# SOME THEOREMS CONCERNING RIESZ'S FIRST MEAN

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**1.** This paper is a study, from a particular point of view, of certain theorems involving the first Riesz's mean. The point of view finds expression in two of my previous papers[1] in which I have shown that all the criteria for the absolute convergence of $\Sigma a_n$ issue from.

Theorem A. *If $\{\lambda_n\}$ is a strictly increasing divergent sequence with $\lambda_{n+1}-\lambda_n=O(1)$ and if*

(a) $$\frac{|a_n|}{\lambda_{n+1}-\lambda_n}\leq F(\lambda_{n+1}),$$

*where $F(x)$ is positive monotone decreasing, then $\Sigma a_n$ is absolutely convergent provided $\int^{\infty} F(x)\,dx$ is convergent;*

and its modification, the generalized Brink convergence test:

Theorem B. *If $\{\lambda_n\}$ is as in Theorem (A) and if*

(b) $$\frac{1}{\lambda_{n+1}-\lambda_n}\log\frac{|a_{n+1}|(\lambda_{n+1}-\lambda_n)}{|a_n|(\lambda_{n+2}-\lambda_{n+1})}\leq g(\lambda_{n+1}),$$

*where $g(x)$ has a continuous derivative $g'(x)$ such that $\int^{\infty}|g'(x)|\,dx$ is convergent, then $\Sigma a_n$ is absolutely convergent provided $\int^{\infty}\exp\left[\int^{x}g(\xi)\,d\xi\right]dx$ is convergent.*

A further twofold enquiry may be pursued. 1° If $\Sigma a_n$ belongs to the special class of series summable $R(\lambda_n, 1)$, how

(1) Rajagopal: [6], [7].

may we relax the restriction of convergence on the „test integrals" in order that hypothesis (a) or (b) shall still ensure the convergence of $\Sigma a_n$? 2° If $\Sigma a_n$ belongs to the wider class of series bounded $R(\lambda_n, 1)$, how may we relate its limits of oscillation to the „test integrals" under the same hypotheses? It is suggested here that the classic theorems relating to Riesz's first mean fall naturally within the scope of this enquiry.

Notation. $\{\lambda_n\}$ is a sequence such that

$$0 < \lambda_0 < \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n \to \infty \qquad (n \to \infty).$$

$\sum_{n=0}^{\infty} a_n$ denotes a real series; $s_n = \sum_{\nu=0}^{n} a_\nu$. $\sigma_n$ is the first Riesz's mean of $\{s_n\}$ with respect to $\{\lambda_n\}$; i.e.,

$$\lambda_{n+1}\sigma_n = \sum_{\nu=0}^{n} (\lambda_{\nu+1} - \lambda_\nu) s_\nu .$$

$$\frac{\overline{s}}{\underline{s}} = \lim_{n=\infty} \frac{\text{sup.}}{\text{inf.}} s_n, \qquad \frac{\overline{\sigma}}{\underline{\sigma}} = \lim_{n=\infty} \frac{\text{sup.}}{\text{inf.}} \sigma_n .$$

**2.** The results in this section are the analogues of Theorems (A) and (B) for series summable $R(\lambda_n, 1)$. The main argument of the section runs on the lines of a simple and direct proof given by N. Higaki[2], of the Hardy-Landau convergence theorem for such summable series.

Theorem 1. *Let $\lambda_{n+1} \sim \lambda_n$. Let $\Sigma a_n$ satisfy the conditions:*

(1) $$\sigma_n = s + o(1),$$

(2a) $$\frac{a_n}{\lambda_{n+1} - \lambda_n} \leq F(\lambda_{n+1}),$$

*where $F(x)$ is monotone decreasing in every interval $(\lambda_n, \lambda_{n+1})$ and $\int^x F(\xi)\,d\xi$ is a slowly decreasing function in the sense that*

(3a) $$\limsup_{x=\infty} \int_x^{x'} F(\xi)\,d\xi \leq 0,$$

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(2) Higaki: [2], 74—75.

