction, 6 for υ-elimination, Lemma 2 (a) for $\sim$-introduction, and finally Lemma 2 (c1) for $\sim$-elimination.

Can this formal system describe an intuitive domain of bivalent propositions? To answer this question, we define as follows the consistency and completeness of a formal system:

Definition 1. (i) The neg.-propositional calculus is *semantically consistent* if every provable formula, interpreted on the model of the calculus — the algebra $\mathfrak{A}$ — is a contradiction.

(ii) The neg.-propositional calculus is *simply consistent* if there is an A such that $\vdash A$ and $\vdash {\sim}A$.

(iii) The neg.-propositional calculus is *syntactically consistent* if a formula is unprovable in it.

THEOREM 4. *The neg.-propositional calculus is* (*semantically, simple, syntactically*) *consistent.*

Proof. On the indicated model, the truth value of all axioms is $\perp$ and the rule of inference keeps this property (semantical consistency). The value $\perp$ can not at the same time belong to a formula and to its negation (simple consistency), and, so, as an example of an unprovable formula we have $A \,\varepsilon \sim A$ (syntactical consistency).

Definition 2. (i) The neg.-propositional calculus is *semantically complete* if every contradiction is provable in it.

(ii) The neg.-propositional calculus is *simply complete* if for every A either $\vdash A$ or $\vdash \sim A$.

(iii) The neg.-propositional calculus is *syntactically complete* if it has the following property: if one adds to the axioms an unprovable formula, one can prove every formula.

Completeness in the first sense implies completeness in other two senses. Then we prove the following:

THEOREM 5. *The neg.-propositional calculus is semantically complete.*

Let us assign some n-tuple of values $\top$, $\perp$, to the propositional letters $A_1, \dots, A_n$ that occur in the formula F. Then by $B_1, \dots, B_n$ we denote a corrected n-tuple, where B_i $(1 \le i \le n)$ is A_i or $\sim A_i$, according as the assigned value is $\perp$ or $\top$.

LEMMA 3. $B_1, \dots, B_n \vdash F$ *if* $\tau(F) = \perp$ *and* $B_1, \dots, B_n \vdash \sim F$ *if* $\tau(F) = \top$.

Proof. Let us denote by d the degree of the formula F, i.e. the number of connectives in F.

Basis $(d = 0)$. Since F is a propositional letter A_i, for $i \in \{1, \dots, n\}$, we have $A_i \vdash A_i$ if $\tau(A_i) = \perp$, or $\sim A_i \vdash \sim A_i$ if $\tau(A_i) = \top$.

Induction step. If the degree of F is $k + 1$, then F has one of the forms: (a) $\sim A$, (b) $A \to B$, (c) $A \,\upsilon\, B$ or (d) $A \,\varepsilon\, B$ with A, B of degree $\le n$.

(a) We must show: (a1) $B_1, \dots, B_n \vdash \sim F$ if $B_1, \dots, B_n \vdash A$ and (a2) $B_1, \dots, B_n \vdash F$ if $B_1, \dots, B_n \vdash \sim A$, or (a1) $A \vdash \sim\sim A$ and (a2) $\sim A \vdash \sim A$. The proof is carried out by Lemma 1.

(b) If F has the form $A \to B$, four subcases arise:

(b1) If $B_1, \dots, B_n \vdash \sim A$ and $B_1, \dots, B_n \vdash \sim B$, then $B_1, \dots, B_n \vdash A \to B$ (and by Lemma 1, it is sufficient to prove $\sim A, \sim B \vdash A \to B$).

(b2) If $B_1, \dots, B_n \vdash \sim A$ and $B_1, \dots, B_n \vdash B$, then $B_1, \dots, B_n \vdash \sim(A \to B)$ $(\sim A, B \vdash \sim(A \to B))$.

(b3) If $B_1, \dots, B_n \vdash A$ and $B_1, \dots, B_n \vdash \sim B$, then $B_1, \dots, B_n \vdash A \to B$ $(A, \sim B \vdash A \to B)$;

(b4) If $B_1, \dots, B_n \vdash A$ and $B_1, \dots, B_n \vdash B$, then $B_1, \dots, B_n \vdash A \to B$ $(A, B \vdash A \to B)$;

SUBLEMMA 1. $A, \sim A \vdash B$.