*for any* $x' > x$ *such that* $x' \sim x$. *Then*

(4) $$s_n = s + o(1).$$

Proof. If $m > q$,

$$\frac{\lambda_{m+1}}{\lambda_{q+1}}\sigma_m - \sigma_q = \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right)s_q + \left[\sum_{\nu=q+1}^{m}(\lambda_{\nu+1} - \lambda_\nu)(s_\nu - s_q)\right] / \lambda_{q+1}$$

(5) $$\leq \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) s_q + \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) \operatorname*{Max}_{q+1 \leq \nu \leq m} (s_\nu - s_q).$$

Summing (2a) for $n = q+1, q+2, \ldots \nu$, we obtain

$$s_\nu - s_q \leq (\lambda_{q-2} - \lambda_{q+1})F(\lambda_{q+2}) + \cdots + (\lambda_{\nu+1} - \lambda_\nu)F(\lambda_{\nu+1})$$

$$\leq \int_{\lambda_{q+1}}^{\lambda_{\nu+1}} F(x)\,dx$$

Hence (5) gives

(6) $$\frac{\lambda_{m+1}}{\lambda_{q+1}}\sigma_m - \sigma_q \leq \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) s_q + \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) \operatorname*{Max}_{q+1 \leq \nu \leq m} \int_{\lambda_{q+1}}^{\lambda_{\nu+1}} F(x)\,dx.$$

Let $q$ be first chosen from an increasing sequence of integers tending to $\infty$ so that $\lim s_q = \underline{s}$. Let $m$ be then chosen from another sequence, so related to the first that for an associated $q$ and $m$

$\lim \frac{\lambda_{m+1}}{\lambda_{q+1}} = \lambda - 0\ (> 1)$. Then making $q \to \infty$ in (6), we obtain

$$(\lambda - 1)s \leq (\lambda - 1)\underline{s} + (\lambda - 1) \limsup \operatorname*{Max}_{q+1 \leq \nu \leq m} \int_{\lambda_{q+1}}^{\lambda_{\nu+1}} F(x)\,dx.$$

Removing the factor $\lambda - 1$ and letting $\lambda \to 1 + 0$, we find, in consequence of (3a)

$$s \leq \underline{s} + \epsilon.$$

$\epsilon > 0$ being arbitrary, it follows that

(7) $$s \leq \underline{s}.$$

Next, $q$ being $< m$,

$$\frac{\lambda_{q+1}}{\lambda_{m+1}}\sigma_q - \sigma_m = -\left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) s_m + \left[\sum_{\nu=q+1}^{m} (\lambda_{\nu+1} - \lambda_\nu)(s_m - s_\nu)\right] / \lambda_{m+1}$$

(8) $$\leqq -\left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) s_m + \left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) \operatorname{Max}_{q+1 \leqq \nu \leqq m} (s_m - s_\nu)$$

(9) $$\leqq -\left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) s_m + \left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) \operatorname{Max}_{q+1 \leqq \nu \leqq m} \int_{\lambda_{\nu+1}}^{\lambda_{m+1}} F(x)\,dx.$$

Suppose now that $m$, $q$ are members of increasing divergent sequences of integers chosen (in the order mentioned) so that $\lim s_m = \bar{s}$ and $\lim \frac{\lambda_{q+1}}{\lambda_{m+1}} = \theta + 0\,(< 1)$. Then, from (9), letting $m \longrightarrow \infty$ first and then $\theta \longrightarrow 1 - 0$ (after removing the factor $1 - \theta$), we get, as a result of (3a),

$$-s \leqq -\bar{s} + \epsilon$$

and hence

(10) $$\bar{s} \leqq s$$

(4) now follows from (7) and (10).

Corollary 1.1. If $F(x) = \frac{H}{x}$, the convergence criterion for series $\Sigma a_n$, summable $R(\lambda_n, 1)$, assumes the form:

$$\frac{a_n}{\lambda_{n+1} - \lambda_n} \leqq \frac{H}{\lambda_{n+1}}. \qquad \text{[Hardy-Landau]}$$

If $F(x) = \frac{a_n}{\lambda_{n+1} - \lambda_n}$ for $\lambda_n < x \leqq \lambda_{n+1}$ $(n = 0, 1, 2, \ldots)$, the criterion assumes the form:

$$\limsup_{n=\infty} (s_{n'} - s_n) \leqq 0$$

for $n' > n$ such that $\lambda_{n'} \sim \lambda_n$ as $n \longrightarrow \infty$. [Schmidt-Karamata][3])

In Theorem 1, (2a) can of course be replaced by

$$\frac{a_n}{\lambda_{n+1} - \lambda_n} \geqq f(\lambda_{n+1}),$$

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(3) Karamata: [4], 33.

where $f(x)$ is monotone increasing in every interval $(\lambda_n, \lambda_{n+1})$ and $\int^x f(\xi)d\xi$ is a slowly decreasing function. The new form of the theorem gives the „left-handed" convergence criteria corresponding to the above.

Corollary 1.2. Theorem 1 can be readily modified so as to bring out its relation to Theorem B. The modification will run as follows:

*If, in Theorem B, we assume the summability $R(\lambda_n, 1)$ of $\Sigma a_n$ and, instead of the convergence of the test integral, the slow increase of $\int^x \exp\left[\int^x g(\xi)d\xi\right]dx$, the convergence of $\Sigma a_n$ follows.*

To establish this, we observe that (b) leads to

$$|a_n| \leqq C \int_{\lambda_n}^{\lambda_{n+1}} \exp\left[\int^x g(\xi)\,d\xi\right] dx \text{ [4]}$$

where $C$ is a constant depending on the lower limit in $\int^x g(\xi)d\xi$, and then proceed on the lines of the proof of Theorem 1.

**3.** The oscillation theorems stated below as lemmas are due in substance to Fejér.[5] They involve two functions, one based on the idea of increase, and the other on the idea of decrease, of a sequence $\{s_n\}$ with respect to the sequence $\{\lambda_n\}$ in Theorem 1. The first function may be defined by

$$\limsup_{n=\infty} \max_{n\leqq\nu\leqq H-1} (s_\nu - s_{n-1}) \leqq W(t) \qquad \text{(say)}$$

for $t>1$, $\lambda_N \leqq t\lambda_n < \lambda_{N+1}$; the second function being

$$-\liminf_{n=\infty} \min_{n\leqq\nu\leqq N-1} (s_\nu - s_{n-1}) \leqq w(t) \qquad \text{(say),}$$

with the same restrictions on $t$, $N$,

The arguments of the preceding section show that if $\bar{\sigma} = \underline{\sigma} = s$, then $\lim_{t=1+0} W(t) = 0$ or $\lim_{t=1+0} w(t) = 0$ implies $\bar{s} = \underline{s} = s$.

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(4) The argument is as in Rajagopal: [7], 119.

(5) Fekete and Winn: [1], 490; Karamata: [3], 20.

They also suggest that if $\overline{\sigma} \neq \underline{\sigma}$, we can obtain the following relations between $\overline{\sigma}$, $\underline{\sigma}$, $\overline{s}$ or $\underline{s}$ and $W(t)$, $w(t)$.

Lemma 1. *Let $\lambda_{n+1} \sim \lambda_n$. Let $W(t)$ be defined as above. Then*

$$\lambda\underline{\sigma} - \overline{\sigma} \leqq (\lambda-1)\underline{s} + (\lambda-1)\, W(\lambda) \qquad (\lambda > 1), \tag{11}$$

$$\theta\underline{\sigma} - \overline{\sigma} \leqq -(1-\theta)\overline{s} + (1-\theta)\, W\left(\frac{1}{\theta}\right) \qquad (0<\theta<1); \tag{12}$$

and, if $\theta = \frac{1}{\lambda}$, from (11) and (12),

$$\overline{s} - \underline{s} \leqq \frac{\lambda+1}{\lambda-1}\,(\overline{\sigma} - \underline{\sigma}) + 2\, W(\lambda). \qquad \text{[Fejér.]}$$

Proof. From (5), with the choice of $q$, $m$ in the first part of the proof of Theorem 1, we get

$$\frac{\lambda_{m+1}}{\lambda_{q+1}}\,\sigma_m - \sigma_q \leqq \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) s_q + \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) W(\lambda) + o(1) \quad (q \to \infty).$$

Proceeding to the limit as $q \to \infty$, we obtain (11) from the last inequality.

Again, from (8) with the choice of $m$, $q$ in the latter part of the Theorem 1, we have

$$\frac{\lambda_{q+1}}{\lambda_{m+1}}\,\sigma_q - \sigma_m \leqq -\left(1 - \frac{\lambda_{q+1}}{\lambda_{m-1}}\right) s_m + \left(1 - \frac{\lambda_{q+1}}{\lambda_{m+1}}\right) W\left(\frac{1}{\theta}\right) + o(1) \quad (m \to \infty)$$

and when $m \to \infty$, we are led to (12).

The above lemma involves $W(t)$, i. e., a „right-handed" condition. It can be converted into a result involving $w(t)$, i. e., a „left-handed" condition, by restating it in terms of $\{s_n'\} \equiv \{-s_n\}$. It can also be modified so as to give the following double-conditioned result.