(b1)	1.	$B, \sim B \vdash A$	Sublemma 1
	2.	$\sim B \vdash A \to B$	$\to$-in. 1
	3.	$\sim A, \sim B \vdash A \to B$	Lemma 1, 2
(b2)	1.	$A \to B,\ B \vdash A$	$\to$-el.
	2.	$A \to B,\ B \vdash B$	Lemma 1
	3.	$A\ \upsilon\ B,\ B \vdash \sim(A \to B)$	υ-el. 1, 2
	4.	$\sim A \vdash A\ \upsilon\ B$	υ-in.
	5.	$\sim A, B \vdash A\ \upsilon\ B$	Lemma 1, 4
	6.	$\sim A, B \vdash B$	Lemma 1
	7.	$\sim A, B \vdash \sim(A \to B)$	Lemma 1, 3, 5, 6

For (b3) and (b4) we proceed similarly.

For (c1), (c2), (c3) we use υ-in.

(c4)	1.	$A \to B,\ B \vdash A$	$\to$-el.
	2.	$A \to B,\ B \vdash B$	Lemma 1
	3.	$B, A\ \upsilon\ B \vdash \sim(A\ \upsilon\ B)$	υ-el. 1,2
	4.	$B, A \to B \vdash \sim(A\ \upsilon\ B)$	Lemma 2, 3
	5.	$A, B \vdash B$	Lemma 1
	6.	$A, B \vdash A \to B$	(b4)
	7.	$A, B \vdash \sim(A\ \upsilon\ B)$	Lemma 1, 5, 6, 3

For (d1) we use ε-in.

SUBLEMMA 2. $\sim A\ \upsilon \sim B \vdash \sim(A\ \varepsilon\ B)$.

(In the proof we use ε-el. and υ-el.)

(d2)	1.	$\sim B \vdash A\ \upsilon\ B$	υ-in.
	2.	$B \vdash \sim A\ \upsilon \sim B$	$A \vdash A,\ B \vdash B,\ 1$
	3.	$\sim A\ \upsilon \sim B \vdash \sim(A\ \varepsilon\ B)$	Sublemma 2
	4.	$B \vdash \sim(A\ \varepsilon\ B)$	Lemma 1, 2, 3
	5.	$\sim A, B \vdash \sim(A\ \varepsilon\ B)$	Lemma 1, 4

Case (d3) is similar to (d2). For (d4) we use (d3).

LEMMA 4. *If for any of 2^n possible sequences of values we always have $B_1, \dots, B_n \vdash F$, then*

$$A_1\ \upsilon \sim A_1, \dots, A_n\ \upsilon \sim A_n \vdash F.$$

Proof. Let us take $n = 2$. By hypothesis, $\sim A_1, \sim A_2 \vdash F$; $\sim A_1, A_2 \vdash F$; $A_1, \sim A_2 \vdash F$ and $A_1, A_2 \vdash F$. So, by a triple application of υ-elimination (using Lemma 2), we obtain:

1.	$\sim A_1, \sim A_2 \vdash F$	hyp.
2.	$\sim A_1, A_2 \vdash F$	hyp.
3.	$\sim A_1, \sim F \vdash \sim\sim A_1 \vdash A_1$	Lemma 2, 1
4.	$\sim A_1, \sim F \vdash \sim A_2$	Lemma 2, 2
5.	$\sim A_1, A_2 \ \upsilon \sim A_2 \vdash F$	υ-el. 3, 4
6.	$A_1, \sim A_2 \vdash F$	hyp.
7.	$A_1, A_2 \vdash F$	hyp.
8.	$A_1, \sim F \vdash A_2$	Lemma 2, 6
9.	$A_1, \sim F \vdash \sim A_2$	Lemma 2, 7
10.	$A_1, A_2 \ \upsilon \sim A_2 \vdash F$	υ-el. 8, 9
11.	$A_1 \ \upsilon \sim A_1, A_2 \ \upsilon \sim A_2 \vdash F$	υ-el. 5, 10

Proof of Theorem 5. Let F be a contradiction and $A_1, \dots, A_n$ all of its letters. By the last lemma:

$$A_1 \ \upsilon \sim A_1, \dots, A_n \ \upsilon \sim A_n \vdash F$$

and, since we have $\vdash A \ \upsilon \sim A$ by Lemma 1, we should finally have $\vdash F$.

This proof follows the method exposed in [**1**].

REFERENCES

[**1**] S. C. Kleene, *Introduction to Metamathematics*, North-Holland, Amsterdam, 1952.

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