Lemma 2. *Let $\lambda_{n+1} \sim \lambda_n$. Let $W(t)$, $w(t)$ be as already defined. Then for $0 < \theta < 1 < \lambda$,*

$$\lambda\overline{\sigma} - \theta\underline{\sigma} \geqq (\lambda-\theta)\overline{s} - (\lambda-1)\, w(\lambda) - (1-\theta)\, W\left(\frac{1}{\theta}\right). \tag{13}$$

$$\theta\overline{\sigma} - \lambda\underline{\sigma} \geqq -(\lambda-\theta)\underline{s} - (\lambda-1)\, W(\lambda) - (1-\theta)\, w\left(\frac{1}{\theta}\right). \tag{14}$$

Proof. If $q<m<p$,

$$\frac{\lambda_{p+1}}{\lambda_{m+1}}\sigma_p-\frac{\lambda_{q+1}}{\lambda_{m+1}}\sigma_q=\left(\frac{\lambda_{p+1}}{\lambda_{m+1}}-\frac{\lambda_{q+1}}{\lambda_{m+1}}\right)s_m+ \tag{15}$$

$$+\left[\sum_{\nu=m+1}^{p}(\lambda_{\nu+1}-\lambda_\nu)(s_\nu-s_m)\right]/\lambda_{m+1}$$

$$-\left[\sum_{\nu=q+1}^{m}(\lambda_{\nu+1}-\lambda_\nu)(s_m-s_\nu)\right]/\lambda_{m+1}.$$

Let $m$ be a member of an increasing sequence of integers tending to $\infty$ so that $\lim s_m=\bar{s}$. Also, let $p$, $q$ be membres of two other increasing sequences, one $p$ and one $q$ being associated with every $m$ so that $\lim\frac{\lambda_{p+1}}{\lambda_{m+1}}=\lambda-0$, $\lim\frac{\lambda_{q+1}}{\lambda_{m+1}}=\theta+0$. Then (15) yields (13) in the limit as $m\to\infty$.

(14) is derived from (13) by considering $\{-s_n\}$ instead of $\{s_n\}$.

We can deduce from Lemma 1 the following result which admits of a generalization involving the $k$-th and $(k+1)$-th Riesz means $(k=1,2,3,\ldots)$.[6])

*If, in Lemma 1,* $s_n=\sum_{\nu=0}^{n}a_n$ *and* $\frac{a_n}{\lambda_{n+1}-\lambda_n}\leq\frac{H}{\lambda_{n+1}}$, *then*

$$\frac{(\sigma-\underline{s})^2}{4H}\leq\bar{\sigma}-\underline{\sigma},\qquad \frac{(\bar{s}-\bar{\sigma})^2}{4H}\leq\bar{\sigma}-\underline{\sigma}.$$

But these relations are merely crude forms of the Fekete-Winn inequalities (19), (20) contained in one of the theorems of the next section.

4. These theorems furnish an answer to the second question raised at the outset. They together with Theorem 1 form a group headed by Theorem A.

Theorem 2. *Let* $\lambda_{n+1}\sim\lambda_n$. *Let* $\Sigma a_n$ *be bounded* $R(\lambda_n, 1)$ *and*

$$\frac{a_n}{\lambda_{n+1}-\lambda_n}\leq F(\lambda_{n+1}), \tag{2á}$$

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(6) Minakshi Sundaram: [5].

where $F(x)$ is monotone decreasing and

$$\limsup_{x=\infty} \operatorname*{Max}_{x \leqq x' \leqq tx} \int_x^{x'} F(\xi)\, d\xi = W_F(t).$$

Then

$$(16) \qquad \lambda \underline{\sigma} - \overline{\sigma} \leqq (\lambda - 1) \underline{s} + \int_1^{\lambda} W_F(t)\, dt \qquad (\lambda > 1),$$

$$(17) \qquad \theta \underline{\sigma} - \overline{\sigma} \leqq -(1-\theta) \overline{s} + \int_\theta^1 W_F\left(\frac{1}{t}\right) dt \qquad (0 < \theta < 1).$$

**Proof.** For $m > q$,

$$(18) \quad \frac{\lambda_{m+1}}{\lambda_{q+1}} \sigma_m - \sigma_q = \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) s_q + \left[\sum_{\nu=q+1}^{m} (\lambda_{\nu+1} - \lambda_\nu)(s_\nu - s_q)\right] \Big/ \lambda_{q+1},$$

where we may suppose that $s_n = \sum_{\nu=0}^{n} a_\nu$ and $q$, $m$ are as in the proof of (11).

As in the proof of Theorem 1,

$$s_\nu - s_q \leqq \int_{\lambda_{q+1}}^{\lambda_{\nu+1}} F(x)\, dx$$

$$\leqq W_F\left(\frac{\lambda_{\nu+1}}{\lambda_{q+1}}\right) + o(1) \qquad (q \to \infty).$$

Hence the second member of the right side of (18)

$$= \sum_{\nu=q+1}^{m} \left(\frac{\lambda_{\nu+1}}{\lambda_{q+1}} - \frac{\lambda_\nu}{\lambda_{q+1}}\right)(s_\nu - s_q)$$

$$\leqq \left[\left(\frac{\lambda_{q+2}}{\lambda_{q+1}} - 1\right) W_F\left(\frac{\lambda_{q+2}}{\lambda_{q+1}}\right) + \left(\frac{\lambda_{q+3}}{\lambda_{q+1}} - \frac{\lambda_{q+2}}{\lambda_{q+1}}\right) W_F\left(\frac{\lambda_{q+3}}{\lambda_{q+1}}\right) + \cdots + \left(\lambda - \frac{\lambda_m}{\lambda_{q+1}}\right) W_F\left(\frac{\lambda_{m+1}}{\lambda_{q+1}}\right)\right] +$$

$$+ \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - \lambda\right) W_F\left(\frac{\lambda_{m+1}}{\lambda_{q+1}}\right) + \left(\frac{\lambda_{m+1}}{\lambda_{q+1}} - 1\right) o(1) \qquad (q \to \infty).$$

As $q \to \infty$, the last expression $\to \int_1^{\lambda} W_F(t)\,dt$ so that the reasoning employed in the proof of (11) now establishes (16).

(17) can be obtained in the same way by refining the proof of (12).

Corollary 2. *If, in Theorem 2, $F(x) = \frac{H}{x}$, then*

$$e^{(\sigma - s)/H} - \frac{\bar{\sigma} - s}{H} - 1 \leqq 0, \tag{19}$$

$$e^{-(\bar{s} - \sigma)/H} + \frac{\bar{s} - \bar{\sigma}}{H} - 1 \leqq 0. \tag{20}$$

The proof of (19) depends on the fact that now $W_F(t) = H \log t$ and (16) can be expressed in the form $\Phi(\lambda) \leqq 0$. Choosing $\lambda$ to the most advantage, i.e., so that $\Phi(\lambda)$ is maximum, we obtain (19).

The proof of (20) is similar.

Remark. It is obvious that Lemma 1 is only a crude form of Theorem 2. Nevertheless, the proof of the lemma lays bare the mechanism behind the proof of the theorem.

It is easy to state a theorem which is related to Theorem 2. in the same way that Corollary 1.2 is related to Theorem 1. But this theorem is of less interest than the following double-conditioned modification of Theorem 2.

Theorem 3. *Let $\lambda_{n+1} \sim \lambda_n$. Let $\Sigma a_n$ be bounded $R(\lambda_n, 1)$ and*

$$f(\lambda_{n+1}) \leqq \frac{a_n}{\lambda_{n+1} - \lambda_n} \leqq F(\lambda_{n+1}), \tag{2'a}$$

*where $F(x)$ is monotone decreasing, $f(x)$ is monotone increasing and*

$$\limsup_{x=\infty} \operatorname{Max}_{x \leqq x' \leqq tx} \int_x^{x'} F(\xi)\,d\xi = W_F(t),$$

$$-\liminf_{x=\infty} \operatorname{Min}_{x \leqq x' \leqq tx} \int_x^{x'} f(\xi)\,d\xi = w_f(t).$$

*Then, for* $0 < \theta < 1 < \lambda$,

$$\lambda\bar{\sigma} - \theta\underline{\sigma} \geqq (\lambda - \theta)\bar{s} - \int_1^\lambda w_f(t)\,dt - \int_\theta^1 W_F\left(\frac{1}{t}\right)dt, \tag{21}$$

$$\theta\bar{\sigma} - \lambda\underline{\sigma} \geqq -(\lambda - \theta)\underline{s} - \int_1^\lambda W_F(t)\,dt - \int_\theta^1 w_f\left(\frac{1}{t}\right)dt. \tag{22}$$

The proofs of (21) and (22) are refined out of the proofs of (13) and (14) in the manner already indicated.

Corollary 3. *If, in Theorem 3,* $F(x) = \frac{H}{x}$ *and* $f(x) = -\frac{K}{x}$, *the most advantageous choice of* $\lambda$, $\theta$ *in (21), (22) gives:*

$$k(e^{(\bar{s}-\bar{\sigma})/K} - 1) + H(e^{-(\bar{s}-\underline{\sigma})/H} - 1) \leqq 0, \tag{23}$$

$$K(e^{-(\bar{\sigma}-\underline{s})/K} - 1) + H(e^{(\underline{\sigma}-\underline{s})/H} - 1) \leqq 0. \tag{24}$$

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